Master's Thesis

Computing Coefficients in Certain Normally Nonsingular Expansions

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March 2023

Abstract

In his paper [Ban20] Banagl derives explicit formulae for the coefficients in the linear combination (for which he coins the term *normally nonsingular expansion*) of the Goresky-MacPherson L-class in real codimension 4 of certain singular Schubert varieties. However, these formulae still contain non-trivial Kronecker products, namely Kronecker products of the pulled back cohomological Hirzebruch L-class of the normal bundle of some transversely intersecting smooth submanifold in an ambient Grassmannian (or cup products in which such L-classes appear) with the fundamental class of the underlying transverse intersection. The aim of this thesis is to develop general techniques to compute these concrete Kronecker products and therefore determine the coefficients in the related normally nonsingular expansions. In particular, we will determine the coefficients in the normally nonsingular expansion of $L_6(X_{3,2})$, i.e. the sixth Goresky-MacPherson L-class of the Schubert variety $X_{3,2}$. A great deal of these techniques will be to transform certain linear bases into one another: One interesting result we obtain in this regard is the novel *Epsilon-Algorithm*, which – as it turns out later – is a weaker version of a well-known theorem in algebraic geometry that Griffiths-Harris [GH78] call *Gauss-Bonnet (I)*.

Zusammenfassung

In seinem Paper [Ban20] leitet Banagl explizite Formeln für die Koeffizienten in der Linearkombination (für welche er den Begriff *normally nonsingular expansion* einführt) der Goresky-MacPherson-L-Klasse in reeller Kodimension 4 gewisser singulärer Schubert-Varietäten her. Allerdings enthalten diese Formeln nicht-triviale Kronecker-Produkte, und zwar Kronecker-Produkte der zurückgezogenen kohomologischen Hirzebruch-L-Klasse des Normalenbündels bestimmter transvers schneidender glatter Untermannigfaltigkeiten in umgebenden Graßmann-Mannigfaltigkeiten (oder Cup-Produkte, in denen solche L-Klassen auftauchen) mit der Fundamentalklasse des zugrunde liegenden transversen Schnittes. Das Ziel dieser Arbeit ist es, allgemeine Methoden zu entwickeln, mit denen man diese konkreten Kronecker-Produkte ausrechnen kann und damit die Koeffizienten in den zugehörigen normally nonsingular expansions bestimmen kann. Insbesondere werden wir die Koeffizienten in der normally nonsingular expansion von $L_6(X_{3,2})$, d.h. der sechsten Goresky-MacPherson-L-Klasse der Schubert-Verietät $X_{3,2}$, bestimmen. Ein bedeutender Teil dieser Methoden wird sein, gewisse lineare Basen ineindander zu transformieren: Ein interessantes Resultat, welches wir diesbezüglich erhalten, ist der gänzlich neue *Epsilon-Algorithmus*, der, wie sich später herausstellen wird, eine schwächere Version eines bekannten Theorems aus der algebraischen Geometrie ist, das Griffiths-Harris [GH78] *Gauss-Bonnet (I)* nennen.

Acknowledgements

First, I would like to express my gratitude to my advisor, Prof. Banagl, who suggested this topic for my master's thesis and introduced me to the wide, previously unknown to me, fields of enumerative geometry and flag manifolds as well as characteristic classes of singular spaces. The interesting combination of them made writing this thesis into an intriguing learning experience and the topic itself turned out to be quite fruitful, so much of the credit to be taken for the final results can be ascribed to his sophisticated advice.
Next, I have to thank Dominik Wrazidlo for the numerous discussions we had and all the useful tips he gave me in them. Thanks for taking the time for that.
Lastly, the whole process of writing a master's thesis of course would not be that fun without the necessary diversion but also – equally important – support (and let it be of moral kind). For that I want to sincerely thank all my friends and family, especially my girlfriend Amrei, my best friend Nils and my mother.

Contents

Chapter 0

Summary and Overview

Here we give a brief overview of what will be accomplished in this thesis and in which order we shall do so.

Chapter 1 will present all the necessary notation to understand the subsequent chapters and recapitulate basic topological and algebro-topological facts about Grassmannians and Schubert varieties as well as introduce the reader to Schubert calculus. In Sections 1.4 and 1.5 we furthermore give a very short outline of the Goresky-MacPherson L-class and Banagl's L-class Gysin formula, although one should consider this to be more of a motivation than a summary. The reason for that is that, even though we heavily rely on Banagl's final results, we will not need the machinery behind them anymore. Finally, in Section 1.5 we will describe the main goal of the thesis, which we nevertheless briefly summarize here as well: In Section 4 of his work [Ban20] Banagl applied one main result of that paper, an L-class Gysin formula, to derive explicit formulae for the coefficients in linear combinations of Goresky-MacPherson L-classes in real codimension 4 of certain (singular) Schubert varieties with respect to the basis of Schubert classes. Banagl introduced the notion of *normally nonsingular expansion* for such a linear combination. However, these formulae still contain non-trivial terms whose value it remains to determine[1]. Concretely, the formulae for the coefficients λ and μ in the normally nonsingular expansion

$$L_6(X_{3,2}) = \lambda\,[X_3] + \mu\,[X_{2,1}]$$

of the sixth Goresky-MacPherson L-class of the Schubert variety $X_{3,2}$ are

$$\begin{aligned}\lambda &= 1 + \langle\, L^1(\nu_{X_3'}|_{X_2''})\,,\,[X_2'']\,\rangle \quad ,\\ \mu &= \frac{2}{3} + \langle\, \omega \smile L^1(\nu_{X_{2,2}'}|_{X_{2,1}''})\,,\,[X_{2,1}'']\,\rangle \quad ,\end{aligned}$$

where the apostrophes in the notation indicate that the corresponding Schubert variety is with respect to a possibly different flag than the standard flag and $\omega \in H^2(X_{2,1}'';\mathbb{Q})$ is defined via

$$\langle\, \omega\,,\,[X_1'']\,\rangle = +1 \quad .$$

[1] The achievement of Banagl in [Ban20, Section 4] was to obtain formulae for the coefficients in the normally nonsingular expansion of the L-class that do not contain any singular characteristic classes anymore but only depend on ordinary cohomological characteristic classes, more specifically on the Hirzebruch L-class.

All of this, however, will be explained in full detail in Section 1.5. What remains is to calculate the concrete Kronecker products in the above formulae. Our main goal in this thesis is precisely to do so and thus to determine the values of λ and μ. The methods and techniques we develop along the way, however, are so general that we could compute arbitrary Kronecker products of similar form at the end. We will more or less bring this up again in Section 10.1 in the Outlook.

Chapter 2 will be the only genuine topological one in this work, aside from small parts in Chapter 8. The maps defined there should be completely standard[2]. Nonetheless, we will work quite carefully here because the considerations in this chapter form the topological basis for the rest of the thesis and will be applied quite often, e.g. in the vital Lemma 27 in Chapter 4 but also in Chapter 9.

In Chapter 3 we derive formulae for the inverse of a cohomological L-class of a smooth manifold and the cohomological L-class of a normal bundle (via the standard split exact sequence of vector bundles for the normal bundle) of a smooth submanifold in an ambient smooth manifold, albeit both of them should actually be well-known and not really new. These formulae will help simplifying the Kronecker products for λ and μ in the subsequent chapter.

We will start dealing with the actual equations for λ and μ in Chapter 4, where we use the previously obtained results from Chapters 2 and 3 to simplify the Kronecker products. The general idea is to "push" everything to the cohomology group of the ambient Grassmannian. The resulting formulae will depend on newly introduced coefficients of the first cohomological L-classes of specific Grassmannians with respect to the linear dual basis of Schubert classes[3] and we still have to determine them in the upcoming chapters. In the case of the equation for μ a further obstacle that remains is the appearing cup product on the left-hand side of the Kronecker product. Besides directly tackling our main goal for the first time, this chapter also provides two lemmata – Lemmata 25 and 27. They are rather easy to prove but turn out to be central in the context of restricting linear duals of Schubert classes from an ambient Grassmannian as base space to a Schubert subvariety (or more generally in the context of "pushing" certain linear duals of Schubert classes in the cohomology ring of a Schubert variety to the cohomology ring of another Schubert variety) and thus yield a more general view on the topic, away from the concrete setting. We will make use of both of them multiple times throughout the rest of the thesis.

In Chapter 5 we will consider a new linear basis of the $2i$-th cohomology group $H^{2i}(G)$ of the Grassmannian G. Up to that point, we will only have dealt with the linear dual basis of Schubert classes, where $H^{2i}(G)$ and $\mathrm{Hom}_{\mathbb{Q}}(H_{2i}(G), \mathbb{Q})$ are identified via the Kronecker isomorphism. Now we also consider the basis consisting of Poincaré duals of Schubert classes in $H_{2\dim(G)-2i}(G)$, as Schubert calculus and the (trivial) Proposition 28 – also from this chapter – allow us to then solve the problem of the appearing cup product in the equation for μ, at least if we were able to transform the linear dual basis of Schubert classes into the Poincaré dual basis and vice versa. We will indeed achieve computing the change-of-basis matrix of those two bases in the vital Corollary 30 [4].

With the significant insights from Chapter 5 we will be able to solve the already simplified equations for λ and μ in Chapter 6 – apart from determining the values of the newly introduced δ-coefficients: These will be defined as coefficients in the linear combination of the first Pontryagin class of (the tangent bundle of) a Grassmannian $G = G_k(\mathbb{C}^n)$ with respect to the Poincaré dual

[2] Although no references to any sources are given, as I still achieved the results by myself.

[3] Here we identify cohomology classes with linear dual maps via the Kronecker isomorphism from the Universal Coefficient Theorem.

[4] Although it turns out later the the bases of linear duals of Schubert classes and the basis of Poincaré duals of Schubert classes are identical, cf. Corollary 62. However, we will not make use of this in the rest of the thesis, as I originally was not aware of one important fact that allows to simplify Corollary 30 so drastically.

basis of Schubert classes:

$$p_1(G) =: \sum_{|\beta|=2} \delta^{\beta}_{k,n}\, PD(\sigma_\beta) \quad ,$$

(the Schubert class σ_β is defined as $[X_b]$ with b the complementary partition to β; cf. Section 1.3 for the definition of the *complementary partition*), see Equation (6.1). As this is a problem completely independent of our original task and setting, we may consider our main goal as solved after this chapter. The final equations for λ and μ are given in (6.7) and (6.11).

Chapter 7 deals with the necessary change-of-coefficients from rationals to integers of homology and cohomology groups since all considerations up to that point will have been with respect to rational coefficients but for determining the δ-coefficients it is indispensable to work with integer ones. Being completely self-contained, this chapter is rather independent of the rest of the thesis and mainly deals with basic algebraic topology. The expert reader or the reader who believes that the change-of-coefficients is fully well-behaved, should have no problem skipping it and directly continue with the next chapter.

It remains to determine the values of the δ-coefficients appearing in the final equations for λ and μ: This will be done in Chapter 8 and turns out to be quite an extensive task, at least if one pursues the classical approach (see below for what the non-classical one is). First, in Section 8.1, we deal with the cohomology groups of Grassmannians which are generated not only by the linear duals and the Poincaré duals of Schubert classes but also by the monomials in Chern classes of the canonical bundle. We present an own proof for Milnor-Stasheff's [MS74] *Problem 14-D* and give a slightly enhanced version of it. This enables us – up to an a priori difficult to determine sign – to transform monomials in Chern classes of the canonical bundle into products of linear duals of Schubert classes and thus (via the results from Chapter 5) into polynomials in Poincaré duals of Schubert classes. In more detail, our Corollary 47 (which is an adapted version of Problem 14-D) states

$$[X_{\underbrace{\underbrace{1,\ldots,1}_{i \text{ times}},0,\ldots,0}_{k \text{ times}}}]^\vee = (-1)^{\varepsilon_i}\, c_i(\gamma^k(\mathbb{C}^n)) \in H^{2i}(G_k(\mathbb{C}^n);\mathbb{Z}) \quad ,$$

where $\gamma^k(\mathbb{C}^n)$ denotes the canonical bundle over the Grassmannian $G_k(\mathbb{C}^n)$ and it remains to determine the sign $(-1)^{\varepsilon_i}$. Then, in Section 8.2, we finally determine the values of the relevant δ-coefficients by expressing the first Pontryagin class as the polynomial in Chern classes

$$p_1(G) = c_1(G)^2 - 2\, c_2(G)$$

and then invoking a formula by Borel-Hirzebruch in [BH58]. This yields the Chern class of a Grassmannian as a polynomial in Chern classes of the canonical bundle and by our adapted version of Milnor-Stasheff's Problem 14-D we thus can express the Chern class of a Grassmannian as polynomial in Poincaré duals of Schubert classes (and therefore, via Schubert calculus, as linear combination of Poincaré duals of Schubert classes), although there still remains the annoying sign $(-1)^{\varepsilon_i}$. In practice, this will only become relevant in one single case and we will then manually show that $(-1)^{\varepsilon_2} = +1$. Having invested all this work, we finally will be able to fully compute the values of λ and μ and it will turn out that

$$\lambda = \mu = \frac{2}{3} \quad ,$$

see Equation (8.13). Having already achieved our main goal, in the last section of this chapter, Section 8.3, we generalize the methods from the previous two sections for arbitrary δ-coefficients, i.e. coefficients defined via

$$p_i(G) = \sum_{|\beta|=2i} \delta^{\beta}_{k,n}\, PD(\sigma_\beta) \quad ,$$

mostly to emphasize that the central ideas in the previous sections do not require our specific setting at all. For that we develop an algorithm (and prove its correctness), Algorithm 50, which computes the coefficients of the j-th Chern class of a given Grassmannian G in the linear combination with respect to its Poincaré dual basis, i.e. it computes the coefficients κ_α in the linear combination

$$c_j(G) = \sum_{|\alpha|=j} \kappa_\alpha\, PD(\sigma_\alpha) \quad .$$

Algorithm 50 relies on another algorithm that computes the annoying sign $(-1)^{\varepsilon_i}$ for input i recursively. We develop this completely new algorithm, the *Epsilon-Algorithm*, in Subsection 8.3.2 and again prove its correctness. The basic ideas behind the Epsilon-Algorithm coincide with the ones for the manual proof of the statement that $(-1)^{\varepsilon_2} = +1$ in the previous section[5]. Unfortunately for me, quite some time after finishing the proof of the Epsilon-Algorithm, I stumbled across an apparently well-known result in classical algebraic geometry that Griffiths-Harris call *Gauss-Bonnet (I)*. It states that in fact $(-1)^{\varepsilon_i} = (-1)^i$ and of course renders the Epsilon-Algorithm useless in some ways. For more details in this regard, see Subsection 8.3.3, in which we also present an enhanced version of Corollary 30 (the change-of-basis lemma in Chapter 5). That is to say, we find out that the linear dual $[X_\alpha]^\vee$ is actually equal to the Poincaré dual $PD(\sigma_\alpha)$, see Corollary 62.

Instead of the more classical methods applied in Chapter 8, one can also determine the δ-coefficients via a theorem by Aluffi-Mihalcea in [AM08]. This is the content of Chapter 9. I only found out about the possibility of this alternative approach after I already had computed the relevant δ-coefficients with the techniques described in the previous chapter, when Dominik Wrazidlo recommended Aluffi-Mihalcea's paper to me[6] and I understood its implications. Thus – consistent with the original chronological order of progress – I decided to put this approach in second place, albeit it is in certain ways more direct than the one from Chapter 8. However, this decision also may be justified by the fact that, while Chapter 8 only relies on standard well-known facts (including Borel-Hirzebruch's formula) and otherwise is self-contained with some quite elaborate proofs of profound theorems, we will just apply Aluffi-Mihalcea's result without gaining deeper insights. Furthermore, Aluffi-Mihalcea work with Chern-Schwartz-MacPherson classes, i.e. the singular, homological equivalent of the Chern classes, and applying them seems a bit too much for our purposes, as we are only interested in the Chern classes of the (smooth) Grassmannians.

Lastly, in Chapter 10, we will present an outlook by examining the question whether our techniques developed in this thesis suit a remaining open problem in Banagl-Wrazidlo's recent work [BW22]: An intriguing secondary result of their Theorem 7.1 is a recursive formula for determining the coefficients in normally nonsingular expansions of arbitrary Gysin coherent characteristic classes in ambient Grassmannians. However, this formula still contains terms that are difficult to evaluate

[5] Even though we switch the ambient Grassmannian in which we work: While in the manual proof (for $i = 2$) we will work in $G_2(\mathbb{C}^5)$, for the Epsilon-Algorithm with arbitrary input i one works in the Grassmannian $G_i(\mathbb{C}^{i+1})$. It actually is an interesting open question whether one could determine the sign in any Grassmannian $G_i(\mathbb{C}^{i+m})$ for $m > 0$; take a look at the proof of the Epsilon-Algorithm why this is considerably harder to see than for the case $m = i + 1$.

[6] and I would like to express my gratitude to him for that

in practice: Besides the *genera of characteristic subvarieties*, which we will not deal with, the terms of *normally nonsingular integration* remain to determine. Since these are expressible as certain Kronecker products (cf. [BW22, Rem. 7.9]), similar to ones in the equations for λ and μ that we will evaluate successfully in this thesis, the question arises whether analogous tricks can be applied to generally compute the normally nonsingular integration. However, in Section 10.3 we will give substantial arguments, including an example, that this is not the case and that it probably requires additional methods or an outright different approach to solve the problem of the unknown terms of normally nonsingular integration; though there is one exemption for which our tricks actually work: This will be discussed first in Section 10.1.

Chapter 1

Introduction and Notation

1.1 Grassmannians and Schubert Varieties

If not otherwise mentioned, in the following all vector spaces are tacitly assumed to be complex and $\dim(V)$ denotes the complex dimension of the complex vector space V. Furthermore, by the dimension of a complex manifold we mean its complex dimension.

Definition 1. Given an n-dimensional complex vector space V and $0 \leq k \leq n$. The *Grassmannian* $G_k(V)$ is defined as

$$G_k(V) := \left\{ W \subseteq V \,\middle|\, W \text{ } k\text{-dimensional linear subspace of } V \right\} \quad .$$

In the following, we are only interested in Grassmannians of the form $G_k(\mathbb{C}^n)$, so V is simply $\mathbb{C}^n$. The *open Stiefel manifold* $V_k(\mathbb{C}^n) \subseteq \mathbb{C}^{n\times k}$ is the set of all linearly independent vectors $(v_1, \ldots, v_k)$ in $\mathbb{C}^n$. It is an open subset of $\mathbb{C}^{n\times k}$. The map

$$\mathrm{quot} : V_k(\mathbb{C}^n) \twoheadrightarrow G_k(\mathbb{C}^n) \quad , \quad (v_1, \ldots, v_k) \mapsto \mathrm{span}(v_1, \ldots, v_k)$$

is surjective. We endow the Grassmannian $G_k(\mathbb{C}^n)$ with the quotient topology induced by quot, where $V_k(\mathbb{C}^n)$ inherits the subspace topology from $\mathbb{C}^{n\times k}$.

Remark 2. The Grassmannian $G_k(\mathbb{C}^n)$ is a compact topological manifold of real dimension $2\,k\,(n-k)$. It carries a unique smooth structure such that $\mathrm{quot} : V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$ becomes a smooth submersion[1]. In fact, $G_k(\mathbb{C}^n)$ has a canonical complex structure[2], turning it into a complex manifold of complex dimension $\dim(G_k(\mathbb{C}^n)) = k\,(n-k)$, with underlying smooth structure the one we just specified.

All of the above facts can easily be found in the literature, cf. [MS74, Chapt. 5, 6, 14], [Lee03, Example 1.36] and [Kar12, pp. 123-136] for a good overview in that regard.

Now let us define a canonical cell subdivision for the Grassmannian $G_k(\mathbb{C}^n)$, with respect to which the Grassmannian is a finite CW-complex. The cells we construct are called *Schubert cells*

[1]Uniqueness of such a smooth structure follows from the universal property of surjective submersions (cf. [Lee03, Thm. 4.31]). Here is a reference for the existence statement (for the analogous real case though): [Kar12, p. 136] (right before the new subsection B.4; π is defined in Equation (B.2), p. 124, and $\mathrm{St}(k, \mathbb{R}^n)$ on p. 123).

[2]see e.g. [MS74, p. 159]

and their closures will play a major role throughout the rest of this thesis. In the following, we adopt the notation from Banagl [Ban20] and Banagl-Wrazidlo [BW22] to later on allow for a direct transfer of our results to their setting of interest.

Definition 3. A *(complete) flag* $F_* = (F_i)_{i=0}^n$ in $\mathbb{C}^n$ is a family of linear subspaces

$$0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_{n-1} \subset F_n = \mathbb{C}^n$$

with $\dim(F_i) = i$ for all $i = 0, \ldots, n$.

Definition 4. The *standard flag* F_*^{std} in $\mathbb{C}^n$ is defined by $F_i^{\text{std}} := \mathbb{C}^i := \mathbb{C}^i \times \{0\}^{n-i} \subseteq \mathbb{C}^n$ for all i.

Remark 5. Given $g \in \mathrm{GL}(\mathbb{C}^n)$ and a flag F_* in $\mathbb{C}^n$, then $g \cdot F_* := \big(g(F_i)\big)_{i=0}^n$ is also a flag in $\mathbb{C}^n$. Conversely, given arbitrary flags F_* and F'_*, choose bases $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ of $\mathbb{C}^n$ such that $F_i = \mathrm{span}(v_1, \ldots, v_i)$ and $F'_i = \mathrm{span}(v'_1, \ldots, v'_i)$ for all i and define $g \in \mathrm{GL}(\mathbb{C}^n)$ via $g(v_i) := v'_i$. Then $g \cdot F_* = F'_*$. Thus the left action of $\mathrm{GL}(\mathbb{C}^n)$ on the set of flags in $\mathbb{C}^n$ is transitive.

Definition 6 (Partitions)**.** A *partition (of length k)* is a sequence $a = (a_1, \ldots, a_k)$ of nonnegative integers such that

$$a_1 \geq a_2 \geq \ldots \geq a_k \geq 0 \quad .$$

We will sometimes also call such a sequence a *nonincreasing, nonnegative sequence of integers.* For such a partition put

$$|a| := a_1 + \ldots + a_k = \sum_{i=1}^{k} a_i \quad .$$

We say that a *is a partition of* m if $|a| = m$.
For partitions a, b of length k we write $b \leq a$ (or $a \geq b$) if $b_i \leq a_i$ for all $i = 1, \ldots, k$.
Lastly, for integers m, $k \geq 0$ we put

$$\mathcal{P}(m,k) := \left\{ a = (a_1, \ldots, a_k) \in \mathbb{Z}^k \,\middle|\, m \geq a_1 \geq \ldots \geq a_k \geq 0 \right\} \quad ,$$

the set of all partitions a of length k with $m \geq a_1$.

Definition 7 (Schubert varieties)**.** For a partition $a \in \mathcal{P}(n-k,k)$ and a flag F_* in $\mathbb{C}^n$ we define the *Schubert cell* $\mathring{X}_a(F_*) \subseteq G_k(\mathbb{C}^n)$ with respect to a and F_* by

$$\mathring{X}_a(F_*) := \left\{ P \in G_k(\mathbb{C}^n) \,\middle|\, \dim(P \cap F_{a_{k+1-i}+i}) = i \,,\, \dim(P \cap F_{a_{k+1-i}+i-1}) = i-1 \;\; \forall\, i = 1, \ldots, k \right\} \quad .$$

Its (topological) closure $X_a(F_*)$ in $G_k(\mathbb{C}^n)$ is called the *Schubert variety* with respect to a and F_* and is explicitly given by

$$X_a(F_*) = \left\{ P \in G_k(\mathbb{C}^n) \,\middle|\, \dim(P \cap F_{a_{k+1-i}+i}) \geq i \quad \forall\, i = 1, \ldots, k \right\} \quad .$$

If the flag in question is the standard flag F_*^{std}, then we often omit the flag in the notation and simply write $\mathring{X}_a$ respectively X_a.

Let us collect a few basic facts about Schubert varieties, which for instance can be found in Milnor-Stasheff [MS74, Chapt. 6] [3] and Aluffi-Mihalcea [AM08, Sections 1, 2.1, 2.2]: The family

[3]For the analogous real case. For the complex case Milnor-Stasheff at least confront the reader with *Problem 14-D*.

$\{\mathring{X}_a(F_*)\}_{a\in\mathcal{P}(n-k,k)}$ contains the open cells of a cell structure on $G_k(\mathbb{C}^n)$, so with precisely those cells the given Grassmannian is a finite CW-complex (cf. [MS74, Thm. 6.4]). The cell $\mathring{X}_a(F_*)$, belonging to the partition a, has cell dimension $2|a|$ (see [MS74, Problem 14-D]). It actually requires some work to see why the closure of a Schubert cell has the above stated form: One possible way is to note that above explicitly given set is the image of some characteristic map for the cell $\mathring{X}_a(F_*)$ [4]. The Schubert variety $X_a(F_*)$ is compact[5] and the (set-theoretic) disjoint union of all cells belonging to $b \leq a$ (cf. [AM08, p. 2]):

$$X_a(F_*) = \bigsqcup_{b\leq a} \mathring{X}_b(F_*) \quad .$$

From this, or from the explicit definition of the Schubert variety above, one directly sees that for partitions $b \leq a$ we have a closed embedding $X_b(F_*) \subseteq X_a(F_*)$. Additionally, above equation shows that any Schubert variety is a subcomplex of the ambient Grassmannian, since it is closed and a union of cells. Therefore a Schubert variety $X_a(F_*)$ is also a CW-complex, with the cell structure given by the collection of cells corresponding to $b \leq a$. Any Schubert variety $X_a(F_*)$ is a closed complex algebraic subvariety of $G_k(\mathbb{C}^n)$ of (complex) dimension $|a|$ (see e.g. [AM08, p. 7]). It is nonsingular as variety (and thus a complex manifold) if and only if the partition a is *rectangular*, i.e. it is of the form $a = (m, \dots, m, 0, \dots, 0)$ for some $m \leq n-k$ [6]. At last, notice that Grassmannians themselves are Schubert varieties: Namely, for the rectangular partition $a := (n-k, \dots, n-k)$ of length k we indeed have $X_a = G_k(\mathbb{C}^n)$.

1.2 Homology and Cohomology of Schubert Varieties

The natural left action of $\mathrm{GL}(\mathbb{C}^n)$ on the Grassmannian $G := G_k(\mathbb{C}^n)$ is given by $g \cdot P := g(P)$. Notice that for any partition $a \in \mathcal{P}(n-k,k)$ and any flag F_* we have $g \cdot X_a(F_*) = X_a(g \cdot F_*)$. Since the natural action of $\mathrm{GL}(\mathbb{C}^n)$ on the set of flags is transitive and $\mathrm{GL}(\mathbb{C}^n)$ is path-connected, given two flags F_*, F'_* we can find a continuous path $g(t)$ in $\mathrm{GL}(\mathbb{C}^n)$ with $g(0) = \mathbb{1}$ and $g(1) \cdot F_* = F'_*$. Hence $X_a(F'_*) = g(1) \cdot X_a(F_*)$, which shows that we can deform $X_a(F_*)$ into $X_a(F'_*)$ via an isotopy $G \times [0,1] \to G$. Thus, by homotopy invariance of homology, the homology classes $[X_a(F_*)]$ and $[X_a(F'_*)]$ in $H_*(G;\mathbb{Z})$ coincide[7]. We may therefore omit the flag in the notation of the homology class and simply write $[X_a]$ for $[X_a(F_*)]$ for any choice of flag F_* [8]. We call $[X_a] \in H_{2|a|}(G;\mathbb{Z})$ the *Schubert class* with respect to a. Applying the change-of-coefficients map $H_*(G;\mathbb{Z}) \to H_*(G;\mathbb{Q})$, we see that the rational homology class of a Schubert variety also does not depend on the flag, so again we will omit them from the notation. Generally speaking, all results we state here for integer (co)homology groups/classes of Grassmannians or general Schubert varieties will analogously hold for rational (co)homology groups/classes as well. Among other things, this will be verified in Chapter 7, although of course all of this is standard. When dealing with rational coefficients, we will also write $[X_a] \in H_{2|a|}(G;\mathbb{Q})$ for the image of the integer Schubert class $[X_a] \in H_{2|a|}(G;\mathbb{Z})$ under aforementioned canonical map $H_{2|a|}(G;\mathbb{Z}) \to H_{2|a|}(G;\mathbb{Q})$.

[4] Characteristic maps for Schubert cells (in the real case) are e.g. constructed explicitly in [MS74, Chapt. 6].

[5] It is the closure of a cell. Alternatively it is a closed subspace in a compact space. The first argument however has the advantage that it also works for the infinite Grassmannian $G_k(\mathbb{C}^\infty)$.

[6] cf. [BW22, Thm. 4.3], where the following result of Lakshmibai & Weyman is quoted: [LW90, Thm. 5.3, p. 203]

[7] cf. Banagl-Wrazidlo [BW22, Section 4.3]

[8] In particular for the choice of the standard flag F_*^{std}. Then this new notation is compatible with the already established one, where the Schubert variety is meant to be with respect to the standard flag if no other flag is specified.

In the following, we consider a fixed but arbitrary flag F_* and write X_a for $X_a(F_*)$ [9]. Since Schubert varieties are CW-complexes, their homology groups can be computed via cellular homology. Because all cells are even-dimensional, the homology groups in odd degrees vanish, i.e. $H_{2*+1}(X_a;\mathbb{Z}) = 0$. The Chow homology $A_*(X_a)$ of the Schubert variety X_a is freely generated by the classes $[X_b]$ for $b \le a$ (cf. [AM08, p. 7]). For any complex variety X there is a canonical *cycle map*

$$\mathrm{cl} : A_i(X) \longrightarrow H_{2i}^{\mathrm{BM}}(X;\mathbb{Z})$$

from Chow homology to Borel-Moore homology. If X is also compact, Borel-Moore homology and singular homology coincide, so this yields a cycle map

$$\mathrm{cl} : A_i(X) \longrightarrow H_{2i}(X;\mathbb{Z})$$

to singular homology, which is an isomorphism for any Schubert variety $X = X_a$ (cf. [BW22, pp. 9-10]), so $A_i(X_a) \cong H_{2i}(X_a;\mathbb{Z})$. Thus $H_{2i}(X_a;\mathbb{Z})$ is freely generated by the classes $[X_b]_{X_a}$, for partitions $b \le a$ with $|b| = i$. Alternatively, this could also be observed by the fact that X_a is a finite CW-complex with cells in only even dimension, so the cellular chain groups coincide with the homology groups and there is exactly one homology generator for each Schubert cell[10]. For $b \le a$, the inclusion $j : X_b \hookrightarrow X_a$ induces an injection on homology level: $j_* : H_{2i}(X_b;\mathbb{Z}) \hookrightarrow H_{2i}(X_a;\mathbb{Z})$ maps $[X_c]_{X_b}$ for $c \le b$, $|c| = i$, to $[X_c]_{X_a}$ [11]. The latter statement follows from the analogous one for Chow homology and naturality of the cycle map[12].

The Universal Coefficient Theorem (abbreviated: UCT) allows us to express any cohomology group of a Schubert Variety X_a as linear dual space of the corresponding homology group: The UCT for base ring $\mathbb{Z}$ states that

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}(H_{i-1}(X_a;\mathbb{Z}),\mathbb{Z}) \longrightarrow H^i(X_a;\mathbb{Z}) \xrightarrow{\mathrm{Kron}} \mathrm{Hom}_{\mathbb{Z}}(H_i(X_a;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0 \tag{1.1}$$

is a split exact sequence (which is also natural). The map from cohomology to the linear dual space of the homology group is called the *Kronecker map*. Since all homology groups of Schubert varieties are freely generated, the Ext-term always vanishes and the Kronecker map gives an isomorphism

$$\mathrm{Kron} : H^i(X_a;\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(H_i(X_a;\mathbb{Z}),\mathbb{Z}) =: H_i(X_a;\mathbb{Z})^\vee \quad ,$$

where we note that $\cdot^\vee$ is our notation for the $\mathbb{Z}$-linear (respectively $\mathbb{Q}$-linear) dual space of a $\mathbb{Z}$-module (respectively $\mathbb{Q}$-vector space). Similarly, the UCT with base ring $\mathbb{Q}$ [13] implies that the rational Kronecker map

$$\mathrm{Kron} : H^i(X_a;\mathbb{Q}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(H_i(X_a;\mathbb{Q}),\mathbb{Q}) =: H_i(X_a;\mathbb{Q})^\vee$$

is an isomorphism. We will make extensive use of this isomorphism throughout this thesis and often tacitly identify the cohomology groups of Schubert varieties (or other topological spaces for which the Ext-term in the UCT vanishes) with the linear dual spaces of the corresponding homology groups via the Kronecker map (both for integer and rational coefficients respectively).

[9] Although normally omitting the flag in the notation means that the flag is the standard one.

[10] And under the canonical isomorphism $H_{2i}^{\mathrm{cell}}(X_a;\mathbb{Z}) \cong H_{2i}(X_a;\mathbb{Z})$ the Schubert cell $\mathring{X}_b$ is mapped to $[X_b]_{X_a}$.

[11] For instance and to give a reference for this claim, this is an easy consequence of the facts stated in Banagl-Wrazidlo [BW22, Section 4.4] and functoriality.

[12] see Fulton [Ful84, Chapt. 19, p. 372] or [Mus, Rem. 3.7], which refers to Fulton but maybe presents the result a bit more accessible

[13] see e.g. [Hat01, pp. 196-197] for the general statement for principal ideal domains

1.3 Schubert Calculus

Throughout this section we will not explicate the coefficient ring $\mathbb{Z}$ or $\mathbb{Q}$ anymore since all mentioned facts will hold for either one. Let M be a closed oriented n-dimensional manifold with fundamental class $[M] \in H_n(M)$. Then Poincaré duality states that capping with the fundamental class gives an isomorphism

$$H^{n-i}(M) \xrightarrow{\sim} H_i(M) \quad , \quad \alpha \mapsto \alpha \frown [M]$$

and we denoty by $PD = PD_i : H_i(M) \xrightarrow{\sim} H^{n-i}(M)$ its inverse. We can therefore define the *intersection product* on homology via

$$\cdot : H_i(M) \otimes H_j(M) \to H_{i+j-n}(M)\ , \quad a \otimes b \mapsto a \cdot b := PD^{-1}_{i+j-n}\left(PD_i(a) \smile PD_j(b)\right) \quad .$$

This product on the graded homology group $H_*(M)$ is associative with identity element $[M]$. Furthermore $a \cdot b = (-1)^{(n-i)(n-j)}\, b \cdot a$ for $a \in H_i(M)\,, b \in H_j(M)$. In particular, the intersection product on $H_*(M)$ is commutative if $n = \dim_{\mathbb{R}}(M)$ is even and all odd dimensional homology groups of M vanish, which is indeed the case for M a nonsingular Schubert variety[14].

The intersection product on the Grassmannian $G := G_k(\mathbb{C}^n)$ is particularly well-behaved and allows for algorithmic computation via the so-called *Schubert calculus.* For this it is recommendable to introduce alternative but standard notation for Schubert classes: Consider the self-inverse bijection

$$\begin{aligned} \mathcal{P}(n-k,k) &\longrightarrow \mathcal{P}(n-k,k) \\ a = (a_1,\dots,a_k) &\longmapsto \alpha = (\alpha_1,\dots,\alpha_k)\ , \quad \alpha_i := (n-k) - a_{k+1-i}\ , \quad i = 1,\dots,k \quad . \end{aligned}$$

α is called the partition *complementary* to a. Notice that $|\alpha| + |a| = k\,(n-k) = \dim(G)$, so $|\alpha|$ is the codimension of the subvariety $X_a(F_*)$ in G.

Definition 8. Let $G = G_k(\mathbb{C}^n)$ be a Grassmannian. For a partition $\alpha \in \mathcal{P}(n-k,k)$ with complementary partition a, i.e. $a_i = (n-k) - \alpha_{k+1-i}$ for $i = 1,\dots,k$, put

$$\sigma_\alpha := [X_a] \in H_{2|a|}(G;\mathbb{Z})$$

respectively $\sigma_\alpha := [X_a] \in H_{2|a|}(G;\mathbb{Q})$ when dealing with rational coefficients.

Later we will sometimes confusingly refer to *the Schubert class correspondent to the partition a* and either mean $[X_a]$ or σ_a. However it will always be clear from the context which one we actually mean. Furthermore, for readability we try to primarily use Greek letters as indices for $\sigma_.$ and Latin ones for $[X_.]$, although of course there is no formal necessity for that. Additionally, we also allow partitions α of length k that do not lie in $\mathcal{P}(n-k,k)$, i.e. $\alpha_1 > n-k$, and set $\sigma_\alpha := 0 \in H_*(G;\mathbb{Z})$ for those ones.

Definition 9. A Schubert class σ_α is called *special* if α is of the form $\alpha = (\alpha_1, 0, \dots, 0)$ [15].

The intersection product of a special Schubert class with an arbitrary one can now be computed via *Pieri's formula*:

[14] e.g. a Grassmannian

[15] This is actually well-defined since the index of a Schubert class is unique, as $\{[X_a]\}_{a \in \mathcal{P}(n-k,k)}$ is a basis of $H_*(G)$.

Theorem 10 (Pieri's formula)**.** *For partitions $\alpha = (\alpha_1, 0, \ldots, 0)$, $\beta = (\beta_1, \ldots, \beta_k)$ of length k we have the following formula for the intersection product $\sigma_\alpha \cdot \sigma_\beta$ in $H_*(G_k(\mathbb{C}^n))$:*

$$\sigma_\alpha \cdot \sigma_\beta = \sum_\gamma \sigma_\gamma \quad ,$$

where $\gamma = (\gamma_1, \ldots, \gamma_k)$ ranges over all partitions of length k such that

$$\beta_1 \leq \gamma_1 \, , \quad \beta_i \leq \gamma_i \leq \beta_{i-1} \quad \forall\, i = 2, \ldots, k \, , \quad |\gamma| = |\alpha| + |\beta| = \alpha_1 + \sum_i \beta_i \quad .$$

Any Schubert class can be expressed as a polynomial in special Schubert classes due to *Giambelli's formula*[16]:

Theorem 11 (Giambelli's formula)**.**

$$\sigma_{\alpha_1,\ldots,\alpha_k} = \det \begin{pmatrix} \sigma_{\alpha_1} & \sigma_{\alpha_1+1} & \sigma_{\alpha_1+2} & \cdots & \cdots & \sigma_{\alpha_1+k-1} \\ \sigma_{\alpha_2-1} & \sigma_{\alpha_2} & \sigma_{\alpha_2+1} & \cdots & \cdots & \sigma_{\alpha_2+k-2} \\ \sigma_{\alpha_3-2} & \sigma_{\alpha_3-1} & \sigma_{\alpha_3} & \cdots & \cdots & \sigma_{\alpha_3+k-3} \\ \vdots & \vdots & \vdots & & & \vdots \\ \sigma_{\alpha_k-(k-1)} & \sigma_{\alpha_k-(k-1)+1} & \sigma_{\alpha_k-(k-1)+2} & \cdots & \cdots & \sigma_{\alpha_k} \end{pmatrix}$$

These two formulae suffice to be able to algorithmically compute arbitrary intersection products of Schubert classes in $H_*(G)$: Given partitions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_k)$, to compute $\sigma_\alpha \cdot \sigma_\beta$ first express σ_α as polynomial in special Schubert classes via Giambelli, then expand via distributive law and apply Pieri's formula (together with the distributive law) iteratively to each arising summand. The expression one obtains is a sum of Schubert classes with no occurring intersection product anymore.

All of the above can e.g. be found in Griffiths-Harris [GH78, Chapt. 1.5, pp. 197-206], especially Pieri's [GH78, p. 203] and Giambelli's [GH78, p. 205] formulae[17]. Our notation is fully compatible with the one there.

1.4 The Goresky-MacPherson L-Class

We will give no thorough introduction into characteristic classes of singular spaces, of which the *Goresky-MacPherson L-class* is a prominent example, because we will not make use of them at all in this thesis (apart from a minor appearance in Chapter 9 where we will explain everything necessary). However, since the main result of this thesis will be about a particular Goresky-MacPherson L-class of some Schubert variety, let us at least motivate them briefly. For a good compromise between a gentle introduction to and (a sketch of) the rigorous construction of those L-classes see Maxim's

[16]Sequences of shorter length than k have to be filled up with zeroes. If an entry of the sequence α is negative, then $\sigma_\alpha := 0$ (similar to the case of partitions with $\alpha_1 > n - k$).

[17]Caveat: It seems that not everything about Grassmannians stated in Griffiths-Harris, Chapter 1.5, is entirely correct. For instance, a major mistake is their definition of the Schubert cell $W_{a_1,\ldots,a_k}$ on p. 195, which plainly is false (on p. 196 it is said that these sets form a cell decomposition of the Grassmannian, but with their definition they are in general not even disjoint). However, the (main) results stated in Section *The Schubert Calculus* should be without flaws.

book *Intersection Homology & Perverse Sheaves* [Max19, Chapt. 3.3-3.4], of which the rest of this section comes from.

By definition, the cohomological Hirzebruch L-class corresponds to the power series $\sqrt{z}/\tanh(\sqrt{z})$ and (by the Hirzebruch signature theorem) its genus is $[M]\otimes 1 \mapsto \sigma(M)$ with $\sigma(M)$ the (ordinary) signature of M. For M a smooth manifold its i-th L-class is defined via its tangent bundle: $L^i(M) := L^i(TM) \in H^{4i}(M;\mathbb{Q})$. This of course makes generalizations to spaces other than smooth manifolds quite complicated. However, there is the following sophisticated trick: Thom constructed homological L-classes $L_{n-4i}(M) \in H_{n-4i}(M;\mathbb{Q})$ (for M a smooth closed oriented n-manifold) such that $L_{n-4i}(M)$ is Poincaré dual to $L^i(M)$, i.e.

$$L_{n-4i}(M) = L^i(M) \frown [M] \quad \forall i \quad .$$

For that he constructed a map

$$\sigma : \pi^{n-4i}(M) = [M, S^{n-4i}] \longrightarrow \mathbb{Z}$$

which is defined as follows[18]: For any homotopy class of maps $M \to S^{n-4i}$ into the sphere take a smooth representative f and a regular value $p \in S^{n-4i}$ of f. Then the homotopy class is mapped to $\sigma(f^{-1}(p))$, the signature of the $4i$-dimensional fiber $f^{-1}(p)$. Now rationalize this group homomorphism to obtain

$$\sigma \otimes \mathbb{Q} : \pi^{n-4i}(M) \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \quad .$$

Composing this with the inverse of the rational Hurewicz map

$$\pi^{n-4i}(M) \otimes \mathbb{Q} \xrightarrow{\sim} H^{n-4i}(M;\mathbb{Q})$$

yields a linear map

$$\sigma \otimes \mathbb{Q} : H^{n-4i}(M;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

and, via the Universal Coefficient Theorem, the desired class $L_{n-4i}(M) \in H_{n-4i}(M;\mathbb{Q})$.

Now, if X is a closed oriented n-dimensional PL[19] Witt space[20] and $f : X \to S^{n-4i}$ is a PL map, then one can show that for "almost all" $p \in S^{n-4i}$ the fiber $f^{-1}(p)$ can be stratified such that it is a closed oriented PL Witt space too (see [Max19, Prop. 3.4.1]). Therefore the Witt signature of $f^{-1}(p)$ is well-defined. This establishes a map

$$\sigma_{\text{Witt}} : \pi^{n-4i}(X) \longrightarrow \mathbb{Z} \quad .$$

Proceeding as before, we obtain an element in the (singular) homology $L_{n-4i}(X) \in H_{n-4i}(X;\mathbb{Q})$, the *Goresky-MacPherson L-class* of X. Since ordinary signature and Witt signature coincide for manifolds[21], the Goresky-MacPherson L-class coincides with the homological Hirzebruch L-class for smooth closed oriented manifolds, which shows that the Goresky-MacPherson L-class really is a generalization of the Hirzebruch L-class to certain non-manifolds[22].

[18] For technical reasons we do not explain here it is assumed that $4i < \frac{n-1}{2}$. However, it is possible to remove the dimension restriction by crossing with a high-dimensional sphere.

[19] i.e. *piecewise linear*

[20] For a definition of *Witt space* see [Max19, Def. 2.7.1].

[21] see [Max19, Prop. 2.7.5 (b)]

[22] E.g. all complex algebraic varieties are Witt spaces (see [Max19, Example 2.7.2]).

1.5 Banagl's L-Class Gysin Formula and its Applications

In [Ban20] Banagl established the following L-class Gysin formula:

Theorem 12 (Thm. 3.18 in [Ban20])**.** *Let $g : Y \hookrightarrow X$ be a (real) codimension c normally nonsingular inclusion*[23] *of closed oriented even-dimensional PL Witt pseudomanifolds with corresponding Gysin map*[24] *$g^! : H_*(X;\mathbb{Q}) \to H_{*-c}(Y;\mathbb{Q})$. Let ν be the topological normal bundle of g. Then*

$$g^! L_*(X) = L^*(\nu) \frown L_*(Y) \quad .$$

A direct consequence is the following result:

Corollary 13 (Example 3.19 in [Ban20])**.**

$$g^![X] = [Y] \quad ,$$

i.e. the Gysin map sends the fundamental class of X to the fundamental class of Y.

which itself allowed for the proof of

Proposition 14 (Prop. 2.5 in [BW22][25])**.** *Let W be an oriented smooth manifold, $K \subseteq X \subseteq W$ Whitney stratified oriented pseudomanifolds with K compact. Let $M \subseteq W$ be an oriented smooth submanifold which is closed as a subset and transverse to the Whitney strata of X and of K. Then the Gysin map*

$$g^! : H_*(X;\mathbb{Q}) \to H_{*-c}(Y;\mathbb{Q})$$

associated to the (real) codimension c normally nonsingular embedding $g : Y = M \cap X \hookrightarrow X$ fulfills

$$g^![K]_X = [K \cap Y]_Y \quad ,$$

where $[K]_X$ and $[K \cap Y]_Y$ are the fundamental classes respectively.

Combining his L-class Gysin formula (Theorem 12) with Proposition 14 and skilfully choosing nonsingular Schubert varieties, transverse to particular subspaces, Banagl was able to compute the *normally nonsingular expansion* of the Goresky-MacPherson L-class of certain (singular) Schubert varieties in certain degrees (Section 4 in [Ban20] or Sections 6-9 in the, unfortunately unpublished but remarkably clearly written, notes [BW21]). More precisely, he showed[26]

$$L_2(X_{2,1}) = \frac{2}{3}\,[X_1] \,\in H_2(X_{2,1};\mathbb{Q})$$

and derived explicit formulae, eliminating all appearing Goresky-MacPherson L-classes, for the coefficients λ, μ in the linear combination

$$\boxed{L_6(X_{3,2}) = \lambda\,[X_3] + \mu\,[X_{2,1}]} \tag{1.2}$$

[23] see [Ban20, Def. 3.3] for a definition

[24] see [Ban20, p. 1280] for a definition

[25] Although not mentioned explicitly in [Ban20], Banagl already had derived this proposition prior to [BW22], namely he applied it in Section 4 of his earlier work [Ban20].

[26] see Equation (4.1) in [Ban20]

of the sixth Goresky-MacPherson L-class of the Schubert variety $X_{3,2}$ [27]. Namely, these formulae are given as follows:
Work inside the Grassmannian $P := G_2(\mathbb{C}^5)$ and put $X := X_{3,2} = X_{3,2}(F_*^{\mathrm{std}}) \subset P$. Then the following two relations hold[28]:

$$\boxed{\begin{array}{l} \lambda = 1 + \langle\, L^1(\nu_{M_1}|_{Y_1})\, ,\, [Y_1]_{Y_1}\, \rangle \\[2em] M_1 := X_3(F_*^1) \;\; , \quad Y_1 := X_2(F_*^2) = M_1 \cap X \end{array}} \tag{1.3}$$

$$\boxed{\begin{array}{l} \mu = \dfrac{2}{3} + \langle\, \omega \smile L^1(\nu_{M_2}|_{Y_2})\, ,\, [Y_2]_{Y_2}\, \rangle \\[2em] M_2 := X_{2,2}(F_*^3) \;\; , \quad Y_2 := X_{2,1}(F_*^4) = M_2 \cap X \\ \omega \in H^2(Y_2;\mathbb{Q}) \text{ defined by } \langle\, \omega\, ,\, [X_1(F_*^4)]_{Y_2}\, \rangle = +1 \end{array}} \tag{1.4}$$

with suitably chosen flags $F_*^1, \ldots, F_*^4$ in $\mathbb{C}^5$ and all spaces are considered subspaces of $P = G_2(\mathbb{C}^5)$. Here ν_{M_i} denotes the normal bundle of the manifold M_i [29] in P and $L^1(\nu_{M_i}|_{Y_i})$ is the (ordinary) first Hirzebruch L-class of the bundle $\nu_{M_i}|_{Y_i} = (Y_i \hookrightarrow M_i)^* \nu_{M_i}$, the pullback of the normal bundle along the inclusion. Furthermore, $\langle\, \cdot\, ,\, \cdot\, \rangle$ stands for the usual Kronecker pairing and ω is uniquely determined by above equation because $H_2(Y_2;\mathbb{Q})$ is a 1-dimensional $\mathbb{Q}$-vector space with basis $[X_1(F_*^4)]_{Y_2}$ and the Kronecker map is an isomorphism due to the UCT.

Main Goal. The main goal of this thesis is to actually compute the values of λ and μ, defined via Equation (1.2), starting from Equations (1.3) and (1.4).

In Chapter 6 we will reduce this problem to the task of determining particular so-called δ-coefficients, which we will introduce later and which are completely independent of the present situation. The resulting formulae (6.7) and (6.11) can already be considered a solution to our main goal. Then the δ-coefficients will be calculated in Chapter 8, with the final values of λ and μ being obtained in Equation (8.13) as

$$\lambda = \mu = \frac{2}{3} \quad .$$

So, in view of Equation (1.2), we have

$$L_6(X_{3,2}) = \frac{2}{3}\,[X_3] + \frac{2}{3}\,[X_{2,1}] \quad .$$

[27] As no flag is specified, all Schubert varieties are meant to be with respect to the standard flag F_*^{std} (in $\mathbb{C}^5$). Obviously, however, the coefficients λ and μ are independent of the flag chosen, which e.g. follows directly from the subsequent Lemma 17 in Chapter 2 and since $\varphi_*\, L_*(X_{3,2}) = L_*(X_{3,2}(F_*))$ for any flag F_*, where $\varphi : X_{3,2} \xrightarrow{\sim} X_{3,2}(F_*)$ is from Lemma 17. More generally, by Corollary 19, the coefficients are independent of both the ambient Grassmannian and the flag.

[28] These can be found at the end of Section 4 in [Ban20].

[29] $(3,0)\,,\,(2,2)$ are rectangular partitions, therefore $M_1\,,\, M_2$ are smooth manifolds.

Chapter 2

Constructing Canonical Maps between Grassmannians

Recall that the Grassmannian $G_k(\mathbb{C}^n)$ is topologized as follows: Consider the *open Stiefel manifold* $V_k(\mathbb{C}^n) \subseteq \mathbb{C}^{n\times k}$. This is the set of linearly independent k-tuples of vectors in $\mathbb{C}^n$ and is an open subset of $\mathbb{C}^{n\times k}$. We have a canonical map quot : $V_k(\mathbb{C}^n) \twoheadrightarrow G_k(\mathbb{C}^n)$ which sends a linearly independent k-tuple of vectors to the subspace it spans in $\mathbb{C}^n$. We endow $G_k(\mathbb{C}^n)$ with the quotient topology given by quot.

Fix $k \le n$ and $0 \le l \le m$. For a partition $a = (a_1, \dots, a_k) \in \mathcal{P}(n-k, k)$, i.e. $a_1 \le n-k$, put

$$(a, \underline{0}) := (a_1, \dots, a_k, \underbrace{0, \dots, 0}_{l \text{ times}}) \in \mathcal{P}((n+m) - (k+l), k+l) \quad .$$

For any such partition a we consider the Schubert varieties $X_a \subseteq G_k(\mathbb{C}^n)$ and $X_{(a,\underline{0})} \subseteq G_{k+l}(\mathbb{C}^{n+m})$ with respect to the standard flag respectively. Additionally, let us introduce the *shift by l*:

$$\mathrm{sh}_l : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+m} \, , \; (x_1, \dots, x_n) \mapsto (\underbrace{0, \dots, 0}_{l \text{ times}}, x_1, \dots, x_n, \underbrace{0, \dots, 0}_{(m-l) \text{ times}}) \quad .$$

This is a $\mathbb{C}$-linear monomorphism. Below we will be slightly informal and write $\mathbb{C}^l \oplus V$, where V is some k-dimensional subspace of $\mathbb{C}^n$, when we speak of $\mathbb{C}^l \oplus \mathrm{sh}_l(V) \subseteq \mathbb{C}^{n+m}$.

Lemma 15 (cf. *Problem 6-C* in [MS74])**.** *$k \le n$ and $0 \le l \le m$ arbitrary. Then*

$$\iota : G := G_k(\mathbb{C}^n) \hookrightarrow G' := G_{k+l}(\mathbb{C}^{n+m}) \; , \quad V \longmapsto \mathbb{C}^l \oplus V$$

is a closed embedding with the property that

$$\iota(X_a) = X_{(a,\underline{0})}$$

for all partitions $a \in \mathcal{P}(n-k, k)$. Consequently

$$\iota_* \, [X_a]_G = [X_{(a,\underline{0})}]_{G'} \;\in H_{2|a|}(G'; \mathbb{Z}) \quad .$$

Proof. First notice that indeed $\dim_{\mathbb{C}}(\mathbb{C}^l \oplus V) = l + k$. Also ι is injective since $\mathrm{pr}(\mathbb{C}^l \oplus V) = V$, where $\mathrm{pr} : \mathbb{C}^{n+m} \to \mathbb{C}^n$ is the projection which maps $(x_1, \ldots, x_{n+m})$ to $(x_{l+1}, \ldots, x_{l+n})$.

Now let us deal with the continuity of ι: We have the following commutative diagram

$$\begin{tikzcd} V_k(\mathbb{C}^n) \arrow[r, "I"] \arrow[d, two heads, "\mathrm{quot}"'] & V_{k+l}(\mathbb{C}^{n+m}) \arrow[d, two heads, "\mathrm{quot}"] \\ G_k(\mathbb{C}^n) \arrow[r, hook, "\iota"'] & G_{k+l}(\mathbb{C}^{n+m}) \end{tikzcd} \tag{2.1}$$

where I is defined as follows: For the k vectors $v_1, \ldots, v_k$ in $\mathbb{C}^n$ let

$$I(v_1, \ldots, v_k) := (e_1, \ldots, e_l, \mathrm{sh}_l(v_1), \ldots, \mathrm{sh}_l(v_k))$$

with $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the i-th standard vector in $\mathbb{C}^{n+m}$. Since quot is a quotient map, it suffices to prove continuity of I. But clearly I is the restriction of the continuous map $\mathbb{C}^{n \times k} \longrightarrow \mathbb{C}^{(n+m)\times(k+l)}$, given by

$$\mathbf{A} = \begin{pmatrix} v_1^1 & \cdots & v_k^1 \\ \vdots & & \vdots \\ v_1^n & \cdots & v_k^n \end{pmatrix} \longmapsto \begin{pmatrix} \mathbb{1}_l & 0 \\ 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & v_1^1 & \cdots & v_k^1 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & v_1^n & \cdots & v_k^n \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ \underbrace{0 \quad \cdots \quad \cdots \quad 0}_{l \text{ columns}} & & & & 0 & \cdots & 0 \end{pmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\0\\\vdots\\0\end{matrix}}\right\} l \text{ rows} \\ \left.\vphantom{\begin{matrix}0\\\vdots\\0\end{matrix}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{matrix}0\\\vdots\\0\end{matrix}}\right\} m-l \text{ rows} \end{array}$$

Since $G = G_k(\mathbb{C}^n)$ is compact and $G' = G_{k+l}(\mathbb{C}^{n+m})$ is Hausdorff, we can immediately conclude that ι is a closed embedding.

Now let us prove $\iota(X_a) = X_{(a,\underline{0})}$ for all nonincreasing, nonnegative sequences $a = (a_1, \ldots, a_k)$ with $a_1 \leq n - k$: Put $a_{k+1} := \ldots := a_{k+l} := 0$.

$$\begin{aligned} X_{a_1,\ldots,a_k,0,\ldots,0} &= \left\{ V \in G_{k+l}(\mathbb{C}^{n+m}) \,\middle|\, \forall i = 1, \ldots, k+l : \dim(V \cap \mathbb{C}^{a_{(k+l)+1-i}+i}) \geq i \right\} \\ &= \left\{ V \in G_{k+l}(\mathbb{C}^{n+m}) \,\middle|\, \begin{array}{l} \forall i = 1, \ldots, l : \dim(V \cap \mathbb{C}^i) \geq i \text{ and} \\ \forall i = l+1, \ldots, l+k : \dim(V \cap \mathbb{C}^{a_{(k+l)+1-i}+i}) \geq i \end{array} \right\} \\ &= \left\{ V \in G_{k+l}(\mathbb{C}^{n+m}) \,\middle|\, \begin{array}{l} \forall i = 1, \ldots, l : \mathbb{C}^i \subseteq V \text{ and} \\ \forall i = 1, \ldots, k : \dim(V \cap \mathbb{C}^{a_{(k+l)+1-(l+i)}+(l+i)}) \geq (l+i) \end{array} \right\} \\ &= \left\{ V \in G_{k+l}(\mathbb{C}^{n+m}) \,\middle|\, \begin{array}{l} \mathbb{C}^l \subseteq V \text{ and} \\ \forall i = 1, \ldots, k : \dim(V \cap \mathbb{C}^{a_{k+1-i}+l+i}) \geq l+i \end{array} \right\} \end{aligned}$$

Suppose $V \in X_a$. Then $\dim(V \cap \mathbb{C}^{a_{k+1-i}+i}) \geq i \ \forall\, i = 1, \ldots, k$. Clearly $\mathbb{C}^l \subseteq \iota(V) = \mathbb{C}^l \oplus V$. For $i = 1, \ldots, k$ we have:

$$\begin{aligned}
\dim(\iota(V) \cap \mathbb{C}^{a_{k+1-i}+l+i}) &= \dim((\mathbb{C}^l \oplus \mathrm{sh}_l(V)) \cap \mathbb{C}^{a_{k+1-i}+l+i}) \\
&= \dim(\mathbb{C}^l \oplus (\mathrm{sh}_l(V) \cap \mathrm{sh}_l(\mathbb{C}^{a_{k+1-i}+i}))) \\
&= l + \dim(\mathrm{sh}_l(V \cap \mathbb{C}^{a_{k+1-i}+i})) \\
&= l + \dim(V \cap \mathbb{C}^{a_{k+1-i}+i}) \qquad (2.2)
\end{aligned}$$

which proves that $\dim(\iota(V) \cap \mathbb{C}^{a_{k+1-i}+l+i}) \geq l + i$. Thus $\iota(V) \in X_{(a,\underline{0})}$.

Conversely suppose $\widetilde{V}$ is in $X_{(a,\underline{0})}$. Define $V := \mathrm{pr}(\widetilde{V})$ with the projection map pr introduced above. Since $\dim(\widetilde{V} \cap \mathbb{C}^{a_1+l+k}) \geq l + k$ and $\widetilde{V}$ is a $(k+l)$-dimensional subspace, we deduce $\widetilde{V} \subseteq \mathbb{C}^{a_1+l+k} \subseteq \mathbb{C}^{n-k+l+k} = \mathbb{C}^{n+l}$. This shows $\mathbb{C}^l = \ker(\mathrm{pr}|_{\widetilde{V}})$ and therefore $\dim V = \dim \widetilde{V} - \dim \ker(\mathrm{pr}|_{\widetilde{V}}) = (k+l) - l = k$, so $V \in G_k(\mathbb{C}^n)$. Clearly $\iota(V) \subseteq \widetilde{V}$ and by equality of dimensions we even have $\iota(V) = \widetilde{V}$. Additionally $V \in X_a$, since by Equation (2.2) we have $l + \dim(V \cap \mathbb{C}^{a_{k+1-i}+i}) = \dim(\widetilde{V} \cap \mathbb{C}^{a_{k+1-i}+l+i}) \geq l + i$ for all $i = 1, \ldots, k$.

It remains to show $\iota_*[X_a]_G = [X_{(a,\underline{0})}]_{G'}$ for all partitions a. Fix $a = (a_1, \ldots, a_k)$ and let $r := |a| = \sum_{i=1}^k a_i$. By what we have proven already, we know that $\iota| : X_a \xrightarrow{\sim} \iota(X_a) = X_{(a,\underline{0})}$ is a homeomorphism. Since the map on homology, induced by the inclusion $X_a \hookrightarrow G$, maps $[X_a]_{X_a}$ to $[X_a] := [X_a]_G$ and analogously for $[X_{(a,\underline{0})}]_{X_{(a,\underline{0})}}$, it suffices to prove $\iota|_*[X_a]_{X_a} = [X_{(a,\underline{0})}]_{X_{(a,\underline{0})}} \in H_{2r}(X_{(a,\underline{0})}; \mathbb{Z})$. Now $H_{2r}(X_a) := H_{2r}(X_a; \mathbb{Z}) \cong \mathbb{Z}$ is generated by $[X_a]_{X_a}$ and similarly $H_{2r}(X_{(a,\underline{0})}) \cong \mathbb{Z}$ is generated by $[X_{(a,\underline{0})}]_{X_{(a,\underline{0})}}$. Since $\iota|_*$ is an isomorphism, this proves $\iota|_*[X_a]_{X_a} = \pm [X_{(a,\underline{0})}]_{X_{(a,\underline{0})}}$. To determine the sign, we switch over to Chow homology: Let $A_*(X)$ denote the Chow group with respect to some variety X. By naturality of the cycle map (see Section 1.2) we have a commutative diagram of the following form[1]

$$\begin{tikzcd}
A_r(X_a) \arrow[r, "\iota|_*"] \arrow[d, "\cong"', "\mathrm{cl}"] & A_r(X_{(a,\underline{0})}) \arrow[d, "\cong"', "\mathrm{cl}"] \\
H_{2r}(X_a) \arrow[r, "\cong", "\iota|_*"'] & H_{2r}(X_{(a,\underline{0})})
\end{tikzcd}$$

Now by definition of the pushforward in Chow homology[2], we have

$$\iota|_*[X_a]_{X_a} = \deg(X_a \,/\, \iota(X_a))\ [\iota(X_a)]_{X_{(a,\underline{0})}}$$

where $\deg(X_a \,/\, \iota(X_a))$ is the field extension degree of the corresponding function fields. In particular this degree is always positive. Comparing coefficients of the free generator $[X_{(a,\underline{0})}]_{X_{(a,\underline{0})}}$ yields $\iota|_*[X_a]_{X_a} = [X_{(a,\underline{0})}]_{X_{(a,\underline{0})}}$, as desired. $\square$

[1] Note that ι is a proper map since G is compact and G' is Hausdorff. Because ι is injective, for $K \subseteq \iota(X_a)$ compact it holds $\iota|^{-1}(K) = \iota^{-1}(K)$, so $\iota|$ is a proper map as well. It follows that $\iota|$ is a proper morphism of varieties, so the pushforward of $\iota|$ with respect to Chow homology really is defined. Alternatively, just look at [Mus, Rem. 3.7]: Every morphism of complete varieties is proper.

[2] see e.g. [Ful84, pp. 11-12] or [Mus, p. 2]

Remark 16. Using the fact that the canonical projection quot : $V_k(\mathbb{C}^n) \twoheadrightarrow G_k(\mathbb{C}^n)$ also is a smooth submersion[3], Diagram (2.1) in the above proof even shows that ι is smooth since the map $I : V_k(\mathbb{C}^n) \to V_{k+l}(\mathbb{C}^{n+m})$ obviously is smooth. Furthermore, ι is an immersion, so the restriction $\iota| : G_k(\mathbb{C}^n) \xrightarrow{\sim} \iota(G_k(\mathbb{C}^n))$ is a diffeomorphism. Hence, for any rectangular partition $a = (a_0, \ldots, a_0, 0 \ldots, 0) \in \mathcal{P}(n-k, k)$ with $a_0 > 0$, the restriction $\iota| : X_a \xrightarrow{\sim} \iota(X_a) = X_{(a,\underline{0})}$ is a diffeomorphism too[4]. To see that ι is an immersion, look again at Diagram (2.1): It suffices to show, that for any matrix $\mathbf{A} \in V_k(\mathbb{C}^n)$ and any tangent vector $\xi \in T_{\mathbf{A}} V_k(\mathbb{C}^n) \cong \mathbb{C}^{nk}$ we have

$$D_{\mathbf{A}}(\mathrm{quot} \circ I)\,\xi = 0 \implies \xi \in \ker(D_{\mathbf{A}}\mathrm{quot}) \quad ,$$

where D denotes the differential. So assume $D_{\mathbf{A}}(\mathrm{quot} \circ I)\,\xi = 0$. The natural right action of $\mathrm{GL}(k)$ on $V_k(\mathbb{C}^n)$ by right-multiplication is smooth and free and its quotient is $G_k(\mathbb{C}^n)$. Hence $\ker(D_{\mathbf{A}}\mathrm{quot}) = T_{\mathbf{A}}(\mathbf{A} \cdot \mathrm{GL}(k))$, where $\mathbf{A} \cdot \mathrm{GL}(k)$ is the orbit through $\mathbf{A}$ [5]. Now choose a path $t \mapsto \mathbf{A}(t)\,,\ \mathbf{A}(0) = \mathbf{A}\,,$ in $V_k(\mathbb{C}^n)$ representing ξ. Since $[t \mapsto I(\mathbf{A}(t))] = D_{\mathbf{A}} I\, \xi \in \ker(D_{I(\mathbf{A})}\mathrm{quot}) = T_{I(\mathbf{A})}(I(\mathbf{A}) \cdot \mathrm{GL}(k+l))$, we can choose a smooth path $t \mapsto \mathbf{M}(t)\,,\ \mathbf{M}(0) = \mathbb{1}_{k+l}\,,$ in $\mathrm{GL}(k+l)$ with

$$\left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \mathbb{1}_l & 0 \\ 0 & \mathbf{A}(t) \\ 0 & 0 \end{pmatrix} = \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \mathbb{1}_l & 0 \\ 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix} \cdot \mathbf{M}(t) \quad .$$

Now write $\mathbf{M}(t)$ as

$$\mathbf{M}(t) = \begin{pmatrix} * & * \\ * & \mathbf{N}(t) \end{pmatrix}$$

with some $k \times k$-matrix $\mathbf{N}(t)$. Because $\mathbf{N}(0) = \mathbb{1}_k$, we can find some $\epsilon > 0$ such that $\mathbf{N}(t) \in \mathrm{GL}(k)\ \ \forall\, t \in (-\epsilon, \epsilon)$. Then the previous equation shows

$$\left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \mathbb{1}_l & 0 \\ 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix} \cdot \mathbf{M}(t) = \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \mathbb{1}_l & 0 \\ 0 & \mathbf{A} \cdot \mathbf{N}(t) \\ 0 & 0 \end{pmatrix}$$

because the derivatives of all entries except those in the middle right block apparently vanish. Since $D_{\mathbf{A}} I$ is injective by definition of I, we deduce that $\xi = [t \mapsto \mathbf{A} \cdot \mathbf{N}(t)] \in T_{\mathbf{A}}(\mathbf{A} \cdot \mathrm{GL}(k))$. This finishes the proof that ι is an immersion.

Recall that the standard flag F_*^{std} in $\mathbb{C}^n$ is given as $F_i^{\mathrm{std}} = \mathrm{span}(e_1, \ldots, e_i)\ \forall\, i$. For any flag F_* in $\mathbb{C}^n$ there are vectors $v_1, \ldots, v_n$ such that $F_i = \mathrm{span}(v_1, \ldots, v_i)$ for all $i = 1, \ldots, n$. Then $(v_1, \ldots, v_n)$ clearly is a basis of $\mathbb{C}^n$.

Lemma 17. *Given an arbitrary flag F_* in $\mathbb{C}^n$ and vectors $v_1, \ldots, v_n$ such that $F_i = \mathrm{span}(v_1, \ldots, v_i)$ for $i = 1, \ldots, n$. Then the (linear) change of basis $\Phi : \mathbb{C}^n \xrightarrow{\sim} \mathbb{C}^n\,,\ e_i \mapsto v_i\,,$ induces a homeomorphism*

$$\varphi : G_k(\mathbb{C}^n) \xrightarrow{\sim} G_k(\mathbb{C}^n)\,, \quad V \mapsto \Phi(V)$$

[3] see Remark 2

[4] a being rectangular implies that X_a and $X_{(a,\underline{0})}$ are nonsingular and thus are smooth submanifolds of their ambient Grassmannians.

[5] Argument in detail: Clearly the action is smooth. It is free since $V_k(\mathbb{C}^n)$ is by definition the set of linearly independent vectors in $\mathbb{C}^n$. Now, by [Lee03, Prop. 7.26 , p. 166], the injection $\mathrm{GL}(k) \hookrightarrow V_k(\mathbb{C}^n)\,,\ \mathbf{M} \mapsto \mathbf{A} \cdot \mathbf{M}\,,$ onto the orbit through $\mathbf{A}$ is an immersion. Clearly the orbit lies inside the fiber of $\mathrm{quot}(\mathbf{A})$. This shows $T_{\mathbf{A}}(\mathbf{A} \cdot \mathrm{GL}(k)) \subseteq \ker(D_{\mathbf{A}}\mathrm{quot})$. The other direction follows by equality of dimensions of these tangent spaces, where we additionally use that quot is a submersion and the orbit map $\mathrm{GL}(k) \hookrightarrow V_k(\mathbb{C}^n)$ is an immersion.

with the property that $\varphi(X_a(F_*^{\mathrm{std}})) = X_a(F_*)$ for all partitions $a \in \mathcal{P}(n-k,k)$. Consequently, for arbitrary partitions $b \le a \in \mathcal{P}(n-k,k)$, we have

$$\varphi|_* \left[X_b(F_*^{\mathrm{std}})\right]_{X_a(F_*^{\mathrm{std}})} = \left[X_b(F_*)\right]_{X_a(F_*)} \in H_{2|b|}(X_a(F_*);\mathbb{Z}) \quad ,$$

where $\varphi| : X_a(F_*^{\mathrm{std}}) \xrightarrow{\sim} X_a(F_*)$ denotes the restriction of φ.

Proof. Obviously, φ is well-defined and bijective with inverse map $W \mapsto \Phi^{-1}(W)$. We prove that φ is a homeomorphism similarly to the preceding lemma: φ is covered by the map $V_k(\mathbb{C}^n) \to V_k(\mathbb{C}^n)$, $(u_1,\dots,u_k) \mapsto (\Phi(u_1),\dots,\Phi(u_k))$, which is well-defined since Φ is a linear isomorphism. The inverse map is given by $(w_1,\dots,w_k) \mapsto (\Phi^{-1}(w_1),\dots,\Phi^{-1}(w_k))$. Since linear maps are continuous, the map $V_k(\mathbb{C}^n) \to V_k(\mathbb{C}^n)$ covering φ is a homeomorphism. Hence, φ is a homeomorphism too.

Given an arbitrary $V \in G_k(\mathbb{C}^n)$, we want to show $V \in X_a(F_*^{\mathrm{std}}) \Leftrightarrow \varphi(V) \in X_a(F_*)$. Notice that $\Phi(F_i^{\mathrm{std}}) = F_i$ for all i. We have

$$\begin{aligned} V \in X_a(F_*^{\mathrm{std}}) &\Longleftrightarrow \forall\, i = 1,\dots,k :\ \dim(V \cap F^{\mathrm{std}}_{a_{k+1-i}+i}) \ge i \\ &\Longleftrightarrow \forall\, i = 1,\dots,k :\ \dim(\Phi(V \cap F^{\mathrm{std}}_{a_{k+1-i}+i})) \ge i \\ &\Longleftrightarrow \forall\, i = 1,\dots,k :\ \dim(\Phi(V) \cap \Phi(F^{\mathrm{std}}_{a_{k+1-i}+i})) \ge i \\ &\Longleftrightarrow \forall\, i = 1,\dots,k :\ \dim(\varphi(V) \cap F_{a_{k+1-i}+i}) \ge i \\ &\Longleftrightarrow \varphi(V) \in X_a(F_*) \end{aligned}$$

To prove the last statement it suffices to check $\varphi|_* \left[X_b(F_*^{\mathrm{std}})\right]_{X_b(F_*^{\mathrm{std}})} = \left[X_b(F_*)\right]_{X_b(F_*)}$ for any partition b. Again, we already know that both classes are generators of their respective top dimensional homology group, so $\varphi|_* \left[X_b(F_*^{\mathrm{std}})\right]_{X_b(F_*^{\mathrm{std}})} = \pm \left[X_b(F_*)\right]_{X_b(F_*)}$. Arguing as in Lemma 15 via Chow homology, we see that the sign has to be positive. □

Remark 18. Analogous to Remark 16 the above proof even shows that φ is a diffeomorphism.

Corollary 19. *Given partitions $a = (a_1,\dots,a_l,0,\dots 0) \in \mathcal{P}(n-k,k)$ and $a' = (a_1,\dots,a_l,0,\dots 0) \in \mathcal{P}(n'-k',k')$ such that $a_i > 0\ \forall\, i = 1,\dots,l$, and flags F_* in $\mathbb{C}^n$ and F'_* in $\mathbb{C}^{n'}$.*
Then $X_a(F_*) \cong X_{a'}(F'_*)$ (homeomorphic and even isomorphic as varieties[6]).

Proof. For a partition $a = (a_1,\dots,a_l,0,\dots 0)$, $a_i > 0\ \forall\, i = 1,\dots l$, of length k, we denote by $a| := (a_1,\dots,a_l)$ the truncated sequence with only positive entries. It suffices to show that for any such partition $a = (a_1,\dots,a_l,0,\dots 0) \in \mathcal{P}(n-k,k)$, $a_i > 0\ \forall\, i = 1,\dots l$, and any flag F_* in $\mathbb{C}^n$ we have an isomorphism $G_k(\mathbb{C}^n) \supseteq X_a(F_*) \cong X_{a|}(F_*^{\mathrm{std}}) \subseteq G_l(\mathbb{C}^{l+a_1})$. Fix $v_1,\dots,v_n \in \mathbb{C}^n$ with $F_i = \mathrm{span}(v_1,\dots,v_i)\ \forall\, i$. Then the isomorphism may be taken as the restriction of $\varphi \circ \iota$, where $\iota : G_l(\mathbb{C}^{l+a_1}) \hookrightarrow G_k(\mathbb{C}^n)$ is defined in Lemma 15 and $\varphi : G_k(\mathbb{C}^n) \xrightarrow{\sim} G_k(\mathbb{C}^n)$ (with respect to $v_1,\dots,v_n$) in Lemma 17. We have

$$\varphi \circ \iota\,(X_{a|}(F_*^{\mathrm{std}})) = \varphi(X_a(F_*^{\mathrm{std}})) = X_a(F_*) \quad .$$

and $\varphi \circ \iota$ is an embedding, so the restriction $(\varphi \circ \iota)|$ is a homeomorphism. Actually, this is an isomorphism of varieties, as one can see by checking the concrete definitions of ι and φ. We will omit this[7]. □

[6]Since we will not thoroughly check that the homeomorphism we construct is an isomorphism of varieties, as we will not need it later on, here is a reference for the second statement: [AM08, Lemma 2.2].

[7]And I have to admit that I have not done it precisely myself.

Remark 20. Notice that we had to choose $v_1, \ldots, v_n$ in the construction of the isomorphism. Different choices of these vectors can lead to different induced maps φ: E.g. take $n = 2$, $k = 1$, $F_* := F_*^{\mathrm{std}}$ and $(v_1, v_2) := (e_1, e_2)$, $(v_1', v_2') := (e_1, e_1 + e_2)$. Then $\varphi(\mathbb{C}\, e_2) \neq \varphi'(\mathbb{C}\, e_2)$. So the above isomorphism is not canonical. However, the last statement from Lemma 17 assures that at least the induced map φ_* on homology is independent of the choice of the $v_1, \ldots, v_n$.

Chapter 3

A Formula for the L-Class of the Normal Bundle

In this chapter we aim at proving the following proposition, which will help eliminating the L-class of the normal bundle in Formulae (1.3) and (1.4).

Proposition 21 (First L-class of normal bundle)**.** *For any smooth manifold P and any smooth submanifold $M \overset{j}{\hookrightarrow} P$ with normal bundle $\nu_M = TP|_M/TM$ we have*

$$L^1(\nu_M) = j^* L^1(P) - L^1(M) \quad . \tag{3.1}$$

Consider the case of an arbitrary graded ring $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ with $A_n = 0$ for almost all n, A being commutative[1]. In particular $A \cong \bigoplus_n A_n = \prod_n A_n$. Subsequently, A will be the graded cohomology ring $H^{4*}(X;\Lambda)$ of some topological space X of finite cohomological dimension, i.e. $H^i(X;\Lambda) = 0$ for almost all i, over some coefficient ring Λ; in our case Λ will simply be $\mathbb{Z}$ or $\mathbb{Q}$. For any $a \in A$ and $n \geq 0$ let us define a_n, so that for all $a \in A$ the sequence $(a_n)_{n\geq 0}$ is the unique sequence with only finitely many nonzero terms such that $a_n \in A_n \ \forall\, n \geq 0$ and $a = \sum_{n\geq 0} a_n$.

Remark 22. Note that $A_0^\times = A_0 \cap A^\times$.

Lemma 23 (Inverses in graded rings)**.** *The units in A are*

$$A^\times = \{a \in \ A \,|\, a_0 \in A_0^\times\} = \{a \in A \,|\, a_0 \in A^\times\}$$

and for $a \in A^\times$ the inverse a^{-1} is given recursively by

$$\begin{aligned} (a^{-1})_0 &= a_0^{-1} \\ (a^{-1})_n &= -\,a_0^{-1} \sum_{i=1}^{n} a_i\,(a^{-1})_{n-i} \quad \forall\, n \geq 1 \quad . \end{aligned} \tag{3.2}$$

[1] Here *commutativity* means ordinary commutativity in the algebraic sense, not graded commutativity.

Proof. If a is a unit, then $1 = (aa^{-1})_0 = a_0\,(a^{-1})_0$ and a_0 is a unit in A_0. Conversely suppose $a_0 \in A_0^\times$. Put $b := \sum_n b_n \in A$, $b_n \in A_n$, where the sequence $(b_n)_{n\geq 0}$ is defined recursively as

$$\begin{aligned} b_0 &= a_0^{-1} \\ b_n &= -\,a_0^{-1} \sum_{i=1}^{n} a_i\, b_{n-i} \quad \forall\, n \geq 1 \quad . \end{aligned}$$

Notice that $b_n = 0$ for almost all n, since $A_n = 0$ for almost all n. Then for $n \geq 1$ we have

$$(ab)_n = \sum_{i=0}^{n} a_i\, b_{n-i} = a_0\, b_n + \sum_{i=1}^{n} a_i\, b_{n-i} = 0$$

and clearly $(ab)_0 = a_0\, b_0 = 1$. Hence b is inverse to a. □

Remark 24. The proof of Lemma 23 works equally fine for $A = \prod_{n\geq 0} A_n$, where this time we do not additionally assume that $A_n = 0$ for almost all n. In this case we would simply define $b := (b_n)_{n\geq 0}$. Thus the statement of the lemma, including the recursive formula (3.2), also holds for rings of this type. An example for such a ring is $A = R[[x_1, \ldots, x_k]]$, the ring of formal power series in the variables $x_1, \ldots, x_k$ over some commutative ring R. Here A_n is the ring of homogeneous polynomials of degree n in $x_1, \ldots, x_k$.

Now consider the situation of Proposition 21, i.e. P a smooth manifold with smooth submanifold $M \overset{j}{\hookrightarrow} P$. We have the canonical exact sequence of vector bundles

$$0 \longrightarrow TM \longrightarrow TP|_M \longrightarrow \nu_M \longrightarrow 0$$

and, as every exact sequence of vector bundles over a paracompact Hausdorff space splits[2], this yields

$$j^*TP = TP|_M \cong TM \oplus \nu_M \quad . \tag{3.3}$$

The L-class of any bundle ξ over an arbitrary topological space of finite cohomological dimension X can be written as

$$L^*(\xi) = \sum_{i\geq 0} L^i(\xi) \,\in\, H^{4*}(X;\mathbb{Q})\;,\; L^i(\xi) \in H^{4i}(X;\mathbb{Q})$$

with $L^0(\xi) = 1$. By Lemma 23, $L^*(\xi)$ is invertible in $H^{4*}(X;\mathbb{Q})$, and thus also is invertible in the ring extension $H^*(X;\mathbb{Q})$. Given $a \in H^*(X;\mathbb{Q})$, $n \geq 0$, and regarding $H^{4*}(X;\mathbb{Q})$ as commutative graded ring, with the notation introduced above for such graded rings, $a_n \in H^{4n}(X;\mathbb{Q})$ is the cohomology component of a in cohomology degree $4n$. If N is a smooth manifold with tangent bundle TN, we also write $L^*(N)$ for $L^*(TN)$. Note that every manifold is of finite cohomological dimension.

Now Equation (3.3) together with naturality and multiplicativity of the L-class and the just mentioned fact, that all L-classes can be inverted, implies that[3]

$$L^*(\nu_M) = L^*(M)^{-1} \smile j^*L^*(P) \quad . \tag{3.4}$$

[2] see e.g. [nLa23, Prop. 2.1]

[3] Here I would like to thank Prof. Banagl, my thesis advisor, who introduced me to this well-known trick how to compute multiplicative characteristic classes of normal bundles.

and therefore for any $i \geq 0$:

$$\begin{aligned}
L^i(\nu_M) &= (L^*(M)^{-1} \smile j^*L^*(P))_i \\
&= \sum_{k=0}^{i} (L^*(M)^{-1})_k \smile j^*(L^*(P))_{i-k} \\
&= \sum_{k=0}^{i} (L^*(M)^{-1})_k \smile j^*(L^*(P)_{i-k}) \\
&= \sum_{k=0}^{i} (L^*(M)^{-1})_k \smile j^*(L^{i-k}(P))
\end{aligned} \tag{3.5}$$

By Equation (3.2) we have $(L^*(M)^{-1})_0 = 1$ and for arbitrary $k \geq 1$:

$$\begin{aligned}
(L^*(M)^{-1})_k &= -(1^{-1}) \sum_{l=1}^{k} L^*(M)_l \smile (L^*(M)^{-1})_{k-l} \\
&= -\sum_{l=1}^{k} L^l(M) \smile (L^*(M)^{-1})_{k-l}
\end{aligned} \tag{3.6}$$

For $i = 1$, Equations (3.5) and (3.6) yield

$$\begin{aligned}
L^1(\nu_M) &= (L^*(M)^{-1})_0 \smile j^*(L^1(P)) + (L^*(M)^{-1})_1 \smile j^*(L^0(P)) \\
&= 1 \smile j^*(L^1(P)) + (-L^1(M) \smile 1) \smile j^*1 \\
&= j^*L^1(P) - L^1(M)
\end{aligned}$$

which finishes the proof of Proposition 21.

Chapter 4

Simplifying the λ- and μ-Equations

From this chapter onward until the end of Chapter 6 we are dealing with homology and cohomology with coefficients in $\mathbb{Q}$, so unless otherwise mentioned $H_i(X) = H_i(X;\mathbb{Q})$ and $H^i(X) = H^i(X;\mathbb{Q})$ for any topological space X.

4.1 Simplifying the λ-Equation

Recall that in the situation of the λ-Equation (1.3) we consider the Schubert variety $X = X_{3,2} = X_{3,2}(F_*^{\mathrm{std}})$ with respect to the standard flag inside the Grassmannian $P = G_2(\mathbb{C}^5)$. Furthermore, we have $M := M_1 = X_3(F_*^1)$ and the intersection $Y := Y_1 = M \cap X = X_2(F_*^2)$. We attempt to present a rather general version of the setting at first, so that one can easily keep track of which necessities arise in each step of our reasoning and possibly adopt similar techniques for analogous calculations.

So, assume we are given a Grassmannian $P = G_k(\mathbb{C}^n)$, a Schubert variety $X = X_a = X_a(F_*^{\mathrm{std}}) \subseteq P$ with respect to the standard flag[1] and some partition $a = (a_1, \ldots, a_k)$, a Schubert variety $M = X_b(F'_*) \subseteq P$ with respect to some flag F'_* and partition $b = (b_1, \ldots, b_k)$, M being nonsingular[2], and a third Schubert variety $Y = X_c(F''_*) = M \cap X$ with respect to some flag F''_* and partition $c = (c_1, \ldots, c_k)$. Furthermore, assume that $c \leq b$ and $|c| = \sum_i c_i = 2$, i.e. the complex dimension of Y is 2. Then it makes sense to take the Kronecker product of $L^1(\nu_M|_Y) \in H^4(Y)$ with $[Y] \in H_4(Y)$ and this lies in $\mathbb{Q}$. Our goal is to compute

$$\langle\, L^1(\nu_M|_Y)\,,\,[Y]\,\rangle \quad .$$

Actually, it turns out that for our purposes it suffices to demand that $Y \subseteq M$, $c \leq b$ and that Y is of dimension 2: We could simple omit the premise $Y = M \cap X$ or even go further and never mention X anyway since it will appear nowhere in the following calculations. We chose this notation however, to take account of every space appearing in our setting. In the end, we will of course apply our result to the special case where $n = 5$, $k = 2$, $a = (3, 2)$, $b = (3, 0)$, $c = (2, 0)$ and the flags are given by $F'_* = F^1_*$, $F''_* = F^2_*$.

[1] We could just as well consider an arbitrary flag F_* in $\mathbb{C}^n$.

[2] Nonsingularity of M is equivalent to b being a rectangular partition.

Let $([X_d]_P^\vee)_{\substack{d\in\mathcal{P}(n-k,k)\\|d|=2}}$ denote the (linear) dual basis of the basis $([X_d]_P)_{\substack{d\in\mathcal{P}(n-k,k)\\|d|=2}}$ of $H_4(P)$ and similarly let $([X_d(F'_*)]_M^\vee)_{\substack{d\leq b\\|d|=2}}$, where the index set is a subset of $\mathcal{P}(n-k,k)$, denote the dual basis of the basis $([X_d(F'_*)]_M)_{\substack{d\leq b\\|d|=2}}$ of $H_4(M)$ [3]. Define $(\mu_d)_{\substack{d\in\mathcal{P}(n-k,k)\\|d|=2}}$ to be the unique family in $\mathbb{Q}$ with

$$L^1(P)=\sum_{\substack{d\in\mathcal{P}(n-k,k)\\|d|=2}}\mu_d\left[X_d\right]_P^\vee\in H^4(P)\cong\mathrm{Hom}_{\mathbb{Q}}(H_4(P),\mathbb{Q})=H_4(P)^\vee \tag{4.1}$$

and $(\lambda_d)_{\substack{d\leq b\\|d|=2}}$ to be the unique family in $\mathbb{Q}$ with

$$L^1(M)=\sum_{\substack{d\leq b\\|d|=2}}\lambda_d\left[X_d(F'_*)\right]_M^\vee\in H^4(M)\cong H_4(M)^\vee \quad . \tag{4.2}$$

Note that we tacitly identify the cohomology groups with the linear dual spaces of the homology groups via the Kronecker map, as explained already in Section 1.2. So, strictly speaking, we should write $\mathrm{Kron}^{-1}([X_d]_P^\vee)$ instead of $[X_d]_P^\vee$ in Equation (4.1), where $\mathrm{Kron}: H^4(P)\xrightarrow{\sim}\mathrm{Hom}_{\mathbb{Q}}(H_4(P),\mathbb{Q})=H_4(P)^\vee$ denotes the Kronecker map, and mutatis mutandis in Equation (4.2).

Lemma 25. *Let $G=G_l(\mathbb{C}^m)$ be a Grassmannian, containing the Schubert variety $Z:=X_r(F_*)$ with respect to an arbitrary partition $r\in\mathcal{P}(m-l,l)$ and a flag F_* in $\mathbb{C}^m$. Let $j_{Z\subset G}:Z\hookrightarrow G$ denote the inclusion. Then*

$$(j_{Z\subset G})^*([X_s]_G^\vee)=[X_s(F_*)]_Z^\vee \quad \forall\, s\leq r \quad . \tag{4.3}$$

Remark 26. In the situation of Lemma 25 notice that $[X_s]_G^\vee=[X_s(F_*)]_G^\vee$ for all $s\in\mathcal{P}(m-l,l)$. This is because $[X_s]_G=[X_s(F_*)]_G$ for all $s\in\mathcal{P}(m-l,l)$, $|s|=i$, where i is arbitrary but fixed. Thus we trivially obtain identity of the respective dual bases: $[X_s]_G^\vee=[X_s(F_*)]_G^\vee$ for all s with $|s|=i$. But i was arbitrary, so the claim follows.

Proof. The preceding remark clarifies that it suffices to show:

$$(j_{Z\subset G})^*([X_s(F_*)]_G^\vee)=[X_s(F_*)]_Z^\vee \quad \forall\, s\leq r \quad .$$

By the naturality property in the Universal Coefficient Theorem we obtain a commutative diagram

$$\begin{tikzcd} H^p(G) \arrow[r, "\cong", "\mathrm{Kron}"'] \arrow[d, "(j_{Z\subset G})^*"'] & \mathrm{Hom}_{\mathbb{Q}}(H_p(G),\mathbb{Q}) \arrow[d, "(j_{Z\subset G})_*^\vee"] \\ H^p(Z) \arrow[r, "\cong", "\mathrm{Kron}"'] & \mathrm{Hom}_{\mathbb{Q}}(H_p(Z),\mathbb{Q}) \end{tikzcd} \tag{4.4}$$

[3] Recall from Section 1.2 that the Schubert classes inside a Grassmannian are independent of the flag, so we may work with Schubert varieties with respect to the standard flag F_*^{std}. On the other hand, since $M=X_b(F'_*)$, we are forced to consider Schubert varieties with respect to F'_* in this case.

for any integer p, where Kron denotes the Kronecker map and $(j_{Z\subset G})_*^\vee$ is the dual map of $(j_{Z\subset G})_* : H_p(Z) \hookrightarrow H_p(G)$.

Equation (4.3) only makes sense if we identify the cohomology group with the dual space of the corresponding homology group via Kron, i.e. we are imprecise and e.g. write $[X_s(F_*)]_Z^\vee$ for $\mathrm{Kron}^{-1}([X_s(F_*)]_Z^\vee)$. Keeping this in mind, we have to show that

$$\mathrm{Kron} \circ (j_{Z\subset G})^* \circ \mathrm{Kron}^{-1}([X_s(F_*)]_G^\vee) = [X_s(F_*)]_Z^\vee \quad .$$

By Diagram (4.4), this is equivalent to $(j_{Z\subset G})_*^\vee([X_s(F_*)]_G^\vee) = [X_s(F_*)]_Z^\vee$. Take an arbitrary $t \le r$ with $|s| = |t|$. Then

$$\begin{aligned}
\big((j_{Z\subset G})_*^\vee([X_s(F_*)]_G^\vee) \big) ([X_t(F_*)]_Z) &= [X_s(F_*)]_G^\vee \circ (j_{Z\subset G})_* ([X_t(F_*)]_Z) \\
&= [X_s(F_*)]_G^\vee ([X_t(F_*)]_G) \\
&= \begin{cases} 1 \text{ if } & s = t \\ 0 \text{ else} & \end{cases}
\end{aligned}$$

which finishes the proof. □

Applying Lemma 25 to our setting with $l := k$, $m := n$, $r := b$, $Z := M$ $G := P$ and $F_* := F'_*$, we obtain

$$(j_{M\subset P})^*([X_d]_P^\vee) = [X_d(F'_*)]_M^\vee \quad \forall\, d \le b \quad .$$

But this equation now directly proves that the element

$$\hat\alpha := \sum_{\substack{d\le b\\ |d|=2}} \lambda_d\, [X_d]_P^\vee \in H^4(P)$$

is mapped to $L^1(M)$ via $(j_{M\subset P})^*$.

Altogether, the preceding results yield

$$\begin{aligned}
\langle L^1(\nu_M|_Y)\,,\,[Y]\,\rangle &= \langle\,(j_{Y\subset M})^* L^1(\nu_M)\,,\,[Y]\,\rangle \\
&= \langle\,(j_{Y\subset M})^*((j_{M\subset P})^*\, L^1(P) - L^1(M))\,,\,[Y]\,\rangle && \text{by (3.1)} \\
&= \langle\,(j_{Y\subset P})^*(L^1(P) - \hat\alpha)\,,\,[Y]\,\rangle \\
&= \langle\,L^1(P) - \hat\alpha\,,\,(j_{Y\subset P})_*[Y]\,\rangle \\
&= \langle\,L^1(P) - \hat\alpha\,,\,[Y]_P\,\rangle \\
&= \langle\,L^1(P) - \hat\alpha\,,\,[X_c(F''_*)]_P\,\rangle \\
&= \langle\,L^1(P) - \hat\alpha\,,\,[X_c]_P\,\rangle \\
&= \sum_{\substack{d\in\mathcal{P}(n-k,k)\\ |d|=2}} \mu_d\,\langle\,[X_d]_P^\vee\,,\,[X_c]_P\,\rangle \\
&\quad - \sum_{\substack{d\le b\\ |d|=2}} \lambda_d\,\langle\,[X_d]_P^\vee\,,\,[X_c]_P\,\rangle && \text{by definition of } \hat\alpha \text{ and (4.1)} \\
&= \sum_{\substack{d\in\mathcal{P}(n-k,k)\\ |d|=2}} \mu_d\,\delta_{d,c} - \sum_{\substack{d\le b\\ |d|=2}} \lambda_d\,\delta_{d,c} && \text{with } \delta_{\cdot,\cdot} \text{ Kronecker delta}
\end{aligned}$$

$$= \mu_c - \lambda_c \qquad |c| = 2\,,\ c \leq b$$

All in all we have computed that

$$\langle\, L^1(\nu_M|_Y)\,,\,[Y]\,\rangle \;=\; \mu_c - \lambda_c \quad , \tag{4.5}$$

where μ_c and λ_c are defined as coefficients in the linear combination of $L^1(P)$ respectively $L^1(M)$ in Equations (4.1) respectively (4.2).

Having Equation (4.5) in mind, we would like to determine the linear combinations of $L^1(P)$ and $L^1(M)$ with respect to the linear dual bases of Schubert classes. Since $P = G_k(\mathbb{C}^n)$ is a Grassmannian, we obviously aim at understanding the linear coefficients η_d

$$L^1(G) = \sum_{\substack{d \in \mathcal{P}(m-l,l) \\ |d|=2}} \eta_d \, [X_d]_G^\vee$$

for various Grassmanians $G = G_l(\mathbb{C}^m)$, where l and m may vary. As $M = X_b(F'_*)$ is a nonsingular Schubert variety, b is rectangular, say

$$b = (\underbrace{b_0, \ldots, b_0}_{l \text{ times}},\ \underbrace{0, \ldots, 0}_{(k-l) \text{ times}}\ )$$

with $b_0 > 0$, and M is isomorphic (as variety) to $G_l(\mathbb{C}^{l+b_0})$ by Corollary 19. Thus it seems reasonable to determine the coefficients of $L^1(G_l(\mathbb{C}^{l+b_0}))$ if one wants to understand the coefficients λ_d of $L^1(M)$. But perhaps we have to be cautious since M is embedded in $G_k(\mathbb{C}^n)$ as Schubert variety with regard to the flag F'_*. However, it will turn out, that we can indeed determine the λ_d's through the coefficients of the Grassmannian $G_l(\mathbb{C}^{b_0+l})$. This is the next lemma.

For a partition $d = (d_1, \ldots, d_k)$ of length k and $l \leq k$ fixed, put $d| := (d_1, \ldots, d_l)$, the truncated sequence of length l. Furthermore, we define $\widetilde{\lambda}_{d'}$ for partitions $d' \in \mathcal{P}(b_0, l)$ with $|d'| = 2$ by

$$L^1(G_l(\mathbb{C}^{b_0+l})) = \sum_{\substack{d' \in \mathcal{P}(b_0,l) \\ |d'|=2}} \widetilde{\lambda}_{d'} \, [X_{d'}]^\vee_{G_l(\mathbb{C}^{l+b_0})} \quad . \tag{4.6}$$

Lemma 27. *Given a partition d of length k with $d \leq b$, $|d| = 2$. Then $\lambda_d = \widetilde{\lambda}_{d|}$, where the $\widetilde{\lambda}_{d'}$'s are defined through Equation* (4.6)*.*

Proof. Fix a basis $(v'_1, \ldots, v'_n)$ of $\mathbb{C}^n$ with $F'_i = \operatorname{span}(v'_1, \ldots, v'_i)$ for $i = 1, \ldots, n$. Consider the map $\varphi : P = G_k(\mathbb{C}^n) \xrightarrow{\sim} G_k(\mathbb{C}^n) = P$ from Lemma 17 with respect to the flag F'_* and the basis $(v'_1, \ldots, v'_n)$. Recall that we omit the flag in the notation of the Schubert variety if the flag is the standard flag F^{std}_*, i.e. $X_d := X_d(F^{\text{std}}_*)$. We have $\varphi(X_b) = X_b(F'_*) = M$ and $\varphi|_*[X_d]_{X_b} = [X_d(F'_*)]_M$ for all $d \leq b$ [4]. By Remark 18, φ is a diffeomorphism and, since M and X_b are nonsingular as b is rectangular, φ restricts to a diffeomorphism $\varphi| : X_b \xrightarrow{\sim} M$. Thus we obtain

$$L^1(X_b) = L^1(TX_b) = L^1(\varphi|^* TM) = \varphi|^* L^1(M) \overset{(4.2)}{=} \sum_{\substack{d \leq b \\ |d|=2}} \lambda_d \, \varphi|^*[X_d(F'_*)]^\vee_M \quad .$$

[4]Notice that, although we formulated this result for homology with integer coefficients in Lemma 17, the analog statement holds for rational coefficients as well. E.g. one can deduce this from the former via naturality of the change-of-coefficients map $H_*(-;\mathbb{Z}) \to H_*(-;\mathbb{Q})$.

We claim that $\varphi|^*[X_d(F'_*)]^\vee_M = [X_d]^\vee_{X_b}$ for all such indices d. The argument is similar to the proof of Lemma 25: For arbitrary partitions $d, d' \leq b$ with $|d| = |d'| = 2$ we have

$$([X_d(F'_*)]^\vee_M \circ \varphi|_*)\,[X_{d'}]_{X_b} = [X_d(F'_*)]^\vee_M\,[X_{d'}(F'_*)]_M = \begin{cases} 1 & \text{if } d = d' \\ 0 & \text{else} \end{cases}$$

and thus $\varphi|^\vee_*\,[X_d(F'_*)]^\vee_M = [X_d]^\vee_{X_b}$. Now, keeping in mind that we identify the cohomology group with the dual space of its respective homology group via the Kronecker map Kron, this yields

$$\begin{aligned} \varphi|^*[X_d(F'_*)]^\vee_M &= \varphi|^* \circ \mathrm{Kron}^{-1}\,([X_d(F'_*)]^\vee_M) && \text{(due to informal notation)} \\ &= \mathrm{Kron}^{-1} \circ \varphi|^\vee_*\,([X_d(F'_*)]^\vee_M) \\ &= \mathrm{Kron}^{-1}\,([X_d]^\vee_{X_b}) \\ &= [X_d]^\vee_{X_b} && \text{(due to informal notation)} \end{aligned}$$

Thus

$$L^1(X_b) = \sum_{\substack{d \leq b \\ |d|=2}} \lambda_d\,\varphi|^*[X_d(F'_*)]^\vee_M = \sum_{\substack{d \leq b \\ |d|=2}} \lambda_d\,[X_d]^\vee_{X_b} \quad .$$

The truncated sequence of $b = (\underbrace{b_0, \ldots, b_0}_{l \text{ times}}, 0, \ldots, 0)$ is $b| = (\underbrace{b_0, \ldots, b_0}_{l \text{ times}})$. Consider the embedding $\iota : G_l(\mathbb{C}^{b_0+l}) \hookrightarrow G_k(\mathbb{C}^n) = P$ from Lemma 15. Recall that we have shown in Remark 16 that $\iota : G_l(\mathbb{C}^{b_0+l}) = X_{b|} \xrightarrow{\sim} \iota(X_{b|}) = X_b$ is a diffeomorphism, so $T\,G_l(\mathbb{C}^{l+b_0}) \cong \iota^*\,T X_b$ as vector bundles. Since for $d' = (d'_1, \ldots, d'_l) \leq b|$ we have $\iota_*[X_{d'}]_{G_l(\mathbb{C}^{b_0+l})} = [X_{(d',\underline{0})}]_{X_b}$, we obtain for arbitrary partitions $d \leq b$, $|d| = 2$, of length k and $d' \leq b|$, $|d'| = 2$, of length l, similarly to before,

$$\begin{aligned} \iota^*[X_d]^\vee_{X_b}\,([X_{d'}]_{G_l(\mathbb{C}^{l+b_0})}) = [X_d]^\vee_{X_b}\,([X_{(d',\underline{0})}]_{X_b}) &= \begin{cases} 1 & \text{if } (d', \underline{0}) = d \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} 1 & \text{if } d' = d| \\ 0 & \text{else} \end{cases} \end{aligned}$$

where the last equation holds because for any $d \leq b$ the last $(k-l)$ entries of d are all zero, so $(d', \underline{0}) = d \Leftrightarrow d' = d| \quad \forall\, d, d'$. Hence $\iota^*[X_d]^\vee_{X_b} = [X_{d|}]^\vee_{G_l(\mathbb{C}^{l+b_0})}$. This gives rise to

$$\begin{aligned} L^1(G_l(\mathbb{C}^{l+b_0})) \overset{\text{def}}{=} L^1(T\,G_l(\mathbb{C}^{l+b_0})) = L^1(\iota^*\,T X_b) = \iota^* L^1(X_b) &= \sum_{\substack{d \leq b \\ |d|=2}} \lambda_d\ \iota^*[X_d]^\vee_{X_b} \\ &= \sum_{\substack{d \leq b \\ |d|=2}} \lambda_d\ [X_{d|}]^\vee_{G_l(\mathbb{C}^{l+b_0})} \\ &= \sum_{\substack{d' \leq b| \\ |d'|=2}} \lambda_{(d',\underline{0})}\ [X_{d'}]^\vee_{G_l(\mathbb{C}^{l+b_0})} \end{aligned}$$

Comparing coefficients with the ones in Equation (4.6), we obtain $\lambda_{(d',\underline{0})} = \widetilde{\lambda}_{d'}$ for all $d' \leq b|$, $|d'| = 2$. This finishes the proof. $\square$

4.2 Simplifying the μ-Equation

Besides λ we would also like to calculate

$$\mu = \frac{2}{3} + \langle\, \omega \smile L^1(\nu_M|_Y)\, ,\, [Y]_Y\, \rangle$$

where $X = X_{3,2}$, $M := M_2 = X_{2,2}(F^3_*)$, $Y := Y_2 = M \cap X = X_{2,1}(F^4_*)$ all lie inside $P = G_2(\mathbb{C}^5)$ and $\omega \in H^2(Y)$ is determined via

$$\langle\, \omega\, ,\, [X_1(F^4_*)]_Y\, \rangle = +1 \quad . \tag{4.7}$$

As in Section 4.1, let us first slightly generalize the situation and clarify notation: We are working inside the Grassmannian $P = G_k(\mathbb{C}^n)$ and consider partitions $a, b, c \in \mathcal{P}(n-k, k)$ such that $X = X_a = X_a(F^{\text{std}}_*)$ is a Schubert variety, $M = X_b(F'_*)$ is a nonsingular Schubert variety (so b is a rectangular partition) and the intersection $Y = M \cap X = X_c(F''_*)$ is itself Schubert with respect to the flag F''_*. Additionally assume $c \le b$ and $|c| = 3$, so $\dim_{\mathbb{C}} Y = 3$ and $[Y]_Y \in H_6(Y)$. The homology group $H_2(Y)$ is generated by $[X_1(F''_*)]_Y$ as 1-dimensional $\mathbb{Q}$-vector space, so there is a unique $\omega \in H^2(Y)$ with

$$\langle\, \omega\, ,\, [X_1(F''_*)]_Y\, \rangle = +1 \quad . \tag{4.8}$$

Then $\omega \smile L^1(\nu_M|_Y) \in H^6(Y)$, so the expression

$$\langle\, \omega \smile L^1(\nu_M|_Y)\, ,\, [Y]_Y\, \rangle$$

is well-defined and lies in $\mathbb{Q}$. Our goal is to determine its value. Concretely we will apply the result to the setting $n = 5$, $k = 2$, $a = (3, 2)$, $b = (2, 2)$, $c = (2, 1)$ and $F'_* = F^3_*$, $F''_* = F^4_*$. Again, one could of course omit X entirely in the following considerations and it would suffice to require $Y \subseteq M$ and $c \le b$, $|c| = 3$.

The striking difference between this setting and the previous one for λ is of course the newly appearing cup product. But before we deal with that, let us try to proceed as similar to Section 4.1 as possible.

First notice that, expressing ω in our notation, we obviously have $\omega = [X_1(F''_*)]^\vee_Y$. Copying the notation from the previous section, let us define

$$L^1(P) = \sum_{\substack{d \in \mathcal{P}(n-k,k) \\ |d|=2}} \mu_d \left[X_d\right]^\vee_P \in H^4(P) \tag{4.9}$$

$$L^1(M) = \sum_{\substack{d \le b \\ |d|=2}} \lambda_d \left[X_d(F'_*)\right]^\vee_M \in H^4(M) \tag{4.10}$$

and, analogously to before, put

$$\hat{\alpha} := \sum_{\substack{d \le b \\ |d|=2}} \lambda_d \left[X_d\right]^\vee_P \in H^4(P) \quad .$$

As before, we apply Lemma 25 to deduce $(j_{M\subset P})^*(\hat{\alpha}) = L^1(M)$. Applying Lemma 25 once again, we obtain $(j_{Y\subset P})^*\,[X_1]_P^\vee = [X_1(F_*'')]_Y^\vee = \omega$. Now we compute:

$$
\begin{aligned}
&\langle\, \omega \smile L^1(\nu_M|_Y)\,,\, [Y]_Y\,\rangle \\
&= \langle\, j^*_{Y\subset P}\,[X_1]_P^\vee \smile j^*_{Y\subset M}\,L^1(\nu_M)\,,\, [Y]_Y\,\rangle \\
&\overset{(3.1)}{=} \langle\, j^*_{Y\subset P}\,[X_1]_P^\vee \smile j^*_{Y\subset P}\,(L^1(P)-\hat{\alpha})\,,\, [Y]_Y\,\rangle \\
&= \langle\, j^*_{Y\subset P}\,([X_1]_P^\vee \smile (L^1(P)-\hat{\alpha}))\,,\, [Y]_Y\,\rangle \\
&= \langle\, [X_1]_P^\vee \smile (L^1(P)-\hat{\alpha})\,,\, [Y]_P\,\rangle \\
&= \langle\, [X_1]_P^\vee \smile \Big(\sum_{\substack{d\in\mathcal{P}(n-k,k)\\ |d|=2}} \mu_d\,\big[X_d\big]_P^\vee - \sum_{\substack{d\leq b\\ |d|=2}} \lambda_d\,\big[X_d\big]_P^\vee \Big)\,,\, [X_c]_P\,\rangle \\
&\qquad \text{(by definition of } \hat{\alpha}\,,\ \text{Equation (4.9) and since } Y = X_c(F_*'')) \\
&= \sum_{\substack{d\in\mathcal{P}(n-k,k)\\ |d|=2}} \mu_d\,\langle\, [X_1]_P^\vee \smile [X_d]_P^\vee\,,\, [X_c]_P\,\rangle - \sum_{\substack{d\leq b\\ |d|=2}} \lambda_d\,\langle\, [X_1]_P^\vee \smile [X_d]_P^\vee\,,\, [X_c]_P\,\rangle
\end{aligned}
\tag{4.11}
$$

Here we perceive a new difficulty: It appears that we have to compute the cup product of linear duals of Schubert classes in terms of linear duals of Schubert classes. In other words, we aim to compute the coefficients κ_t in expressions of the form

$$
[X_r]^\vee \smile [X_s]^\vee = \sum_{|t|=|r|+|s|} \kappa_t\,[X_t]^\vee \quad ,
$$

where from now on we omit the space in the index of the Schubert class respectively the linear dual of a Schubert class if we are working inside an implicitly understood Grassmannian, i.e. $[X_r] := [X_r]_P$ and $[X_r]^\vee := [X_r]_P^\vee$. With these cup products understood, we could determine the above terms:

$$
\langle\, [X_r]^\vee \smile [X_s]^\vee\,,\, [X_u]\,\rangle = \sum_{|t|=|r|+|s|} \kappa_t\,\langle\, [X_t]^\vee\,,\, [X_u]\,\rangle = \sum_{|t|=|r|+|s|} \kappa_t\,\delta_{t,u} = \kappa_u \quad .
$$

In the next chapter we will solve the problem of calculating the coefficients κ_t. The coefficients μ_d and λ_d can be obtained through the same methods that we will develop to determine the corresponding coefficients in Section 4.1 [5] (however, we will accomplish this only later in Chapter 8). Having all of above accomplished, practically speaking, we were able to compute the value of $\langle \omega \smile L^1(\nu_M|_Y), [Y]_Y \rangle$ by means of (4.11).

[5] As the coefficients μ_d , λ_d defined via Equations (4.9) respectively (4.10) are the same as the ones defined via Equations (4.1) respectively (4.2) for fixed P and M.

Chapter 5

Transforming the Linear Dual Basis into the Poincaré Dual Basis

Recall that given a closed oriented n-manifold M, for any i we have the Poincaré duality isomorphism

$$PD = PD_i : H_i(M) \xrightarrow{\sim} H^{n-i}(M)$$

which is a ring isomorphism with respect to intersection and cup product, cf. Section 1.3:

$$H^{2n-i-j}(M) \ni PD(a \cdot b) = PD(a) \smile PD(b) \quad \forall\, a \in H_i(M)\,,\, b \in H_j(M) \quad .$$

But this directly yields the following result, where $\varepsilon_* : H_0(M) = H_0(M;\mathbb{Q}) \to \mathbb{Q}$ is the augmentation map:

Proposition 28. *M closed oriented n-dimensional manifold, $a \in H_{n-i}(M)\,,\, b \in H_j(M)$. Then*

$$PD_{n-i}(a) \frown b = a \cdot b \;\in H_{j-i}(M)$$

and consequently for $a \in H_{n-i}(M)\,,\, b \in H_i(M)$:

$$\langle\, PD_{n-i}(a)\,,\, b\,\rangle = \varepsilon_*(a \cdot b) \;\in \mathbb{Q} \quad .$$

Proof.

$$\begin{aligned} PD_{n-i}(a) \frown b &= PD_{n-i}(a) \frown (PD_j(b) \frown [M]) \\ &= (PD_{n-i}(a) \smile PD_j(b)) \frown [M] \\ &= PD_{j-i}^{-1}(PD_{n-i}(a) \smile PD_j(b)) \\ &= a \cdot b \end{aligned}$$

□

In Section 4.2 we aimed at computing the coefficients κ_t for the cup product of (linear) duals of Schubert classes:

$$[X_r]^\vee \smile [X_s]^\vee = \sum_{|t|=|r|+|s|} \kappa_t\, [X_t]^\vee \quad .$$

There is however a more convenient choice of basis on the cohomology[1] $H^{2(\dim(P)-q)}(P)\,,\ 0 \leq$

[1] $\dim(P) := \dim_{\mathbb{C}}(P)$ unless otherwise mentioned.

$q \leq \dim(P)$ arbitrary, of the Grassmannian $P = G_k(\mathbb{C}^n)$ when dealing with cup products: Unsurprisingly, this is the basis $\{PD([X_a])\}_{|a|=q}$ of Poincaré duals of the ordinary basis $\{[X_a]\}_{|a|=q}$ of $H_{2q}(P)$ because in this case the cup product of two basis vectors simply is

$$PD([X_r]) \smile PD([X_s]) = PD([X_r] \cdot [X_s])$$

and the intersection product $[X_r] \cdot [X_s]$ in terms of Schubert classes can be computed algorithmically via Schubert calculus, see Section 1.3. Then, using linearity of PD, we are able to express $PD([X_r]) \smile PD([X_s])$ as a linear combination of the $PD([X_t])$, $|t| = |r| + |s| - \dim(P)$. Recall from Section 1.3, that for efficient calculations within Schubert calculus we adopt the Griffiths-Harris notation for Schubert classes: To a partition $a = (a_1, \ldots, a_k)$ we associate the partition $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_i := n - k - a_{k+1-i}$ and write σ_α for $[X_a]$. Note that σ_α lies in $H_{2(\dim(P)-|\alpha|)}(P)$, where as usual $|\alpha| = \sum_i \alpha_i$. Let us recapitulate the two pivotal formulae for Schubert calculus: Pieri's formula (Theorem 10) computes the intersection product of a special Schubert class with an arbitrary one and Giambelli's formula (11) allows to express arbitrary Schubert classes as polynomial in special Schubert classes. Any intersection product of Schubert classes can be computed algorithmically, in the strict sense, by combining Pieri's and Giambelli's formulae.

Keeping this at the back of our mind, it seems desirable to be able to transform the Poincaré dual basis of a cohomology group of a Grassmannian $P = G_k(\mathbb{C}^n)$ into the (linear) dual basis[2] of Schubert classes and vice versa. This is accomplishable via Proposition 28. Let us first prove a trivial statement from linear algebra[3]:

Lemma 29. *Let V be an n-dimensional ($\mathbb{Q}$-)vector space with basis $\{e_i\}_{i=1,\ldots n}$ and $V^\vee$ its dual vector space with dual basis $\{e^i\}_{i=1,\ldots,n}$ with respect to the basis $\{e_i\}_{i=1,\ldots n}$. Let $\{f^i\}_{i=1,\ldots,n}$ be another arbitrary basis of $V^\vee$. Then the matrix $A := \left(f^j(e_i)\right)_{i,j=1,\ldots,n}$ is invertible and the change-of-basis matrix from $\{f^i\}_i$ to $\{e^i\}_i$, i.e.*

$$f^j = \sum_{i=1}^{n} A_{i,j}\, e^i \quad \forall\, j = 1, \ldots, n \quad .$$

Proof. For fixed $j = 1, \ldots, n$ we have

$$f^j(e_k) = \sum_{i=1}^{n} f^j(e_i)\, e^i(e_k) = \left(\sum_{i=1}^{n} f^j(e_i)\, e^i \right) (e_k) \quad \forall\, k = 1, \ldots, n \quad .$$

Since the linear maps f^j and $\sum_{i=1}^n f^j(e_i)\, e^i$ coincide on the basis $\{e_k\}_k$, they are equal, so

$$f^j = \sum_{i=1}^{n} f^j(e_i)\, e^i \quad \forall\, j = 1, \ldots, n \quad ,$$

which proves that the change-of-basis matrix from $\{f^i\}_i$ to $\{e^i\}_i$ is given by $A = \left(f^j(e_i) \right)_{i,j}$. Because change-of-basis matrices are invertible, A is invertible. $\square$

[2] By a *dual basis* (without further specification) we always mean a linear dual basis.

[3] and note that, although we formulate the result for vector spaces over $\mathbb{Q}$, we could just as well work with finitely generated free modules over an arbitrary commutative ring, in particular over $\mathbb{Z}$

We now apply this fact from linear algebra to the case $V := H_{2(\dim(P)-q)}(P)$, so

$$V^\vee = H_{2(\dim(P)-q)}(P)^\vee \underset{\text{Kron}}{\cong} H^{2(\dim(P)-q)}(P) \quad .$$

Put $m := \dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} H^{2(\dim(P)-q)}(P)$. Then, by Poincaré duality, $m = \dim_{\mathbb{Q}} H_{2q}(P)$. Clearly, $\{\sigma_\alpha\}_{|\alpha|=q}$ is a basis of the $\mathbb{Q}$-vector space V and $\{\sigma_\beta\}_{|\beta|=\dim(P)-q}$ is a basis for $H_{2q}(P)$ and thus

$$m = \#\{\alpha \in \mathcal{P}(n-k,k) : |\alpha| = q\} = \#\{\beta \in \mathcal{P}(n-k,k) : |\beta| = \dim(P) - q\} \quad .$$

Now consider fixed enumerations of the index sets $\{\alpha : |\alpha| = q\} = \{\alpha^i\}_{i=1,\dots,m}$ and $\{\beta : |\beta| = \dim(P) - q\} = \{\beta^i\}_{i=1,\dots,m}$ and let $e_i := \sigma_{\alpha^i} \in V$ as well as $f^i := \text{Kron}(PD(\sigma_{\beta^i})) \in V^\vee$. Then $\{e_i\}_i$ is a basis of V (see above) and $\{f^i\}_i$ is a basis of $V^\vee$ (because Kron and PD are $\mathbb{Q}$-linear isomorphisms). In the notation of Lemma 29 we have $e^i = \sigma^\vee_{\alpha^i}$. Furthermore, by Proposition 28 we obtain

$$f^j(e_i) = \langle\, PD(\sigma_{\beta^j})\,,\, \sigma_{\alpha^i}\,\rangle \overset{28}{=} \sigma_{\beta^j} \cdot \sigma_{\alpha^i} = \sigma_{\alpha^i} \cdot \sigma_{\beta^j} \quad ,$$

where we omit the augmentation map ε_* from now on since $P = G_k(\mathbb{C}^n)$ is connected and thus ε_* is an isomorphism. So, in view of Lemma 29, the change-of-basis matrix from $\{f^i\}_i$ to $\{e^i\}_i$ is given by $(\sigma_{\alpha^i} \cdot \sigma_{\beta^j})_{i,j}$. Lastly, apply the inverse of the Kronecker isomorphism $\text{Kron}^{-1} : V^\vee \xrightarrow{\sim} H^{2(\dim(P)-q)}(P)$ to map everything into the cohomology group of P. Then altogether we have proven the following corollary:

Corollary 30 (Base change from PD duals to linear duals)**.** *Let $P := G_k(\mathbb{C}^n)$ be a Grassmannian and $0 \le q \le \dim(P)$. For arbitrary $\beta \in \mathcal{P}(n-k,k)$ with $|\beta| = \dim(P) - q$ we have*

$$PD(\sigma_\beta) = \sum_{\substack{\alpha \in \mathcal{P}(n-k,k)\\ |\alpha|=q}} (\sigma_\alpha \cdot \sigma_\beta)\, \sigma^\vee_\alpha \quad ,$$

where we tacitly omit the augmentation map ε_*.
Furthermore, with respect to fixed enumerations $\{\alpha^i\}_{i=1,\dots,m} = \{\alpha : |\alpha| = q\}$ and $\{\beta^i\}_{i=1,\dots,m} = \{\beta : |\beta| = \dim(P) - q\}$, where $m := \dim_{\mathbb{Q}} H^{2(\dim(P)-q)}(P)$, the change-of-basis matrix from the basis $\{PD(\sigma_{\beta^i})\}_{i=1,\dots,m}$ of the $\mathbb{Q}$-vector space $H^{2(\dim(P)-q)}(P)$ to the basis $\{\sigma^\vee_{\alpha^i}\}_{i=1,\dots,m}$ is given by $(\sigma_{\alpha^i} \cdot \sigma_{\beta^j})_{i,j=1,\dots,m}$.

Remark 31. As it was our convention in Chapter 4 before, in the preceding corollary we write $\sigma^\vee_\alpha \in H^*(P)$ when we tacitly mean $\text{Kron}^{-1}(\sigma^\vee_\alpha)$.

Remark 32. Let $\alpha \in \mathcal{P}(n-k,k)$ be the complementary partition to $a \in \mathcal{P}(n-k,k)$, i.e. $\alpha_i = n - k - a_{k+1-i}$ for all i. Then $\sigma^\vee_\alpha = [X_a]^\vee$ in $H^*(G_k(\mathbb{C}^n))$ because for arbitrary $b \in \mathcal{P}(n-k,k)$ with $|b| = |a|$ and complementary partition β we have

$$\sigma^\vee_\alpha([X_b]) = \sigma^\vee_\alpha(\sigma_\beta) = \delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{else} \end{cases}$$

since $a = b \iff \alpha = \beta$.

Remark 33. Note that all statements in this chapter still hold true if we consider homology respectively cohomology with coefficients in $\mathbb{Z}$. Each proof works equally fine with $\mathbb{Z}$ in place of $\mathbb{Q}$, in particular we obtain the same formula for the change of basis in the finitely generated free abelian group $H^{2(\dim(P)-q)}(P;\mathbb{Z})$. This actually shows that the change-of-basis matrix $(\sigma_{\alpha^i} \cdot \sigma_{\beta^j})_{i,j}$ has entries in $\mathbb{Z}$ and is invertible as matrix with coefficients in $\mathbb{Z}$, i.e. lies in $\text{GL}(m,\mathbb{Z})$ with $m = \dim_{\mathbb{Q}} H^{2(\dim(P)-q)}(P;\mathbb{Q}) = \dim_{\mathbb{Q}} H_{2(\dim(P)-q)}(P;\mathbb{Q}) = \text{rk}_{\mathbb{Z}} H_{2(\dim(P)-q)}(P;\mathbb{Z})$.

Chapter 6

Solving the λ- and μ-Equations

Being able to transform the two canonical bases of the cohomology group of a Grassmannian into one another, we are now in the position to finalize the calculations we began in Chapter 4. At the end of this chapter we will have derived formulae for λ and μ, depending only on some coefficients which can be determined completely independently from the current setting, the one presented in Section 1.5. Thus we may as well consider our original problem as solved then. The rest of this thesis will be concerned with determining the values of those coefficients. Namely, we are going be interested in particular coefficients $\delta^{\beta}_{k,n}$, which we now introduce: For any Grassmannian $G = G_k(\mathbb{C}^n)$ we consider its first (rational) Pontryagin class $p_1(G) \overset{\text{def}}{=} p_1(TG) \in H^4(G;\mathbb{Q}) = H^4(G)$. Then bases of $H^4(G)$ are given by $\{PD(\sigma_\beta)\}_{|\beta|=2}$ and also $\{\sigma^\vee_\alpha\}_{|\alpha|=\dim(G)-2}$, see our considerations before Corollary 30 in Chapter 5. By said corollary, we are capable of transforming those two bases into one another and de facto it is of no relevance anymore which one we choose in the following, it solely is a matter of taste. We define the rational numbers[1] $\delta^{\beta}_{k,n}$, where $\beta = (\beta_1, \dots, \beta_k)$ ranges over all partitions in $\mathcal{P}(n-k,k)$, as the unique family of coefficients in the following linear combination:

$$p_1(G_k(\mathbb{C}^n)) = \sum_{\substack{\beta\in\mathcal{P}(n-k,k)\\|\beta|=2}} \delta^{\beta}_{k,n}\, PD_{G_k(\mathbb{C}^n)}(\sigma_\beta) \quad . \tag{6.1}$$

6.1 Solving the λ-Equation

As we have already assigned in Section 4.1, we put $P = G_2(\mathbb{C}^5)$, $n = 5$, $k = 2$, $a = (3,2)$, $b = (3,0)$, $c = (2,0)$ and $X = X_a = X_{3,2}$ with respect to the flag F^{std}_* as well as $M = X_b(F'_*) = X_3(F^1_*)$ [2] and $Y = X_c(F''_*) = X_2(F^2_*)$ with respect to flags $F'_* = F^1_*$ respectively $F''_* = F^2_*$. It is $\dim(P) = \dim_{\mathbb{C}}(G_k(\mathbb{C}^n)) = k\,(n-k) = 6$. We have

$$\lambda = 1 + \langle\, L^1(\nu_M|_Y)\,,\, [Y]\,\rangle$$
$$\langle\, L^1(\nu_M|_Y)\,,\, [Y]\,\rangle \;=\; \mu_c - \lambda_c$$

[1] Actually these numbers will always be integers as we will explain in detail in Chapter 7, but the advanced reader probably grasps this directly from the definition.

[2] The correct notation would be $X_{3,0}(F^1_*)$ of course, but we tend to leave out trailing zeroes.

by Equations (1.3) and (4.5), where the above coefficients μ_c, λ_c are defined via Equations (4.1) and (4.2).

Computing μ_c: We are considering the fourth cohomology group of P, so put $q := 4$ so that $2\,(\dim(P) - q) = 4$. We want to invoke Corollary 30, so we first fix orderings of $\{\alpha : |\alpha| = 4\}$ and of $\{\beta : |\beta| = 2\}$. We choose the orderings $(\alpha^1, \alpha^2) := ((3,1)\,,\,(2,2))$ [3] and $(\beta^1, \beta^2) = ((1,1)\,,\,(2,0))$. Corollary 30 then yields

$$PD(\sigma_{\beta^j}) = \sum_{i=1}^{2} (\sigma_{\alpha^i} \cdot \sigma_{\beta^j})\, \sigma_{\alpha^i}^{\vee} \quad \forall\, j = 1, 2 \quad .$$

So now we have to compute the values of $\sigma_{\alpha^i} \cdot \sigma_{\beta^j}$, $i, j = 1, 2$, via Schubert calculus. This is done in Appendix A.1.1. We obtain

$$\sigma_{3,1} \cdot \sigma_{1,1} = 0 \qquad\qquad \sigma_{3,1} \cdot \sigma_2 = 1$$
$$\sigma_{2,2} \cdot \sigma_{1,1} = 1 \qquad\qquad \sigma_{2,2} \cdot \sigma_2 = 0$$

and thus

$$PD(\sigma_{1,1}) = \sigma_{2,2}^{\vee} = [X_{1,1}]^{\vee} \qquad\qquad PD(\sigma_2) = \sigma_{3,1}^{\vee} = [X_2]^{\vee} \quad , \tag{6.2}$$

where we refer to the (trivial) Remark 32 for the second equality respectively. Let us state Equation (4.1) again with respect to the current setting:

$$L^1(P) = \sum_{\substack{d \in \mathcal{P}(3,2) \\ |d| = 2}} \mu_d\, [X_d]^{\vee} = \mu_{1,1}\, [X_{1,1}]^{\vee} + \mu_{2,0}\, [X_{2,0}]^{\vee} \quad .$$

On the other hand, by definition of the Hirzebruch L-class we have $L^1(\xi) = \frac{1}{3}\, p_1(\xi) \in H^4(B)$ for every vector bundle $\xi : E \to B$, in particular for ξ the tangent bundle $TP \to P$ we obtain

$$\begin{aligned} L^1(P) &= \frac{1}{3}\, p_1(P) = \frac{1}{3}\, p_1(G_k(\mathbb{C}^n)) \\ &= \frac{1}{3}\, (\, \delta_{2,5}^{1,1}\ PD(\sigma_{1,1}) \,+\, \delta_{2,5}^{2,0}\ PD(\sigma_{2,0})\,) && \text{by (6.1)} \\ &= \frac{1}{3}\, \delta_{2,5}^{1,1}\ [X_{1,1}]^{\vee} \,+\, \frac{1}{3}\, \delta_{2,5}^{2,0}\ [X_{2,0}]^{\vee} && \text{by (6.2)} \end{aligned}$$

and comparing coefficients yields

$$\mu_{1,1} = \frac{1}{3}\, \delta_{2,5}^{1,1} \qquad\qquad \mu_{2,0} = \frac{1}{3}\, \delta_{2,5}^{2,0} \quad . \tag{6.3}$$

In particular

$$\mu_c = \mu_{2,0} = \frac{1}{3}\, \delta_{2,5}^{2,0} \quad . \tag{6.4}$$

Computing λ_c: Recall that

$$L^1(M) = \sum_{\substack{d \le b \\ |d| = 2}} \lambda_d\, \left[X_d(F'_*)\right]_M^{\vee} = \lambda_{2,0}\, [X_{2,0}(F'_*)]_M^{\vee}$$

[3] Notice that $4 > n - k = 3$, so $\alpha = (4,0)$ is not in $\mathcal{P}(n-k, k)$. Also note that, since $\#\{\alpha : |\alpha| = q\} = 2$, apparently $H^4(P)$ is a 2-dimensional $\mathbb{Q}$-vector space.

by definition, cf. Equation (4.2). We have $M = X_b(F'_*)$ with $b = (3,0)$, so $l = 1\,,\, b_0 = 3\,$, in the notation of Section 4.1 since there is precisely 1 nonzero entry in b and it has the value 3. Now Lemma 27 allows us to determine λ_d for $d \leq b = (3,0)\,,\, |d| = 2$: We consider the Grassmannian $G := G_l(\mathbb{C}^{l+b_0}) = G_1(\mathbb{C}^4) \cong \mathbb{C}P^3$. Equation (4.6), defining the $\widetilde{\lambda}_{d'}$'s, in the current case is

$$L^1(G) = \sum_{\substack{d' \in \mathcal{P}(3,1) \\ |d'|=2}} \widetilde{\lambda}_{d'}\, [X_{d'}]_G^\vee = \widetilde{\lambda}_2\, [X_2]_G^\vee \in H^4(G) \quad .$$

By Lemma 27 we have

$$\lambda_c = \lambda_{2,0} = \widetilde{\lambda}_{(2,0)|} = \widetilde{\lambda}_2 \quad ,$$

so it suffices to determine $\widetilde{\lambda}_2$. This is now achieved analogously to before, but pay attention that we now are working inside $G = G_1(\mathbb{C}^4)$ and not $P = G_2(\mathbb{C}^5)$ anymore. We have $\sigma_1 = [X_2]\,,\, \sigma_2 = [X_1]$. Put $q := 1$, so that $2\,(\dim(G) - q) = 4$. We aim for applying Corollary 30, so let us compute the intersection product $\sigma_1 \cdot \sigma_2$. By Pieri $\sigma_1\, \sigma_2 = \sigma_3 = [\text{pt.}] = 1$, so

$$PD(\sigma_2) = \sigma_1^\vee \quad . \tag{6.5}$$

We have

$$\begin{aligned}
L^1(G) &= \widetilde{\lambda}_2\, [X_2]^\vee \overset{\text{Rem. 32}}{=} \widetilde{\lambda}_2\; \sigma_1^\vee \\
L^1(G) &= \frac{1}{3}\, p_1(G) = \frac{1}{3}\, p_1(G_1(\mathbb{C}^4)) = \frac{1}{3}\, (\delta_{1,4}^2\, PD(\sigma_2)) && \text{by (6.1)} \\
&= \frac{1}{3}\, \delta_{1,4}^2\; \sigma_1^\vee && \text{by (6.5)}
\end{aligned}$$

which yields $\widetilde{\lambda}_2 = \frac{1}{3}\, \delta_{1,4}^2$ by comparing coefficients. Altogether

$$\lambda_c = \lambda_{2,0} = \widetilde{\lambda}_2 = \frac{1}{3}\, \delta_{1,4}^2 \quad . \tag{6.6}$$

Hence, we have simplified the equation

$$\lambda = 1 + \langle\, L^1(\nu_M|_Y)\,,\, [Y]\,\rangle = 1 + (\mu_c - \lambda_c)$$

for λ to

$$\boxed{\lambda = 1 + \frac{1}{3}\, (\delta_{2,5}^{2,0} - \delta_{1,4}^2)} \quad . \tag{6.7}$$

6.2 Solving the μ-Equation

The steps in our calculation for μ will now be analogous, or at least similar, to the ones in the calculation for λ, but now we additionally have to compute a cup product. We will achieve this via the technique already explained in Chapter 5.

Similar to Section 6.1 before, we first recapitulate already introduced notation, this time from Section 4.2: We are working inside $P = G_2(\mathbb{C}^5)$, so $k = 2\,,\, n = 5\,$, let $X = X_a = X_a(F_*^{\text{std}})$ with

$a = (3,2)$, $M = X_b(F'_*)$ with $b = (2,2)$, $F'_* = F^3_*$ and $Y = M \cap X = X_c(F''_*)$ with $c = (2,1)$, $F''_* = F^4_*$. In Equation (4.11) we already computed that

$$\begin{aligned}
&\langle\, \omega \smile L^1(\nu_M|_Y)\,,\, [Y]_Y \,\rangle \\
&= \sum_{\substack{d \in \mathcal{P}(3,2) \\ |d|=2}} \mu_d \,\langle\, [X_{1,0}]^\vee \smile [X_d]^\vee \,,\, [X_{2,1}] \,\rangle - \sum_{\substack{d \le (2,2) \\ |d|=2}} \lambda_d \,\langle\, [X_{1,0}]^\vee \smile [X_d]^\vee \,,\, [X_{2,1}] \,\rangle \\
&= \quad \mu_{1,1} \,\langle\, [X_{1,0}]^\vee \smile [X_{1,1}]^\vee \,,\, [X_{2,1}] \,\rangle + \mu_{2,0} \,\langle\, [X_{1,0}]^\vee \smile [X_{2,0}]^\vee \,,\, [X_{2,1}] \,\rangle \\
&\quad - \lambda_{1,1} \,\langle\, [X_{1,0}]^\vee \smile [X_{1,1}]^\vee \,,\, [X_{2,1}] \,\rangle - \lambda_{2,0} \,\langle\, [X_{1,0}]^\vee \smile [X_{2,0}]^\vee \,,\, [X_{2,1}] \,\rangle
\end{aligned}$$

We already know the values of $\mu_{1,1}$, $\mu_{2,0}$ in terms of the δ-coefficients by Equation (6.3), namely $\mu_{1,1} = \frac{1}{3}\,\delta^{1,1}_{2,5}$, $\mu_{2,0} = \frac{1}{3}\,\delta^{2,0}_{2,5}$. By Equation (6.2) we also know that $PD(\sigma_{1,1}) = \sigma^\vee_{2,2} = [X_{1,1}]^\vee$ and $PD(\sigma_{2,0}) = \sigma^\vee_{3,1} = [X_{2,0}]^\vee$. In addition we have

$$PD(\sigma_{1,0}) = \sigma^\vee_{3,2} = [X_{1,0}]^\vee \in H^2(P) \quad . \tag{6.8}$$

This follows from Corollary 30 since $\sigma_{3,2} \cdot \sigma_{1,0} \overset{\text{Pieri}}{=} \sigma_{3,3} + \sigma_{4,2} = \sigma_{3,3} + 0 = 1$.

Combining all of these results, we get

$$\begin{aligned}
\langle\, [X_{1,0}]^\vee \smile [X_{1,1}]^\vee \,,\, [X_{2,1}] \,\rangle &= \langle\, PD(\sigma_{1,0}) \smile PD(\sigma_{1,1}) \,,\, \sigma_{2,1} \,\rangle \\
&= \langle\, PD(\sigma_{1,0} \cdot \sigma_{1,1}) \,,\, \sigma_{2,1} \,\rangle \\
&= (\sigma_{1,0} \cdot \sigma_{1,1}) \cdot \sigma_{2,1} && \text{by Proposition 28} \\
&= \sigma_1 \cdot (\sigma_1^2 - \sigma_2) \cdot \sigma_{2,1} && \text{cf. Appendix A.1.1} \\
&= (\sigma_{3,1} + \sigma_{2,2}) \cdot (\sigma_1^2 - \sigma_2) && \text{by Pieri} \\
&= (\sigma_{3,3} + \sigma_{3,3}) - (\sigma_{3,3} + 0) && \text{by Pieri} \\
&= 1 \\
\langle\, [X_{1,0}]^\vee \smile [X_{2,0}]^\vee \,,\, [X_{2,1}] \,\rangle &= \langle\, PD(\sigma_{1,0}) \smile PD(\sigma_{2,0}) \,,\, \sigma_{2,1} \,\rangle \\
&= \langle\, PD(\sigma_{1,0} \cdot \sigma_{2,0}) \,,\, \sigma_{2,1} \,\rangle \\
&= (\sigma_{1,0} \cdot \sigma_{2,0}) \cdot \sigma_{2,1} && \text{by Proposition 28} \\
&= (\sigma_{3,0} + \sigma_{2,1}) \cdot \sigma_{2,1} && \text{by Pieri} \\
&= \sigma_{3,0} \cdot \sigma_{2,1} + (\sigma_{1,0} \cdot \sigma_{1,1}) \cdot \sigma_{2,1} && \text{by Pieri} \\
&= (\sigma_{1,0} \cdot \sigma_{1,1}) \cdot \sigma_{2,1} && \text{by Pieri} \\
&= 1 && \text{see above}
\end{aligned}$$

Hence the above equation simplifies to

$$\begin{aligned}
\langle\, \omega \smile L^1(\nu_M|_Y)\,,\, [Y]_Y \,\rangle &= \mu_{1,1} + \mu_{2,0} - \lambda_{1,1} - \lambda_{2,0} \\
&= \frac{1}{3}\,(\delta^{1,1}_{2,5} + \delta^{2,0}_{2,5}) - \lambda_{1,1} - \lambda_{2,0} \quad .
\end{aligned}$$

It remains to determine $\lambda_{1,1}$ and $\lambda_{2,0}$. By the defining Equation (4.10) we have

$$L^1(M) = \lambda_{1,1}\,[X_{1,1}(F'_*)]^\vee_M + \lambda_{2,0}\,[X_{2,0}(F'_*)]^\vee_M \quad .$$

With the terminology of Lemma 27 we have $l = k = 2$, $b_0 = 2$, since $b = (2,2)$. Putting $G := G_l(\mathbb{C}^{b_0+l}) = G_2(\mathbb{C}^4)$, the defining Equation (4.6) becomes

$$L^1(G) = \widetilde{\lambda}_{1,1}\,[X_{1,1}]_G^\vee + \widetilde{\lambda}_{2,0}\,[X_{2,0}]_G^\vee$$

and by Lemma 27 we have $\lambda_{1,1} = \widetilde{\lambda}_{1,1}$, $\lambda_{2,0} = \widetilde{\lambda}_{2,0}$. Now the values of $\widetilde{\lambda}_{1,1}$, $\widetilde{\lambda}_{2,0}$ can be computed analogously to before: We have to express $[X_{1,1}]_G^\vee = \sigma_{1,1}^\vee$, $[X_{2,0}]_G^\vee = \sigma_{2,0}^\vee$ in terms of $PD(\sigma_{1,1})$, $PD(\sigma_{2,0})$, where we are now temporarily working inside $G = G_2(\mathbb{C}^4)$. This can again be achieved via Corollary 30, which yields

$$\begin{aligned} PD(\sigma_{1,1}) &= (\sigma_{1,1}\cdot\sigma_{1,1})\,\sigma_{1,1}^\vee + (\sigma_{2,0}\cdot\sigma_{1,1})\,\sigma_{2,0}^\vee \\ PD(\sigma_{2,0}) &= (\sigma_{1,1}\cdot\sigma_{2,0})\,\sigma_{1,1}^\vee + (\sigma_{2,0}\cdot\sigma_{2,0})\,\sigma_{2,0}^\vee \quad . \end{aligned}$$

The appendant Schubert calculus is transferred to Appendix A.1.2, in which we calculate (see Equations (A.1), (A.2), (A.3)) that

$$\sigma_2^2 = 1 \qquad\qquad \sigma_{1,1}^2 = 1 \qquad\qquad \sigma_{1,1}\cdot\sigma_2 = 0$$

and hence

$$PD(\sigma_{1,1}) = \sigma_{1,1}^\vee = [X_{1,1}]_G^\vee \qquad\qquad PD(\sigma_{2,0}) = \sigma_{2,0}^\vee = [X_{2,0}]_G^\vee \quad . \tag{6.9}$$

Now

$$\begin{aligned} L^1(G) &= \frac{1}{3}\,p_1(G) = \frac{1}{3}\,p_1(G_2(\mathbb{C}^4)) \\ &= \frac{1}{3}\,(\delta_{2,4}^{1,1}\,PD(\sigma_{1,1}) + \delta_{2,4}^{2,0}\,PD(\sigma_{2,0})) && \text{by Equation (6.1)} \\ &= \frac{1}{3}\,\delta_{2,4}^{1,1}\,[X_{1,1}]_G^\vee + \frac{1}{3}\,\delta_{2,4}^{2,0}\,[X_{2,0}]_G^\vee && \text{by Equation (6.9)} \end{aligned}$$

Comparing coefficients, we see that $\widetilde{\lambda}_{1,1} = \frac{1}{3}\,\delta_{2,4}^{1,1}$ and $\widetilde{\lambda}_{2,0} = \frac{1}{3}\,\delta_{2,4}^{2,0}$, and thus

$$\lambda_{1,1} = \frac{1}{3}\,\delta_{2,4}^{1,1} \qquad\qquad \lambda_{2,0} = \frac{1}{3}\,\delta_{2,4}^{2,0} \quad . \tag{6.10}$$

Inserting this into above formula yields

$$\begin{aligned} \langle\, \omega \smile L^1(\nu_M|_Y)\,,\,[Y]_Y\,\rangle &= \frac{1}{3}\,(\delta_{2,5}^{1,1} + \delta_{2,5}^{2,0}) - \lambda_{1,1} - \lambda_{2,0} \\ &= \frac{1}{3}\,(\delta_{2,5}^{1,1} + \delta_{2,5}^{2,0} - \delta_{2,4}^{1,1} - \delta_{2,4}^{2,0}) \end{aligned}$$

and combining this with the initial equation for μ, Equation (1.4), we finally obtain the simplified formula for μ:

$$\boxed{\mu = \frac{1}{3}\left(2 + \delta_{2,5}^{1,1} + \delta_{2,5}^{2,0} - \delta_{2,4}^{1,1} - \delta_{2,4}^{2,0}\right)} \quad . \tag{6.11}$$

Chapter 7

On Changing Coefficients from Rationals to Integers

Until now all homology respectively cohomology groups, we were dealing with, had rational coefficients (except in Chapter 2 where we considered integer coefficients and when explicitly stated otherwise). When we determine the δ-coefficients in the following two chapters however, it will be vital to work with $\mathbb{Z}$-coefficients, mostly because of arguments containing generators of the free abelian groups of interest. Thus, first of all we will carefully examine the behavior under change of coefficients from integers to rationals. For the most part, this chapter's content will be a recapitulation of standard knowledge in algebraic topology, so the familiar reader might be pleased to be given a short summary and then skip the rest: This is that everything behaves well. In particular, (linear) duals of Schubert classes with respect to $\mathbb{Z}$-coefficients are mapped[1] onto the respective (linear) duals with respect to $\mathbb{Q}$-coefficients, this is Equation (7.5), and similarly for Poincaré duals, this is Corollary 36. The latter fact has been exploited implicitly already in some nonessential side notes in previous chapters: Namely in Footnote (1), Chapter 6, regarding the definition of the δ-coefficients, and a bit more subtly in Remark 33.

Let us start by giving the basic definitions to avoid any misunderstandings: We change notation, from now on omitting the coefficient ring in the notation of homology respectively cohomology means that we are working with integer coefficients. Given any topological space X, let the i-th chain group be defined as $C_i(X) = C_i(X;\mathbb{Z}) := \bigoplus_\sigma \mathbb{Z}\,\sigma$, where σ ranges over all continuous functions $\Delta^i \to X$. That is, $C_i(X) = \mathbb{Z}^{(\{\sigma:\Delta^i\to X \text{ cont.}\})}$. Let $Z_i(X) = Z_i(X;\mathbb{Z})$ denote the i-cycles and $H_i(X) = H_i(X;\mathbb{Z})$ the i-th homology group with respect to $C_*(X)$. Analogously for $\mathbb{Q}$: Put $C_i(X;\mathbb{Q}) := \bigoplus_\sigma \mathbb{Q}\,\sigma = \mathbb{Q}^{(\{\sigma:\Delta^i\to X \text{ cont.}\})}$ and $Z_i(X;\mathbb{Q})$ and $H_i(X;\mathbb{Q})$ are the i-cycles respectively the i-th homology group with respect to the rationals. Note that we have a canonical isomorphism $C_i(X;\mathbb{Q}) \cong C_i(X)\otimes\mathbb{Q}$ [2] of $\mathbb{Q}$-vector spaces for any i. There is the following commutative diagram

[1] via the canonical map $\mathrm{can}^* : H^*(X_a;\mathbb{Z}) \to H^*(X_a;\mathbb{Q})$, whose construction will be repeated below

[2] $\otimes$ without any further specification means tensoring as abelian groups, i.e. $\otimes_\mathbb{Z}$

of chain complexes of abelian groups:

$$\begin{tikzcd} C_*(X) \arrow[r, hook] \arrow[dr, hook] & C_*(X;\mathbb{Q}) \\ & C_*(X)\otimes\mathbb{Q} \arrow[u, "\cong"'] \end{tikzcd}$$

where the horizontal map sends every σ to σ [3]. The horizontal chain map induces on homology level the canonical change-of-coefficients map $\mathrm{can}_i = \mathrm{can}_i^{\mathbb{Z}\to\mathbb{Q}} : H_i(X;\mathbb{Z}) \to H_i(X;\mathbb{Q})$. Obviously, can_i is natural in X. We have the canonical ($\mathbb{Q}$-linear) identification

$$\begin{array}{ccccc} H_i(X)\otimes\mathbb{Q} & \xrightarrow{\cong} & H_i(C_*(X)\otimes\mathbb{Q}) & \xrightarrow{\cong} & H_i(X;\mathbb{Q}) \\ [\,c\,]\otimes q & \longmapsto & [\,c\otimes q\,] & \longmapsto & [\,q\,c\,] \end{array} \tag{7.1}$$

where the first map is an isomorphism via the Algebraic Universal Coefficient Theorem for Homology[4]. If $H_i(X)$ is a finitely generated free abelian group of rank r with $\mathbb{Z}$-basis $[c_1],\ldots,[c_r]$, then $[c_1]\otimes 1,\ldots,[c_r]\otimes 1$ is a $\mathbb{Q}$-basis of $H_i(X)\otimes\mathbb{Q}$, so via the isomorphism from (7.1) also $[c_1] = \mathrm{can}_i\,[c_1],\ldots,[c_r] = \mathrm{can}_i\,[c_r]$ is a $\mathbb{Q}$-basis of $H_i(X;\mathbb{Q})$. This applies to the case of a Schubert variety X_a (with respect to a fixed flag F_* [5]) since its integer homology group $H_{2i}(X_a,\mathbb{Z})$ is freely generated by the classes $[X_b]$ for $b \le a$, $|b| = i$ [6]. So a $\mathbb{Q}$-basis of $H_{2i}(X_a;\mathbb{Q})$ is given by $\mathrm{can}_{2i}([X_b])$, $b \le a$, $|b| = i$. So far, we denoted these basis elements simply by $[X_b] \in H_{2i}(X_a;\mathbb{Q})$ and we will continue to do so, if the coefficients are understood.

Now let us treat the cohomological case: For any topological space X define $C^i(X) := C^i(X;\mathbb{Z}) := \mathrm{Hom}_{\mathbb{Z}}(C_i(X;\mathbb{Z}),\mathbb{Z})$ and $C^i(X;\mathbb{Q}) := \mathrm{Hom}_{\mathbb{Q}}(C_i(X;\mathbb{Q}),\mathbb{Q})$. Taking the i-th homology of the corresponding chain complexes yields $H^i(X) := H^i(X;\mathbb{Z})$ respectively $H^i(X;\mathbb{Q})$. We now construct a canonical map $C^i(X;\mathbb{Z}) \hookrightarrow C^i(X;\mathbb{Q})$: First of all there is a natural injection $\mathrm{Hom}_{\mathbb{Z}}(C_i(X),\mathbb{Z}) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(C_i(X),\mathbb{Q})$ via composing with the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. By tensor-hom adjunction we have a $\mathbb{Q}$-linear isomorphism $\mathrm{Hom}_{\mathbb{Z}}(C_i(X),\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(C_i(X)\otimes\mathbb{Q},\mathbb{Q})$. Then composing these yields the canonical map:

$$\begin{tikzcd} C^i(X;\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(C_i(X),\mathbb{Z}) \arrow[r, hook] \arrow[ddrr, hook] & \mathrm{Hom}_{\mathbb{Z}}(C_i(X),\mathbb{Q}) \arrow[r, equal, "\sim", "\text{tensor-hom}"'] & \mathrm{Hom}_{\mathbb{Q}}(C_i(X)\otimes\mathbb{Q},\mathbb{Q}) \arrow[d, equal, "\wr"'] \\ & & \mathrm{Hom}_{\mathbb{Q}}(C_i(X;\mathbb{Q}),\mathbb{Q}) \arrow[d, equal] \\ & & C^i(X;\mathbb{Q}) \end{tikzcd}$$

[3]Strictly speaking of course, it maps every function with codomain $\mathbb{Z}$ of the form $\tau \mapsto \begin{cases} 1 \text{ if } \tau = \sigma \\ 0 \text{ else} \end{cases}$ to the function with codomain $\mathbb{Q}$: $\tau \mapsto \begin{cases} 1 \text{ if } \tau = \sigma \\ 0 \text{ else} \end{cases}$.

[4]see [Bre93, Chapt. V, Thm. 7.4]

[5]In this chapter no specified flag in the notation of a Schubert variety means that it is with respect to the fixed flag F_*.

[6]$[X_b]$ is the image of $[X_b] \in A_i(X_a)$ under the cycle map, where $A_*(-)$ denotes Chow homology.

Since tensor-hom adjunction is natural and $C_*(X) \otimes \mathbb{Q} \xrightarrow{\cong} C_*(X \otimes \mathbb{Q})$ is a chain map, $C^*(X) \to C^*(X;\mathbb{Q})$ is a composition of chain maps and thus a chain map too. Then taking the i-th homology gives a canonical homomorphism of abelian groups $\mathrm{can}^i = \mathrm{can}^i_{\mathbb{Z}\to\mathbb{Q}} : H^i(X;\mathbb{Z}) \to H^i(X;\mathbb{Q})$. Similar to the argument that $C^*(X) \to C^*(X;\mathbb{Q})$ is a chain map, one sees that $C^j(X) \to C^j(X;\mathbb{Q})$ is natural in X for all j, so can^i is natural in X as well.

In the following we will work with the definitions of cup and cap product given in Hatcher [Hat01, pp. 206, 239], although this leads to subtle problems regarding the sign convention, see Remark 69 in the Appendix. Let us (temporarily) keep track of the coefficient ring if we take cup and cap products, e.g. we write $\frown_{\mathbb{Q}}$ for the cap product with respect to $\mathbb{Q}$-coefficients. Then the following two (often implicitly used) facts, for which a proof is presented in Appendix A.2, hold:

Fact 34. *For $\alpha \in H^i(X)$, $\beta \in H^j(X)$ we have*

$$\mathrm{can}^{i+j}(\alpha \smile_{\mathbb{Z}} \beta) = \mathrm{can}^i(\alpha) \smile_{\mathbb{Q}} \mathrm{can}^j(\beta) \ \in H^{i+j}(X;\mathbb{Q}) \quad .$$

Fact 35. *For $\alpha \in H^i(X)$, $b \in H_k(X)$ we have*

$$\mathrm{can}_{k-i}(\alpha \frown_{\mathbb{Z}} b) = \mathrm{can}^i(\alpha) \frown_{\mathbb{Q}} \mathrm{can}_k(b) \ \in H_{k-i}(X;\mathbb{Q}) \quad .$$

In more algebraic terms, Fact 34 states that $\mathrm{can}^* : H^*(X) \to H^*(X;\mathbb{Q})$ is a graded ring homomorphism and Fact 35 states that $\mathrm{can}_* : H_*(X) \to H_*(X;\mathbb{Q})$ is a homomorphism of graded $H^*(X)$-modules[7], where $H_*(X;\mathbb{Q})$ is understood as graded $H^*(X)$-module via the graded ring homomorphism $\mathrm{can}^* : H^*(X) \to H^*(X;\mathbb{Q})$.

Now consider an n-dimensional closed oriented manifold M with fundamental class $[M]_{\mathbb{Z}}$. Then its image under the change-of-coefficients map $[M]_{\mathbb{Q}} := \mathrm{can}_n [M]_{\mathbb{Z}}$ is the fundamental class of M with respect to $\mathbb{Q}$-coefficients[8]. We thus have Poincaré duality isomorphisms

$$PD_{\mathbb{Z}}^{-1} : H^i(M;\mathbb{Z}) \xrightarrow{\sim} H_{n-i}(M;\mathbb{Z}), \ \alpha \mapsto \alpha \frown_{\mathbb{Z}} [M]_{\mathbb{Z}}$$
$$PD_{\mathbb{Q}}^{-1} : H^i(M;\mathbb{Q}) \xrightarrow{\sim} H_{n-i}(M;\mathbb{Q}), \ \alpha \mapsto \alpha \frown_{\mathbb{Q}} [M]_{\mathbb{Q}} \quad .$$

An immediate consequence of Fact 35 is that $\mathrm{can}_{n-i} PD_{\mathbb{Z}}^{-1}(\alpha) = PD_{\mathbb{Q}}^{-1}\mathrm{can}^i(\alpha)$ for all $\alpha \in H^i(M;\mathbb{Z})$. Thus we have proven the following two results:

Corollary 36. $PD_{\mathbb{Q}}\,\mathrm{can}_* = \mathrm{can}^*\,PD_{\mathbb{Z}} \quad .$

Corollary 37. *$a \in H_i(M;\mathbb{Z})$, $b \in H_j(M;\mathbb{Z})$ arbitrary. Then*

$$\mathrm{can}_i(a) \cdot_{\mathbb{Q}} \mathrm{can}_j(b) = \mathrm{can}_{i+j-n}(a \cdot_{\mathbb{Z}} b) \quad ,$$

where $\cdot_{\mathbb{Z}}$ respectively $\cdot_{\mathbb{Q}}$ are the intersection products with integer respectively rational coefficients.

Proof of Corollary 37. For the intersection product $\cdot$ we have

$$a \cdot b = PD^{-1}(PD(a) \smile PD(b)) \quad .$$

Now apply Corollary 36 and Fact 34. $\square$

[7] Here we (temporarily) consider $H^*(X)$ with grading $i \mapsto H^{-i}(X)$ and analogously for $H^*(X;\mathbb{Q})$.

[8] Here M is oriented with respect to rational coefficients via its given orientation (with respect to integer coefficients), mapped into homology with rational coefficients via $\mathrm{can}_n : H_n(M, M-x;\mathbb{Z}) \to H_n(M, M-x;\mathbb{Q})$ for $x \in M$.

We are interested in Corollary 36 since in the case of a Grassmannian $P = G_k(\mathbb{C}^n)$ it particularly implies that for any partition $b \in \mathcal{P}(n-k,k)$ the integer cohomology class $PD_{\mathbb{Z}}\,[X_b]$ is mapped to $PD_{\mathbb{Q}}\,[X_b]$ via the canonical map $\mathrm{can}^* : H^*(P;\mathbb{Z}) \to H^*(P;\mathbb{Q})$. Hence we can simply switch from $\mathbb{Z}$- to $\mathbb{Q}$-coefficients and vice versa when considering Poincaré duals of Schubert classes. To give an example, this allowed us to deduce in Footnote (1), Chapter 6, that the δ-coefficients actually have to be integers since we can define integer coefficients for the first integer Pontryagin class analogously to (6.1). Then Corollary 36 directly shows that these integer coefficients also fulfill Equation (6.1), so they have to coincide with the rational coefficients $\delta^{\beta}_{k,n}$ [9].

Corollary 37, on the other hand, allows for computations within Schubert calculus without paying attention to whether we work with integer or rational coefficients, since $\sigma_\alpha \cdot_{\mathbb{Q}} \sigma_\beta = \mathrm{can}_*(\sigma_\alpha \cdot_{\mathbb{Z}} \sigma_\beta)$ for all Schubert classes σ_α , σ_β , where on the left-hand side we of course mean the images of the respective integer Schubert classes under can_*. In particular, if $|\alpha| + |\beta| = \dim_{\mathbb{C}} P$, then $\varepsilon_*(\sigma_\alpha \cdot_{\mathbb{Q}} \sigma_\beta) = \varepsilon_*(\sigma_\alpha \cdot_{\mathbb{Z}} \sigma_\beta) \in \mathbb{Z}$, because $\varepsilon_*\,\mathrm{can}_0 = (\mathbb{Z} \hookrightarrow \mathbb{Q}) \circ \varepsilon_*$.

We now aim at deriving similar results with regard to linear dual bases. For this assume that X is a topological space such that all integer homology groups of X are free and finitely generated $\mathbb{Z}$-modules. Then, by the UCT[10], also all cohomology groups are free and finitely generated.

Our goal is to establish a commutative diagram of the following form:

$$\begin{tikzcd} H^i(X;\mathbb{Z}) \arrow[r, "\cong", "\mathrm{Kron}"'] \arrow[d, "\mathrm{can}^i"'] & \mathrm{Hom}_{\mathbb{Z}}(H_i(X;\mathbb{Z}),\mathbb{Z}) \arrow[d, "\mathrm{nat}"] \\ H^i(X;\mathbb{Q}) \arrow[r, "\cong", "\mathrm{Kron}"'] & \mathrm{Hom}_{\mathbb{Q}}(H_i(X;\mathbb{Q}),\mathbb{Q}) \end{tikzcd} \tag{7.2}$$

Here the horizontal arrows are the ordinary Kronecker maps and can^i was defined above. We now give the definition of the *natural map* nat:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}}(H_i(X),\mathbb{Z}) \arrow[r, hook] \arrow[drr, "\mathrm{nat}"'] & \mathrm{Hom}_{\mathbb{Z}}(H_i(X),\mathbb{Q}) \arrow[r, equal, "\sim", "\mathrm{tensor-hom}"'] & \mathrm{Hom}_{\mathbb{Q}}(H_i(X)\otimes\mathbb{Q},\mathbb{Q}) \arrow[d, equal, "\wr"', "(7.1)"] \\ & & \mathrm{Hom}_{\mathbb{Q}}(H_i(X;\mathbb{Q}),\mathbb{Q}) \end{tikzcd} \tag{7.3}$$

Proposition 38. *Diagram* (7.2) *commutes, where* nat *is defined in* (7.3).

The proof of this proposition is again transferred to Appendix A.2. We introduce the following notation, which indeed has been established partially already and will play a role in the upcoming chapters too: For any free abelian group H put $H^\vee := \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Z})$ and for any $\mathbb{Q}$-vector space W put $W^\vee := \mathrm{Hom}_{\mathbb{Q}}(W,\mathbb{Q})$, where strictly speaking we should of course also keep track of the underlying ring in the notation, e.g. we should write $H^{\vee,\mathbb{Z}}$ or something similar, but we will make sure to avoid confusion in the following.

We want to apply Proposition 38 to the setting we are interested in: Namely, let $X = X_a$ be a Schubert variety with partition $a = (a_1,\ldots,a_k)$. Then a $\mathbb{Z}$-basis of the finitely generated free abelian group $H_{2j}(X)$ is given by the Schubert classes $[X_b]$, $b \le a$, $|b| = j$. Let $[X_b]_{\mathbb{Q}} := \mathrm{can}_{2j}[X_b]$ be the

[9] Recall that by definition the rational Pontryagin class is simply the image of the integer Pontryagin class under can^*.

[10] Universal Coefficient Theorem

corresponding Schubert classes with respect to rational coefficients. Then $[X_b]_{\mathbb{Q}}$, $b \leq a$, $|b| = j$, form a $\mathbb{Q}$-basis of $H_{2j}(X;\mathbb{Q})$. Furthermore let $\{[X_b]^\vee\}_{b\leq a\,,\,|b|=j}$ denote the integer dual basis of $\{[X_b]\}_{b\leq a\,,\,|b|=j}$, i.e. $[X_b]^\vee \in H_{2j}(X)^\vee$ and $[X_b]^\vee\,[X_c] = \delta_{b,c}$ for $c \leq a$, $|c| = j$, where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta. Analogously let $\{[X_b]_{\mathbb{Q}}^\vee\}_{b\leq a\,,\,|b|=j}$ denote the rational dual basis of $\{[X_b]_{\mathbb{Q}}\}_{b\leq a\,,\,|b|=j}$, i.e. $[X_b]_{\mathbb{Q}}^\vee \in H_{2j}(X;\mathbb{Q})^\vee$ and $[X_b]_{\mathbb{Q}}^\vee\,[X_c]_{\mathbb{Q}} = \delta_{b,c}$ for $c \leq a$, $|c| = j$. We now make the following claim:

$$\mathrm{nat}\,[X_b]^\vee = [X_b]_{\mathbb{Q}}^\vee \quad . \tag{7.4}$$

Proof of Equation (7.4). We show $(\mathrm{nat}\,[X_b]^\vee)\,[X_c]_{\mathbb{Q}} = \delta_{b,c}$ for all $c \leq a$, $|c| = j$. This indeed holds since

$$\begin{aligned}
&(\mathrm{nat}\,[X_b]^\vee)\,[X_c]_{\mathbb{Q}} \\
&= \mathrm{tensor}-\mathrm{hom}((\mathbb{Z}\hookrightarrow\mathbb{Q})\circ[X_b]^\vee)\ ([X_c]\otimes 1) && \text{by (7.1) and (7.3)} \\
&= (\mathbb{Z}\hookrightarrow\mathbb{Q})\circ[X_b]^\vee\ ([X_c]) \\
&= \delta_{b,c}
\end{aligned}$$

□

The commutativity of Diagram (7.2) together with Equation (7.4) is exactly the behavior we would have hoped for, as it allows us to transfer relations about integer duals of Schubert classes to the corresponding rational setting. We often identify the cohomology $H^{2j}(X)$ with $H_{2j}(X)^\vee$ and $H^{2j}(X;\mathbb{Q})$ with $H_{2j}(X;\mathbb{Q})^\vee$ via the Kronecker map respectively and thus obtain bases $\{[X_b]^\vee\}_{b\leq a\,,\,|b|=j}$ of $H^{2j}(X)$ as $\mathbb{Z}$-module respectively $\{[X_b]_{\mathbb{Q}}^\vee\}_{b\leq a\,,\,|b|=j}$ of $H^{2j}(X;\mathbb{Q})$ as $\mathbb{Q}$-vector space. By (7.2) and (7.4) we see that

$$\mathrm{can}^{2j}\,[X_b]^\vee = [X_b]_{\mathbb{Q}}^\vee \quad . \tag{7.5}$$

Let us give an example to demonstrate the utility of Equation (7.5) (in combination with Corollary 36): Let us return to Remark 33. There it is observed that the following two analogous identities hold:

$$\begin{aligned}
PD_{\mathbb{Z}}(\sigma_\beta) &= \sum_{|\alpha|=q} \varepsilon_*(\sigma_\alpha \cdot_{\mathbb{Z}} \sigma_\beta)\,\sigma_\alpha^\vee \ \in H^{2(\dim(P)-q)}(P;\mathbb{Z}) \quad \forall\,\beta \text{ with } |\beta| = \dim(P)-q \\
PD_{\mathbb{Q}}(\sigma_\beta) &= \sum_{|\alpha|=q} \varepsilon_*(\sigma_\alpha \cdot_{\mathbb{Q}} \sigma_\beta)\,\sigma_\alpha^\vee \ \in H^{2(\dim(P)-q)}(P;\mathbb{Q}) \quad \forall\,\beta \text{ with } |\beta| = \dim(P)-q
\end{aligned}$$

Notice that in the first line we mean integer Schubert classes $\sigma_\alpha \in H_*(P;\mathbb{Z})$ and integer linear duals of Schubert classes while in the second line we mean rational Schubert classes $\sigma_\alpha \in H_*(P;\mathbb{Q})$ respectively rational linear duals of Schubert classes. Now apply the map $\mathrm{can}^{2(\dim(P)-q)}$ to the first line. Then Corollary 36, Equation (7.5) and comparing coeffients yields

$$\mathbb{Z} \ni \varepsilon_*(\sigma_\alpha \cdot_{\mathbb{Z}} \sigma_\beta) = \varepsilon_*(\sigma_\alpha \cdot_{\mathbb{Q}} \sigma_\beta) \quad ,$$

which was claimed in the second part of Remark 33. One of course might object that this example, to demonstrate the utility of Formula (7.5), is not very convincing since we derived the same result above without invoking Equation (7.5). Nonetheless, this formula is highly advantageous for us, in the sense that from now on we are allowed to switch coefficients from $\mathbb{Z}$ to $\mathbb{Q}$ and vice versa, knowing that everything behaves as it should. Thus we can drastically simplify notation and omit the underlying coefficient ring completely, as it was done already in the previous chapters.

Chapter 8

Determining the δ-Coefficients

From now on, unless explicitly mentioned otherwise, we are always considering homology and cohomology with integer coefficients, so $H_i(X) := H_i(X;\mathbb{Z})$ and $H^i(X) := H^i(X;\mathbb{Z})$ for any topological space X and any integer i.

In Chapter 6 we reduced our problem of determining λ and μ to the following two equations (see Equations (6.7) and (6.11)):

$$\lambda = 1 + \frac{1}{3}\,(\delta^{2,0}_{2,5} - \delta^{2}_{1,4})$$

$$\mu = \frac{1}{3}\left(2 + \delta^{1,1}_{2,5} + \delta^{2,0}_{2,5} - \delta^{1,1}_{2,4} - \delta^{2,0}_{2,4}\right)$$

It remains to figure out the values of $\delta^{1,1}_{2,5}$, $\delta^{2,0}_{2,5}$, $\delta^{1,1}_{2,4}$, $\delta^{2,0}_{2,4}$ and $\delta^{2}_{1,4}$. In this and the next chapter we present two approaches, independent of one another, to determine these particular δ-coefficients. While the second one surely is in some ways more direct, this is only due to the fact that we will apply strong, recent results by Aluffi & Mihalcea [AM08], which we will simply quote and for which we do not repeat any proofs. Mainly, this is due to the fact that their results go far beyond than what we actually need: Aluffi-Micalhea compute the homological Chern-Schwartz-MacPherson classes for arbitrary Schubert varieties but we are only interested in (the Poincaré duals of) those classes for nonsingular Schubert varieties, i.e. the ordinary Chern classes of Grassmannians, which have been understood for a much longer time. The first approach, presented in this chapter, on the other hand relies only on the highly classical works of Borel & Hirzebruch [BH58, Section 16.2] and Milnor & Stasheff [MS74]. Perhaps the reader is already familiar with them and, if so, might be more content with this approach since, based upon these few well-known results, the rest of the argument will be completely self contained. As this was also the path I originally pursued and how I determined the δ-coefficients at first, I decided to put it in first place here.

8.1 Preliminaries

But before we actually determine the δ-coefficients, some preliminary work has to be done. Namely, we have to prove only one, however nontrivial, result which will later allow us to determine a particular sign. This result can be found as *Problem 14-D* in Milnor-Stasheff [MS74] but no solutions are given there, so for the sake of completeness I would like to present a proof developed by myself.

For that, although not necessary, it turns out to be rather useful to formulate the result for infinite Grassmannians (as it is done in [MS74]), so let us first deal with them briefly.

To get a good overview over infinite Grassmannians it is recommended to read Chapters 5, 6 and a short paragraph in Chapter 14 of Milnor-Stasheff [MS74]. Complex infinite Grassmannians behave as similar to real infinite Grassmannians as the complex (finite) Grassmannians behave to the real finite ones. In the following, let $k \geq 0$ be fixed. By Lemma 15 we have canonical closed embeddings $G_k(\mathbb{C}^n) \hookrightarrow G_k(\mathbb{C}^{n+m})$ for all $n \geq k$, $m \geq 0$, defined by $V \mapsto V$. Obviously these form a direct system, so we may take their direct limit. There is a canonical bijection between $G_k := G_k(\mathbb{C}^\infty)$ and $\lim_{m\to\infty} G_k(\mathbb{C}^{k+m})$ and we topologize $G_k(\mathbb{C}^\infty)$ via the direct limit topology. By [MS74, Lemma 5.5] this is the same as the quotient topology induced by the infinite open Stiefel manifold $V_k(\mathbb{C}^\infty)$, where $\mathbb{C}^\infty$ [1] is given the direct limit topology of the inclusions $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+m}$. As in the real case (cf. [MS74, Thm. 6.4]), the definition of the Schubert cells extends analogously from the finite to the infinite case: For any partition $a = (a_1, \dots, a_k)$ of length k we consider the Schubert variety X_a which consists of exactly the k-dimensional subspaces V in $\mathbb{C}^\infty$ such that $\dim_{\mathbb{C}}(V \cap \mathbb{C}^{a_{k+1-i}}) \geq i \quad \forall\, i = 1, \dots, k$. The Schubert cell belonging to a is given by $\mathring{X}_a = X_a - (\bigcup_{b<a} X_b)$. It is a topological open cell of dimension $2|a|$. Thus $G_k(\mathbb{C}^\infty)$ is an infinite CW-complex filtered by the subcomplexes $G_k(\mathbb{C}^k) \subset G_k(\mathbb{C}^{k+1}) \subset \dots G_k(\mathbb{C}^{k+m}) \subset \dots$, where the inclusions $G_k(\mathbb{C}^{k+m}) \hookrightarrow G_k(\mathbb{C}^\infty)$ are closed embeddings[2].

The reason we are interested in the infinite Grassmannian $G_k(\mathbb{C}^\infty)$ is that its cohomology ring is easier to describe than that of finite Grassmannians. But for a sufficiently small degree p the $2p$-th cohomology group of the infinite Grassmannian coincides with the $2p$-th cohomology group of a given finite Grassmannian:

Fact 39. *Let $\iota : G_k(\mathbb{C}^{k+m}) \hookrightarrow G_k(\mathbb{C}^\infty)$ be the inclusion. Then for all $p \leq m$ the homomorphisms $\iota_* : H_{2p}(G_k(\mathbb{C}^{k+m})) \to H_{2p}(G_k(\mathbb{C}^\infty))$ on homology and $\iota^* : H^{2p}(G_k(\mathbb{C}^\infty)) \to H^{2p}(G_k(\mathbb{C}^{k+m}))$ on cohomology are isomorphisms.*

Compare this also with [MS74, Problem 6-B], where the analogous result is stated for the real case.

Proof of Fact 39. Notice that both $G_k(\mathbb{C}^{k+m})$ and $G_k(\mathbb{C}^\infty)$ only possess cells of even dimension. Thus, using cellular homology, the odd homology groups are zero and the even ones are simply the cellular chain groups which are all finitely generated free abelian groups. By the UCT and the five lemma it suffices to show that all homology homomorphisms ι_* are isomorphisms for $p \leq m$. Then the cohomology maps for $p \leq m$ are isomorphisms as well. Since $G_k(\mathbb{C}^{k+m})$ is a subcomplex of $G_k(\mathbb{C}^\infty)$ via the inclusion $\iota : G_k(\mathbb{C}^{k+m}) \hookrightarrow G_k(\mathbb{C}^\infty)$, the induced map

$$\iota_* : H_{2p}(G_k(\mathbb{C}^{k+m})) = C_{2p}^{\text{cell}}(G_k(\mathbb{C}^{k+m})) \hookrightarrow C_{2p}^{\text{cell}}(G_k(\mathbb{C}^\infty)) = H_{2p}(G_k(\mathbb{C}^\infty))$$

is simply the inclusion for all p. For any p, the cells of dimension $2p$ in $G_k(\mathbb{C}^{k+m})$ are exactly the $\mathring{X}_a$ for partitions $a = (a_1, \dots, a_k)$ of length k with $a_1 \leq m$ and $|a| = p$. The cells of dimension $2p$ in $G_k(\mathbb{C}^\infty)$ are exactly the $\mathring{X}_a$ for partitions $a = (a_a, \dots, a_k)$ of length k with $|a| = p$. Now for $p \leq m$, every partition a with $|a| = p$ also fulfills $a_1 \leq m$. Hence ι_* is an isomorphism. $\square$

[1] As in [MS74] we denote by $\mathbb{C}^\infty$ the $\mathbb{C}$-vector space $\mathbb{C}^{(\mathbb{N})}$

[2] One easily sees that $G_k(\mathbb{C}^\infty)$ is Hausdorff since finite Grassmannians are. Then the inclusions are closed embeddings since $G_k(\mathbb{C}^{k+m})$ is compact. Now analogously to Lemma 15 one checks that for the inclusion $\iota : G_k(\mathbb{C}^{k+m}) \hookrightarrow G_k(\mathbb{C}^\infty)$ and any partition $a = (a_1, \dots, a_k) \in \mathcal{P}(m,k)$ one has $\iota(X_a) = X_a$. Hence the inclusion ι maps every cell $\mathring{X}_a$ homeomorphically to the cell $\mathring{X}_a$. Thus $G_k(\mathbb{C}^\infty)$ is indeed a CW-complex and every $G_k(\mathbb{C}^{k+m})$ is a finite subcomplex of $G_k(\mathbb{C}^\infty)$ via the inclusion ι.

Remark 40. The proof of Fact 39 shows that for arbitrary p (not necessarily $p \leq m$) and any partition $a = (a_1, \ldots, a_k)$ with $a_1 \leq m$, $|a| = p$, the map ι_* sends $[X_a]$ to $[X_a]$. Thus, for any partition $a = (a_1, \ldots, a_k)$ with $|a| = p$ its dual map ι^* sends $[X_a]^\vee$ to $[X_a]^\vee$ if $a_1 \leq m$ and to zero otherwise. In particular, if $p \leq m$, then for all partitions a with $|a| = p$ we have $\iota^* [X_a]^\vee = [X_a]^\vee$.

The last ingredient for our proof below is the *canonical bundle*: As in [MS74] we denote the canonical bundle over $G_k(\mathbb{C}^n)$ by $\gamma^k(\mathbb{C}^n)$. Its total space $E(\gamma^k(\mathbb{C}^n))$ carries the subspace topology from $G_k(\mathbb{C}^n) \times \mathbb{C}^n$ and consists of all pairs of the form (V, x) with $x \in V$. Similarly the canonical bundle over $G_k(\mathbb{C}^\infty)$ will be denoted by $\gamma^k(\mathbb{C}^\infty)$ or just γ^k. Its total space $E(\gamma^k)$ consists of all pairs of the form (V, x) with $V \subseteq \mathbb{C}^\infty$ a k-dimensional $\mathbb{C}$-linear subspace and $x \in \mathbb{C}^\infty$ such that $x \in V$. Again we give the total space $E(\gamma^k) \subseteq G_k(\mathbb{C}^\infty) \times \mathbb{C}^\infty$ the subspace topology. Then we have the following important theorem:

Fact 41 (Theorem 14.5 in [MS74])**.** *The cohomology ring $H^*(G_k(\mathbb{C}^\infty))$ is the polynomial ring over $\mathbb{Z}$ generated by the Chern classes $c_1(\gamma^k), \ldots, c_k(\gamma^k)$. There are no polynomial relations between these k generators.*

Furthermore, the following fact is trivial but will be beneficial for us later on in this chapter:

Fact 42. *For $n \geq k$ let $\iota : G_k(\mathbb{C}^n) \hookrightarrow G_k(\mathbb{C}^\infty)$ be the inclusion. Then $\iota^*\gamma^k(\mathbb{C}^\infty) \cong \gamma^k(\mathbb{C}^n)$ as complex vector bundles.*

Proof. The map $\gamma^k(\mathbb{C}^n) \to \gamma^k(\mathbb{C}^\infty)$, $(V, x) \mapsto (\iota(V), (x, 0, 0, \ldots))$ covers ι and its restriction to the fiber of $V \in G_k(\mathbb{C}^n)$ is a $\mathbb{C}$-linear isomorphism. The result follows from the complex analog of [MS74, Lemma 3.1]. $\square$

Corollary 43. *For $n \geq k$ and any p it holds $c_p(\gamma^k(\mathbb{C}^n)) = \iota^* c_p(\gamma^k(\mathbb{C}^\infty))$, where $\iota : G_k(\mathbb{C}^n) \hookrightarrow G_k(\mathbb{C}^\infty)$ denotes the inclusion.*

We are now ready to prove the following preliminary, nonetheless essential, proposition:

Proposition 44 (Problem 14-D in [MS74])**.** *For all $k \in \mathbb{N}$ and all $1 \leq p \leq k$ we have*

$$[X_{\underbrace{\underbrace{1, \ldots, 1}_{p \text{ times}}, 0, \ldots, 0}_{\text{length } k}}]^\vee = (-1)^{\varepsilon_p}\, c_p(\gamma^k(\mathbb{C}^\infty)) \in H^{2p}(G_k(\mathbb{C}^\infty); \mathbb{Z}) \quad ,$$

where $\varepsilon_p \in \{0, 1\}$ is uniquely determined by

$$[X_{\underbrace{1, \ldots, 1}_{p \text{ times}}}]^\vee = (-1)^{\varepsilon_p}\, c_p(\gamma^p(\mathbb{C}^\infty)) \in H^{2p}(G_p(\mathbb{C}^\infty); \mathbb{Z})$$

for all p.

Remark 45. Let us briefly comment on the notation in the preceding proposition: All homology groups of $G_k = G_k(\mathbb{C}^\infty)$ are finitely generated and free abelian groups, so

$$\mathrm{Kron} : H^{2p}(G_k(\mathbb{C}^\infty); \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(H_{2p}(G_k(\mathbb{C}^\infty)), \mathbb{Z}) = H_{2p}(G_k(\mathbb{C}^\infty))^\vee$$

is an isomorphism of abelian groups. As already done several times before, we omit Kron respectively Kron^{-1} in the notation. Additionally, as before, $\{[X_a]^\vee\}_a$ denotes the $\mathbb{Z}$-dual basis, i.e.: For fixed p we have a finite $\mathbb{Z}$-basis of $H_{2p}(G_k)$ of the form $\{\, [X_a]\}_{|a|=p}$, indexed by partitions a of length k with $|a| = p$. Then $\{[X_a]^\vee\}_{|a|=p}$ is its $\mathbb{Z}$-dual basis in $H_{2p}(G_k)^\vee \cong H^{2p}(G_k)$. This explains the notation in the above proposition. Furthermore, a significant part of the proof will be, to actually show that $[X_{1,\dots,1}]^\vee$ is, up to sign, equal to $c_p(\gamma^p)$ for all p, so the reader should not be concerned with that for now. Finally, let us clarify in which way the above statement is a modification of *Problem 14-D* in [MS74]: As already mentioned, the cellular chain groups in even degree are the homology groups of the same degree since there are no cells of odd dimension. Thus we may identify $H^{2p}(G_k) \cong H_{2p}(G_k)^\vee$ with $C^{2p}_{\text{cell}}(G_k) = C^{\text{cell}}_{2p}(G_k)^\vee$, the group of cellular $2p$-cocycles. Since the Schubert cell $\mathring{X}_a \in C^{\text{cell}}_{2p}(G_k) = H^{\text{cell}}_{2p}(G_k)$ is identified with $[X_a] \in H_{2p}(G_k)$ under the canonical isomorphism $H^{\text{cell}}_{2p}(G_k) \cong H_{2p}(G_k)$, the cellular cocyle that assigns 1 to $\mathring{X}_{1,\dots,1,0,\dots,0}$ and 0 to all other cells of the same dimension (cf. [MS74, Problem 14-D]) is identified with $[X_{1,\dots,1,0,\dots,0}]^\vee$.

Remark 46. It is actually easy to see that ε_1 is well-defined and $\varepsilon_1 = 1$, i.e. $[X_1]^\vee = -c_1(\gamma^1)$:
By Fact 41, $c_1(\gamma^1)$ is a generator of $H^2(G_1) \cong \mathbb{Z}$. Since $[X_1]$ is a generator of $H_2(G_1) \cong \mathbb{Z}$, the dual $[X_1]^\vee$ is a generator of $H^2(G_1)$ too. Thus $[X_1]^\vee = \pm c_1(\gamma^1)$. To see that $[X_1]^\vee = -c_1(\gamma^1)$, choose n odd and large enough[3]. By Fact 39 the inclusion $\iota : \mathbb{C}P^n = G_1(\mathbb{C}^{n+1}) \hookrightarrow G_1$ induces an isomorphism on cohomology in degree 2 and clearly $\iota^* [X_1]^\vee = [X_1]^\vee$ (by Remark 40) as well as $\iota^* c_1(\gamma^1) = c_1(\gamma^1(\mathbb{C}^{n+1}))$ (by Corollary 43), so it suffices to show $[X_1]^\vee = -c_1(\gamma^1(\mathbb{C}^{n+1})) =: a$. By [MS74, Remark, p. 170] we have $\langle\, a^n\,,\, [\mathbb{C}P^n]\,\rangle = +1$. Hence $a^n = PD(1)$. On the other hand we have $[X_1]^\vee = PD(\sigma_1)$ because $\langle\, PD(\sigma_1)\,,\, [X_1]\,\rangle = \sigma_1 \cdot \sigma_{n-1} = \sigma_n = 1$ by Proposition 28 and Pieri. This shows $([X_1]^\vee)^n = PD(\sigma_1^n) \overset{\text{Pieri}}{=} PD(\sigma_n) = PD(1) = a^n \in H^{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$. Since $[X_1]^\vee = \pm a$ and n is odd, we deduce $[X_1]^\vee = a$.

Proof of Proposition 44. Part I: For any k define ι_k by

$$\iota_k : G_k(\mathbb{C}^\infty) \hookrightarrow G_{k+1}(\mathbb{C}^\infty)\,,\;\; V \mapsto \mathbb{C} \oplus V \quad .$$

This map is induced by the maps $G_k(\mathbb{C}^n) \hookrightarrow G_{k+1}(\mathbb{C}^{n+1})\,,\, n \geq k\,,$ from Lemma 15 by taking the direct limit $n \to \infty$. Again applying Lemma 15, ι_k maps the Schubert variety $X_{a_1,\dots,a_k} \subseteq G_k(\mathbb{C}^\infty)$ onto the Schubert variety $X_{a_1,\dots,a_k,0} \subseteq G_{k+1}(\mathbb{C}^\infty)$ for any partition $a = (a_1, \dots, a_k)$ [4]. This implies that $\iota_{k\,*} : H_{2|a|}(G_k) \to H_{2|a|}(G_{k+1})$ maps $[X_a]$ to $[X_{a,0}]$ [5].

Part II: Now let us show that indeed

$$[X_{\underbrace{1,\dots,1}_{k \text{ times}}}]^\vee = \pm\, c_k(\gamma^k) \tag{8.1}$$

for all k and thus the ε_k are well-defined. The case $k = 1$ was covered in Remark 46, so let us assume $k > 1$ [6]. Let $E := E(\gamma^k)$ denote the total space of the bundle γ^k and $E_0 = E_0(\gamma^k) \subset E$ the complement of the zero section of γ^k with $\pi_0 : E_0 \to G_k$ the restriction of the projection

[3] Actually every odd n works.

[4] Notice that $X_a \subseteq G_k(\mathbb{C}^\infty)$ is exactly the image of $X_a \subseteq G_k(\mathbb{C}^n)$ under the inclusion $G_k(\mathbb{C}^n) \hookrightarrow G_k(\mathbb{C}^\infty)$ for any n with $a_1 \leq n - k$: For $V \in X_a(G_k(\mathbb{C}^\infty))$ we have $\dim(V \cap \mathbb{C}^{a_1+k}) \geq k$ and thus $V \subseteq \mathbb{C}^{a_1+k} \subseteq \mathbb{C}^n$.

[5] $[X_a]$ corresponds to the cellular cycle $\mathring{X}_a \in C^{\text{cell}}_{2|a|}(G_k)$ and we have just seen that ι_k maps the Schubert cell $\mathring{X}_a$ onto the Schubert cell $\mathring{X}_{a,0}$ which, as cellular cycle, corresponds to $[X_{a,0}]$. Alternatively this can be deduced from Lemma 15, where above statement was proven for the finite-dimensional case (and one additionally uses that $[X_a] \in H_{2|a|}(G_k(\mathbb{C}^n))$ is mapped to $[X_a] \in H_{2|a|}(G_k)$ via the map on homology induced from the inclusion $G_k(\mathbb{C}^n) \hookrightarrow G_k$).

[6] Although the subsequent proof also works for $k = 1$ without any problems. However this allows us to be less careful with notation.

$\pi = \pi(\gamma^k) : E(\gamma^k) \to G_k$ to E_0[7]. Furthermore put $c_p := c_p(\gamma^k)$ for any p. The long exact Gysin sequence for γ^k is given by[8]

$$\ldots \longrightarrow \underbrace{H^0(G_k)}_{\cong\mathbb{Z}} \xrightarrow{-\cup c_k} H^{2k}(G_k) \xrightarrow{\pi_0^*} H^{2k}(E_0) \longrightarrow H^1(G_k) \xrightarrow{-\cup c_k} \ldots$$

We will show that $[X_{1,\ldots,1}]^\vee \in \ker\,\pi_0^*$ in Part III of the proof. Exactness of the Gysin sequence yields $[X_{1,\ldots,1}]^\vee = b\,c_k$ for some $b \in \mathbb{Z}$. Then $1 = [X_{1,\ldots,1}]^\vee\,[X_{1,\ldots,1}] = b\,\underbrace{c_k\,[X_{1,\ldots,1}]}_{\in\mathbb{Z}}$. This shows that $b \in \mathbb{Z}^\times = \{\pm 1\}$ and thus $[X_{1,\ldots,1}]^\vee = \pm\, c_k(\gamma^k)$.

Part III: Our goal is to show that $[X_{1,\ldots,1}]^\vee \in \ker\,\pi_0^*$. By naturality in the UCT we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{2k-1}(G_k),\mathbb{Z}) & \longrightarrow & H^{2k}(G_k) & \xrightarrow[\text{Kron}]{\cong} & \mathrm{Hom}(H_{2k}(G_k),\mathbb{Z}) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \pi_0^*} & & \big\downarrow{\scriptstyle (\pi_{0\,*})^\vee} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{2k-1}(E_0),\mathbb{Z}) & \longrightarrow & H^{2k}(E_0) & \xrightarrow[\text{Kron}]{} & \mathrm{Hom}(H_{2k}(E_0),\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

We of course already know that the upper Ext term is zero. Directly at the beginning of the proof of Theorem 14.5 in [MS74] Milnor and Stasheff show that $H^*(E_0) \cong H^*(G_{k-1})$. Thus all cohomology groups of E_0 are finitely generated and free. Hence all homology groups of E_0 are finitely generated as well[9]. This shows that the lower Ext term in the above diagram is a torsion group. Taking advantage of the fact that the sequence in the UCT splits and that $H^{2k}(E_0)$ is free, the lower Ext term vanishes and the lower Kronecker map $H^{2k}(E_0) \xrightarrow{\cong} \mathrm{Hom}(H_{2k}(E_0,\mathbb{Z}))$ is an isomorphism as well. Hence

$$\underbrace{[X_{1,\ldots,1}]^\vee}_{H^{2k}(G_k)} \in \ker\,\pi_0^* \Longleftrightarrow (\pi_{0\,*})^\vee\,\underbrace{[X_{1,\ldots,1}]^\vee}_{\in H_{2k}(G_k)^\vee} = 0$$

and it suffices to prove $[X_{1,\ldots,1}]^\vee \circ \pi_{0\,*} = 0$ which is equivalent to $[X_{1,\ldots,1}]^\vee|_{\mathrm{im}\,\pi_{0\,*}} = 0$. We now have

$$\begin{aligned}
\mathrm{rk}\ \mathrm{im}\,\pi_{0\,*} \le \mathrm{rk}\ H_{2k}(E_0) &= \mathrm{rk}\ H^{2k}(E_0) = \mathrm{rk}\ H^{2k}(G_{k-1}) \\
&= \mathrm{rk}\ H_{2k}(G_{k-1}) = \mathrm{rk}\ H_{2k}(G_k) - 1
\end{aligned}$$

where the last equation follows by counting Schubert cells[10]. In Part IV of the proof we will show that $[X_{a_1,\ldots,a_{k-1},0}] \in \mathrm{im}\,\pi_{0\,*}$ for all partitions $(a_1,\ldots,a_{k-1},0)$ of k [11]. We assume this result for now. Now consider an arbitary $\alpha \in \mathrm{im}\,\pi_{0\,*}$. We have to show that $[X_{1,\ldots,1}]^\vee(\alpha) = 0$. Assume by contradiction $[X_{1,\ldots,1}]^\vee(\alpha) \neq 0$. Write

$$\alpha = \eta_{1,\ldots,1}\,[X_{1,\ldots,1}] + \sum_{(a_1,\ldots,a_{k-1})} \eta_{a_1,\ldots,a_{k-1},0}\,[X_{a_1,\ldots,a_{k-1},0}] \quad ,$$

[7] This notation is chosen to be compatible with the one in Milnor-Stasheff [MS74].

[8] cf. [MS74, p. 143]

[9] By well-known results in algebraic topology, e.g. by [Hat01, Proposition 3F.12].

[10] There is exactly one partition $a = (a_1,\ldots,a_k)$ with $|a| = \sum_i a_i = k$ and where the last entry a_k is non-zero, namely $(1,\ldots,1)$.

[11] i.e. $\sum_{i=1}^{k-1} a_i = k$

where the sum runs over all partitions $(a_1,\ldots,a_{k-1})$ of k, so this is the unique linear combination of α with respect to the Schubert basis of $H_{2k}(G_k)$. We thus have

$$\eta_{1,\ldots,1}\,[X_{1,\ldots,1}] = \underbrace{\alpha}_{\in \operatorname{im}\pi_{0\,*}} - \sum_{(a_1,\ldots,a_{k-1})} \eta_{a_1,\ldots,a_{k-1},0}\,\underbrace{[X_{a_1,\ldots,a_{k-1},0}]}_{\in \operatorname{im}\pi_{0\,*}} \in \operatorname{im}\pi_{0\,*} \quad .$$

In addition $\eta_{1,\ldots,1} = [X_{1,\ldots,1}]^\vee(\alpha) \neq 0$ and thus $\{\eta_{1,\ldots,1}\,[X_{1,\ldots,1}]\} \cup \{[X_{a_1,\ldots,a_{k-1},0}]\}_{(a_1,\ldots,a_{k-1})}$ is a linearly independent family. But this means that we have found a linearly independent family in $\operatorname{im}\pi_{0\,*}$ of cardinality $\operatorname{rk} H_{2k}(G_k)$, contradicting the fact that $\operatorname{rk}\operatorname{im}\pi_{0\,*} < \operatorname{rk} H_{2k}(G_k)$ as we have seen above. This shows that $[X_{1,\ldots,1}]^\vee|_{\operatorname{im}\pi_{0\,*}} = 0$.

Part IV: As announced in Part III, we now show that $[X_{a_1,\ldots,a_{k-1},0}] \in \operatorname{im}\pi_{0\,*}$ for all partitions $(a_1,\ldots,a_{k-1},0)$ of k. Consider the following commutative diagram

$$\begin{tikzcd} & E_0(\gamma^k) \arrow[d, "\pi_0"] \\ G_{k-1}(\mathbb{C}^\infty) \arrow[ur, "g"] \arrow[r, "\iota_{k-1}"'] & G_k(\mathbb{C}^\infty) \end{tikzcd}$$

where g is defined as $g(V) := (\mathbb{C}\oplus V, e_1)$ with $e_1 = (1,0,0,\ldots)$. Clearly g is continuous, as $E_0(\gamma^k)$ is endowed with the subspace topology of $G_k \times \mathbb{C}^\infty$. By Part I we have $[X_{a_1,\ldots,a_{k-1},0}] = \iota_{k-1\,*}[X_{a_1,\ldots,a_{k-1}}] = \pi_{0\,*}\,g_*\,[X_{a_1,\ldots,a_{k-1}}] \in \operatorname{im}\pi_{0\,*}$. This finishes the extensive proof of the claim (8.1) in Part II.

Part V: Let $k \geq 1$ and p be arbitrary. As in [MS74] we (temporarily) denoty by ε^j the trivial complex vector bundle of complex rank j. We then have a canonical bundle homomorphism, fiberwise being an isomorphism (cf. [MS74, Problem 6-C]):

$$\begin{tikzcd} \varepsilon^1 \oplus \gamma^k(\mathbb{C}^\infty) \arrow[r] \arrow[d] & \gamma^{k+1}(\mathbb{C}^\infty) \arrow[d] \\ G_k(\mathbb{C}^\infty) \arrow[r, hook, "\iota_k"] & G_{k+1}(\mathbb{C}^\infty) \end{tikzcd}$$

where the upper horizontal map sends $(V,z,v) \in G_k(\mathbb{C}^\infty)\times\mathbb{C}\times\mathbb{C}^\infty$ to $(\mathbb{C}\oplus V,(z,v))$. This shows[12] that $\iota_k^*\gamma^{k+1} \cong \varepsilon^1 \oplus \gamma^k$ as complex vector bundles, which allows for the following computation (where $a = (a_1,\ldots,a_k)$ is any partition with $|a| = p$):

$$\begin{aligned} c_p(\gamma^{k+1})\,[X_{a,0}] &= c_p(\gamma^{k+1})\,(\iota_{k\,*}[X_a]) && \text{(by Part I)} \\ &= \langle\, c_p(\gamma^{k+1})\,,\, \iota_{k\,*}[X_a]\,\rangle \\ &= \langle\, \iota_k^* c_p(\gamma^{k+1})\,,\, [X_a]\,\rangle \\ &= \langle\, c_p(\varepsilon^1 \oplus \gamma^k)\,,\, [X_a]\,\rangle \\ &= \langle\, c_p(\gamma^k)\,,\, [X_a]\,\rangle \\ &= c_p(\gamma^k)\,[X_a] \end{aligned}$$

So in total we have shown that

$$c_p(\gamma^{k+1}(\mathbb{C}^\infty))\,[X_{a_1,\ldots,a_k,0}] = c_p(\gamma^k(\mathbb{C}^\infty))\,[X_{a_1,\ldots,a_k}] \tag{8.2}$$

[12] For example by the complex analog of [MS74, Lemma 3.1].

for all $k \geq 1$, all p and all partitions $a = (a_1, \dots, a_k)$ with $|a| = p$.

Part VI: In the final part of this proof let us now show the actual statement via an induction on k. The base case, $k = 1$, is trivial. In the induction step now assume the statement holds true for $k - 1$. We have to show

$$[X_{\underbrace{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0}_{\text{length } k}}]^\vee = (-1)^{\varepsilon_p}\, c_p(\gamma^k) \quad \forall\, p \leq k \quad .$$

There is nothing to show for $p = k$. Now let $p < k$ be arbitrary but fixed. Consider an arbitrary partition $a = (a_1, \dots, a_k)$ of p. We need to prove that

$$c_p(\gamma^k)\,[X_a] = \begin{cases} (-1)^{\varepsilon_p} & \text{if } a = \underbrace{(\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0)}_{\text{length } k} \\ 0 & \text{otherwise} \end{cases}$$

Observe that $a_k = 0$ since $|a| = \sum_{i=1}^k a_i = p < k$. Thus $a = (a_1, \dots, a_{k-1}, 0)$ and Equation (8.2) yields $c_p(\gamma^k)\,[X_a] = c_p(\gamma^{k-1})\,[X_{a_1,\dots,a_{k-1}}]$. By induction hypothesis the latter expression is $(-1)^{\varepsilon_p}$ if $a = \underbrace{(\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0)}_{\text{length } k}$ and 0 otherwise. □

In practice, we will mainly apply the following version of *Problem 14-D* for finite Grassmannians:

Corollary 47 (Problem 14-D for finite Grassmannians)**.** *For $n > k$ and $1 \leq p \leq k$ we have*

$$[X_{\underbrace{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0}_{\text{length } k}}]^\vee = (-1)^{\varepsilon_p}\, c_p(\gamma^k(\mathbb{C}^n)) \;\in H^{2p}(G_k(\mathbb{C}^n);\mathbb{Z}) \quad .$$

Proof. Let $\iota : G_k(\mathbb{C}^n) \hookrightarrow G_k(\mathbb{C}^\infty)$ be the inclusion and $a := (a_1, \dots, a_k) := (\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0)$. By Remark 40 we have $\iota^*[X_a]^\vee = [X_a]^\vee \in H^{2p}(G_k(\mathbb{C}^n))$ since $a_1 = 1 \leq n - k$. Furthermore, by Corollary 43 we have $\iota^* c_p(\gamma^k(\mathbb{C}^\infty)) = c_p(\gamma^k(\mathbb{C}^n))$. Proposition 44 yields $\iota^*[X_a]^\vee = (-1)^{\varepsilon_p}\,\iota^* c_p(\gamma^k(\mathbb{C}^\infty))$. All in all

$$[X_a]^\vee = \iota^*[X_a]^\vee = (-1)^{\varepsilon_p}\,\iota^* c_p(\gamma^k(\mathbb{C}^\infty)) = (-1)^{\varepsilon_p}\, c_p(\gamma^k(\mathbb{C}^n)) \quad .$$

□

8.2 Determining the δ-Coefficients

Now let us actually compute the values of $\delta^{1,1}_{2,5}$, $\delta^{2,0}_{2,5}$, $\delta^{1,1}_{2,4}$, $\delta^{2,0}_{2,4}$ and $\delta^2_{1,4}$. We claim that

$$\delta^2_{1,4} = 4 \tag{8.3}$$

$$\delta^{1,1}_{2,4} = 2 \qquad \delta^{2,0}_{2,4} = 2 \tag{8.4}$$

$$\delta^{1,1}_{2,5} = 1 \qquad \delta^{2,0}_{2,5} = 3 \tag{8.5}$$

As already mentioned, we now consider homology and cohomology with integer coefficients instead of rational ones, so we alter notation and put $H_*(X) := H_*(X;\mathbb{Z})$ for any topological space X and analogously for cohomology. Additionally, p_1 now means the first integer Pontryagin class and the Poincaré duality isomorphism PD is with respect to integer coefficients. By Equation (6.1) [13] we then have

$$p_1(G_k(\mathbb{C}^n)) = \sum_{\substack{\beta\in\mathcal{P}(n-k,k)\\ |\beta|=2}} \delta_{k,n}^{\beta}\, PD(\sigma_\beta) \;\in\; H^4(G_k(\mathbb{C}^n);\mathbb{Z}) \quad , \tag{8.6}$$

where the sum runs over all partitions $\beta = (\beta_1, \ldots, \beta_k)$ of 2 with $\beta_1 \leq n-k$. This is due to Corollary 36, as already explained in Chapter 7. Furthermore, by Equation (7.5) and Corollary 36 in Chapter 7, all of the base-change coefficients between the dual basis $\{\sigma_\alpha^\vee\}_{|\alpha|=k(n-k)-i}$ and the Poincaré dual basis $\{PD(\sigma_\beta)\}_{|\beta|=i}$ are independent of the underlying coefficient ring $\mathbb{Z}$ or $\mathbb{Q}$[14]. In particular, all base-change coefficients we already computed in Chapter 6 (with underlying coefficient ring $\mathbb{Q}$) equal their respective base-change coefficient with underlying coefficient ring $\mathbb{Z}$. We thus may switch between those two coefficient rings without further mentioning it.

Determining $\delta_{1,4}^2$:
Clearly $G_1(\mathbb{C}^4) \cong \mathbb{C}P^3$. By [MS74, p. 178] the total Pontryagin class of $\mathbb{C}P^3$ is given by $p(\mathbb{C}P^3) = 1 + 4\,a^2$, where $a = -c_1(\gamma^1(\mathbb{C}^4))$. Both a and $PD(\sigma_1)$ are generators of $H^2(G_1(\mathbb{C}^4)) \cong \mathbb{Z}$, hence $\pm\, a = PD(\sigma_1)$ [15] and $a^2 = PD(\sigma_1^2) = PD(\sigma_2)$. This already shows that $\delta_{1,4}^2 = 4$.

Determining $\delta_{2,4}^{1,1}$ and $\delta_{2,4}^{2,0}$:
Let us (temporarily) set $G := G_2(\mathbb{C}^4)$. Since the tangent space TG is a complex vector bundle, we have the following well-known relation[16] between the first Pontryagin class of TG and its Chern classes:

$$p_1(G) = p_1(TG) = c_1(TG)^2 - 2\,c_2(TG) = c_1(G)^2 - 2\,c_2(G) \quad . \tag{8.7}$$

A brief comment on notation: It seems that some authors define the Chern class of a complex manifold M as the Chern class of its holomorphic tangent bundle $T'M$ [17]. We however stick to the notational convention $c_*(M) := c_*(TM)$. Even if we were too careless and wrongly applied sources for which the Chern class of a complex manifold is defined as $c_*(T'M)$, our results still would be correct, because the tangent bundle TM and the holomorphic tangent bundle $T'M$ are isomorphic as real vector bundles so their Pontryagin classes coincide.

We want to apply classical results from Borel & Hirzebruch in [BH58, Section 16.2]: Borel-Hirzebruch write

$$\mathbf{W}(m,n) = U(m+n)/(U(m)\times U(n)) \cong G_m(\mathbb{C}^{m+n})$$

for the Grassmannian, so in our case $m = n = 2$. Additionally in the notation of Borel-Hirzebruch σ_r stands for "the r-th Chern class of the canonical principal $U(m)$-bundle of $\mathbf{W}(m,n)$"[18], which translated to our notation is just $c_r(\gamma^m(\mathbb{C}^{m+n}))$. Furthermore let I be the ideal in the ring

[13] Remember that in Chapter 6 we implicitly worked with rational coefficients.

[14] Alternatively this follows because the formula for the base change in Corollary 30 both holds for integer and rational coefficients, as stated in Remark 33, and the appearing intersection products with integer and rational coefficients are the same (by Corollary 37).

[15] Actually we already know the sign: $a = [X_1]^\vee$ by Remark 46 and Corollary 47 and $[X_1]^\vee = \sigma_2^\vee = PD(\sigma_1)$ by Corollary 30.

[16] cf. [MS74, Corollary 15.5]

[17] e.g. Griffiths & Harris do so, cf. [GH78, p. 407]

[18] see [BH58, p. 522]

$S\{x_1,\dots,x_m\} \otimes S\{x_{m+1},\dots,x_{m+n}\}$ [19], generated by the symmetric power series in the variables $x_1,\dots,x_{m+n}$ without constant term. Then $H^*(\mathbf{W}(m,n);\mathbb{Z})$ can be identified with[20]

$$S\{x_1,\dots,x_m\} \otimes S\{x_{m+1},\dots,x_{m+n}\} \mod I$$

and σ_r can equivalently be viewed as the r-th elementary symmetric function in the x_i , $1 \le i \le m$,[21] but we will take advantage of this alternative characterization only later. Borel-Hirzebruch now derive the following formulae[22]:

$$c(\mathbf{W}(m,n)) = \prod_{i=1}^{m}(1-x_i)^{m+n} \prod_{1\le i\le j\le m}(1-(x_i-x_j)^2)^{-1} \mod I \tag{8.8}$$

$$c_1(\mathbf{W}(m,n)) = -(m+n)\,\sigma_1 \tag{8.9}$$

$$c_2(\mathbf{W}(m,n)) = \left(\binom{m+n}{2} + m - 1\right)\sigma_1^2 + (n-m)\,\sigma_2 \tag{8.10}$$

Translating (8.9) and (8.10) into our notation, we obtain

$$\begin{aligned} c_1(G) &= -(2+2)\,c_1(\gamma^2(\mathbb{C}^{2+2})) = -4\,c_1(\gamma^2(\mathbb{C}^4)) \\ c_2(G) &= \left(\binom{2+2}{2} + 2 - 1\right) c_1(\gamma^2(\mathbb{C}^{2+2}))^2 + (2-2)\,c_2(\gamma^2(\mathbb{C}^{2+2})) \\ &= 7\,c_1(\gamma^2(\mathbb{C}^4))^2 \end{aligned}$$

By Fact 41, Fact 39 and Corollary 43 we see that $c_1(\gamma^2(\mathbb{C}^4))$ is a generator of $H^2(G) \cong \mathbb{Z}$. Furthermore both $PD(\sigma_{1,0})$ and $[X_{1,0}]^\vee$ are generators as well, cf. Corollary 30 in Chapter 5. Thus they all coincide up to sign[23] and we deduce $c_1(\gamma^2(\mathbb{C}^4))^2 = ([X_{1,0}]^\vee)^2 = PD(\sigma_{1,0})^2 = PD(\sigma_{1,0}^2)$. By Pieri $\sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0}$, hence $c_1(\gamma^2(\mathbb{C}^4))^2 = PD(\sigma_{1,1}) + PD(\sigma_{2,0})$. In total

$$\begin{aligned} p_1(G) &= c_1(G)^2 - 2\,c_2(G) = 16\,c_1(\gamma^2(\mathbb{C}^4))^2 - 2\,(7\,c_1(\gamma^2(\mathbb{C}^4))^2) \\ &= 2\,PD(\sigma_{1,1}) + 2\,PD(\sigma_{2,0}) \end{aligned}$$

which, in view of Equation (8.6), shows $\delta_{2,4}^{1,1} = \delta_{2,4}^{2,0} = 2$.

Determining $\delta_{2,5}^{1,1}$ and $\delta_{2,5}^{2,0}$:
This is by far the most laborious of the three cases we have to consider. The reason for it will become obvious further below. Set $G := G_2(\mathbb{C}^5)$. Applying Equations (8.9) and (8.10) for $m = 2$, $n = 3$ yields

$$\begin{aligned} c_1(G) &= -5\,c_1(\gamma^2(\mathbb{C}^5)) \\ c_2(G) &= 11\,c_1(\gamma^2(\mathbb{C}^5))^2 + c_2(\gamma^2(\mathbb{C}^5)) \quad . \end{aligned}$$

[19] This is the smallest subring of the ring of power series in the variables $x_1,\dots,x_{m+n}$ that contains both the power series which are symmetric in the $x_1,\dots,x_m$ as well as the ones which are symmetric in the $x_{m+1},\dots,x_{m+n}$.

[20] see [BH58, p. 521]

[21] see [BH58, p. 522]

[22] again [BH58, p. 522]

[23] Actually with our techniques developed so far we are able to determine these signs. However this will not be necessary here.

Together with Equation (8.7) we obtain

$$
\begin{aligned}
p_1(G) &= 25\, c_1(\gamma^2(\mathbb{C}^5))^2 - 2\left(11\, c_1(\gamma^2(\mathbb{C}^5))^2 + c_2(\gamma^2(\mathbb{C}^5))\right) \\
&= 3\, c_1(\gamma^2(\mathbb{C}^5))^2 - 2\, c_2(\gamma^2(\mathbb{C}^5)) \quad .
\end{aligned}
$$

The remaining task is to express the first and second Chern classes of the canonical bundle in terms of Poincaré duals of Schubert classes. The distinctive difficulty to before is that now the c_2-term of the canonical bundle does not vanish, so we really have to invest some more additional work to determine the appropriate sign. The second cohomology group $H^2(G)$ is isomorphic to $\mathbb{Z}$ and all of the following elements are generators of it:

$$
[X_{1,0}]^\vee = \sigma_{3,2}^\vee \ , \qquad\qquad PD(\sigma_{1,0}) \ , \qquad\qquad c_1(\gamma^2(\mathbb{C}^5)) \quad .
$$

Thus they have to coincide up to sign[24], hence

$$
c_1(\gamma^2(\mathbb{C}^5))^2 = PD(\sigma_{1,0}^2) \overset{\text{Pieri}}{=} PD(\sigma_{1,1}) + PD(\sigma_{2,0}) \quad .
$$

Now let us treat the term with the second Chern class of the tautological bundle: By Corollary 47, i.e. our version of Milnor-Stasheff's *Problem 14-D* for finite Grassmannians, we have

$$
[X_{1,1}]^\vee = (-1)^{\varepsilon_2}\, c_2(\gamma^2(\mathbb{C}^5)) \quad .
$$

We claim that $\varepsilon_2 = 0$ and thus $[X_{1,1}]^\vee = c_2(\gamma^2(\mathbb{C}^5))$. This requires some work as well as a clever trick and will be done below. But accepting this for now we are finished, as $c_2(\gamma^2(\mathbb{C}^5)) = [X_{1,1}]^\vee = PD(\sigma_{1,1})$ by Equation (6.2). Combining all these results we obtain

$$
\begin{aligned}
p_1(G) &= 3\,(PD(\sigma_{1,1}) + PD(\sigma_{2,0})) - 2\, PD(\sigma_{1,1}) \\
&= PD(\sigma_{1,1}) + 3\, PD(\sigma_{2,0}) \quad ,
\end{aligned}
$$

hence $\delta_{2,5}^{1,1} = 1$ and $\delta_{2,5}^{2,0} = 3$, which shows Equation (8.5).

Proving $\varepsilon_2 = 0$:
We continue to work inside $G = G_2(\mathbb{C}^5)$. Altering notation, we now mean $\gamma^2(\mathbb{C}^5)$ if we write γ^2, instead of $\gamma^2(\mathbb{C}^\infty)$ as before. This will lead to no confusion. We already know

$$
PD(\sigma_{1,1}) \overset{(6.2)}{=} [X_{1,1}]^\vee \overset{\text{Cor.47}}{=} (-1)^{\varepsilon_2}\, c_2(\gamma^2) \ \in H^4(G) \cong \mathbb{Z}^2 \quad .
$$

Our approach will be inspired by the method used by Milnor-Stasheff to show that the generator $a = -c_1(\gamma^1(\mathbb{C}^{n+1}))$ of $H^2(\mathbb{C}P^n)$ raised to the power of n is compatible with the natural orientation on $\mathbb{C}P^n$, i.e. that $\langle\, a^n\, ,\, [\mathbb{C}P^n]\,\rangle = +1$ [25]. We make the following ansatz: Suppose we have found certain integer coefficients $\alpha_1, \ldots, \alpha_4$ such that in $H^{2\dim(G)}(G) = H^{12}(G)$ we have

$$
c_6(G) = \alpha_1\, c_1(\gamma^2)^6 + \alpha_2\, c_1(\gamma^2)^4\, c_2(\gamma^2) + \alpha_3\, c_1(\gamma^2)^2\, c_2(\gamma^2)^2 + \alpha_4\, c_2(\gamma^2)^3 \quad .
$$

In general such coefficients surely exist since the cohomology ring $H^*(G_2(\mathbb{C}^\infty))$ is the integer polynomial ring in $c_1(\gamma^2(\mathbb{C}^\infty))$, $c_2(\gamma^2(\mathbb{C}^\infty))$, by Fact 41 and the homomorphism $H^{12}(G_2(\mathbb{C}^\infty)) \twoheadrightarrow H^{12}(G)$

[24] The sign can be derived easily with our so far developed techniques: By Corollary 30 we have $PD(\sigma_{1,0}) = \sigma_{3,2}^\vee$ and Remark 46 and Corollary 47 yield $[X_{1,0}]^\vee = -c_1(\gamma^2(\mathbb{C}^5))$.

[25] see [MS74, Remark on p. 170]

is surjective[26]. In fact the above linear combination is not even unique[27]. However we will demand a further requirement below which makes it considerably trickier to find such coefficients. The relation between the Chern classes of the tautological bundle γ^2 and the Poincaré duals of Schubert classes is given by

$$\begin{aligned} c_1(\gamma^2)^2 &= PD(\sigma_1^2)\,, & c_1(\gamma^2)^4 &= PD(\sigma_1^4)\,, & c_1(\gamma^2)^6 &= PD(\sigma_1^6) \\ c_2(\gamma^2) &= (-1)^{\varepsilon_2} PD(\sigma_{1,1})\,, & c_2(\gamma^2)^2 &= PD(\sigma_{1,1}^2)\,, & c_2(\gamma^2)^3 &= (-1)^{\varepsilon_2} PD(\sigma_{1,1}^3) \end{aligned}$$

Since $c_6(G)$ is the highest Chern class of the tangent bundle TG, we have $c_6(G) = e(TG)$ the Euler class of the oriented bundle TG. It is a well-known fact[28] that

$$\chi(G) = \langle\, e(TG)\,,\, [G]\,\rangle$$

where $\chi(G)$ is the Euler characteristic of G. On the other hand, counting Schubert cells of $G = G_2(\mathbb{C}^5)$ yields $\chi(G) = 10$. Hence

$$\begin{aligned} 10 = \chi(G) &= \langle\, e(TG)\,,\, [G]\,\rangle = \langle\, c_6(G)\,,\, \sigma_0\,\rangle \\ &= \alpha_1 \langle\, PD(\sigma_1^6)\,,\, \sigma_0\,\rangle + (-1)^{\varepsilon_2}\,\alpha_2 \langle\, PD(\sigma_1^4\cdot\sigma_{1,1})\,,\, \sigma_0\,\rangle \\ &\quad + \alpha_3 \langle\, PD(\sigma_1^2\cdot\sigma_{1,1}^2)\,,\, \sigma_0\,\rangle + (-1)^{\varepsilon_2}\,\alpha_4 \langle\, PD(\sigma_{1,1}^3)\,,\, \sigma_0\,\rangle \\ &= \alpha_1\,\sigma_1^6 + (-1)^{\varepsilon_2}\,\alpha_2\,(\sigma_1^4\cdot\sigma_{1,1}) + \alpha_3\,(\sigma_1^2\cdot\sigma_{1,1}^2) + (-1)^{\varepsilon_2}\,\alpha_4\,\sigma_{1,1}^3 \end{aligned}$$

where the last equation holds due to Proposition 28. In Appendix A.3.1 we apply Schubert calculus to compute

$$\sigma_1^6 = 5 \qquad \sigma_1^4\cdot\sigma_{1,1} = 2 \qquad \sigma_1^2\cdot\sigma_{1,1}^2 = 1 \qquad \sigma_{1,1}^3 = 1 \quad . \tag{8.11}$$

If we now additionally require that $2\,\alpha_2 + \alpha_4 \neq 0$, then the above computation shows that

$$(-1)^{\varepsilon_2} = \frac{10 - 5\alpha_1 - \alpha_3}{2\,\alpha_2 + \alpha_4} \quad , \tag{8.12}$$

which enables us to determine ε_2 once such suitable coefficients $\alpha_1, \ldots, \alpha_4$ have been found. To actually find them, we again fall back on Borel-Hirzebruch [BH58, Section 16.2]: As before, we temporarily adopt their notation and write $G = G_2(\mathbb{C}^5) = \mathbf{W}(m,n)$ with $m = 2\,, n = 3$. In addition we now also write σ_r for the r-th Chern class of the canonical bundle $\gamma^2 = \gamma^2(\mathbb{C}^5)$ and identify it with the r-th elementary symmetric polynomial in the variables $x_i\,,\ 1 \leq i \leq m$ [29]. This means we consider power series in the variables $x_1, \ldots, x_5$ and $\sigma_1 = x_1 + x_2\,,\ \sigma_2 = x_1x_2$ are the elementary symmetric polynomials in the variables x_1, x_2. Then Borel-Hirzebruch's formula (8.8) in this concrete setting is

$$\begin{aligned} c(G) \prod_{1\leq i\leq j\leq 2} (1 - (x_i - x_j)^2) &\equiv \prod_{i=1}^{2} (1 - x_i)^5 && \mod I \\ c(G)\left(1 - x_1^2 - x_2^2 + 2x_1x_2\right) &\equiv \left(\sum_{i=0}^{5} \binom{5}{i} (-x_1)^i\right) \left(\sum_{i=0}^{5} \binom{5}{i} (-x_2)^i\right) && \mod I \end{aligned}$$

[26]Since $H_{12}(G) \to H_{12}(G_2(\mathbb{C}^\infty))$ maps each Schubert class for $G_2(\mathbb{C}^5)$ to the respective Schubert class for $G_2(\mathbb{C}^\infty)$; the map on cohomology is linearly dual to this map and thus surjective.

[27]i.e. the homomorphism $H^{12}(G_2(\mathbb{C}^\infty)) \twoheadrightarrow H^{12}(G)$ is no isomorphism

[28]e.g. see [MS74, Corollary 11.12]

[29]Using the letter sigma again will lead to no confusion since we avoid any Schubert classes during the rest of the argument.

Since $\dim G = \dim_{\mathbb{C}} G = 2 \cdot (5-2) = 6$, all cohomology groups $H^{2i}(G)$, $i > 6$, are zero. Via above identification this means that all homogeneous polynomials in $S\{x_1, x_2\} \otimes S\{x_3, x_4, x_5\}$ with degree > 6 are zero mod I. Keeping this in mind and expanding the right-hand side, we obtain

$$\begin{aligned} c(G)\big(1 - x_1^2 - x_2^2 + 2x_1x_2\big) \equiv \;& 5x_1^5x_2 - x_1^5 \\ & + 50x_1^4x_2^2 - 25x_1^4x_2 + 5x_1^4 \\ & + 100x_1^3x_2^3 - 100x_1^3x_2^2 + 50x_1^3x_2 - 10x_1^3 \\ & + 50x_1^2x_2^4 - 100x_1^2x_2^3 + 100x_1^2x_2^2 - 50x_1^2x_2 + 10x_1^2 \\ & + 5x_1x_2^5 - 25x_1x_2^4 + 50x_1x_2^3 - 50x_1x_2^2 + 25x_1x_2 - 5x_1 \\ & - x_2^5 + 5x_2^4 - 10x_2^3 + 10x_2^2 - 5x_2 + 1 \end{aligned}$$

modulo I. Comparing the terms of degree 2, 4 and 6 on both sides yields

$$\begin{aligned} c_2(G) - x_1^2 - x_2^2 + 2x_1x_2 &\equiv 10x_1^2 + 25x_1x_2 + 10x_2^2 \\ c_4(G) + c_2(G)(-x_1^2 - x_2^2 + 2x_1x_2) &\equiv 5x_1^4 + 50x_1^3x_2 + 100x_1^2x_2^2 + 50x_1x_2^3 + 5x_2^4 \\ c_6(G) + c_4(G)(-x_1^2 - x_2^2 + 2x_1x_2) &\equiv 5x_1^5x_2 + 50x_1^4x_2^2 + 100x_1^3x_2^3 + 50x_1^2x_2^4 + 5x_1x_2^5 \end{aligned}$$

modulo I [30]. This allows us to successively compute $c_2(G)$, then $c_4(G)$ and finally $c_6(G)$. We obtain

$$c_6(G) \equiv 16\,x_1^6 + 24\,x_1^5x_2 + 40\,x_1^4x_2^2 + 50\,x_1^3x_2^3 + 40\,x_1^2x_2^4 + 24\,x_1x_2^5 + 16\,x_2^6$$

modulo I. The right-hand side is a symmetric polynomial in x_1, x_2 and thus can be written as a polynomial in the elementary symmetric polynomials σ_1, σ_2. In Appendix A.3.2 we show that with the coefficients

$$\alpha_1 = 16 \qquad \alpha_2 = -72 \qquad \alpha_3 = 88 \qquad \alpha_4 = -14$$

the polynomial

$$\alpha_1\,\sigma_1^6 + \alpha_2\,\sigma_1^4\,\sigma_2 + \alpha_3\,\sigma_1^2\,\sigma_2^2 + \alpha_4\,\sigma_2^3$$

equals the above polynomial on the right-hand side. Since the elementary symmetric polynomials σ_1, σ_2 correspond to the first respectively second Chern class of the canonical bundle γ^2 and $2\,\alpha_2 + \alpha_4 \neq 0$, we have found the coefficients we required for our ansatz. Inserting these values into Equation (8.12) shows that

$$(-1)^{\varepsilon_2} = \frac{10 - 5 \cdot 16 - 88}{2 \cdot (-72) + (-14)} = 1$$

and thus $\varepsilon_2 = 0$.

Values for λ and μ:
With the values of the δ-coefficients in Equations (8.3)-(8.5) together with Formulae (6.7) and (6.11) we can finally determine the values of λ and μ:

$$\lambda = \frac{2}{3} \qquad\qquad \mu = \frac{2}{3}\quad . \tag{8.13}$$

[30] This step may seem a bit imprecise at first, which is due to Borel-Hirezbruch's notation. We will present the precise argument in detail however in Subsection 8.3.1, cf. Equation (8.17) and Algorithm 49.

The linear combination of the sixth Goresky-MacPherson L-class of the Schubert variety $X_{3,2}$ in terms of its Schubert classes is thus

$$\boxed{L_6(X_{3,2}) = \frac{2}{3}\,[X_3] + \frac{2}{3}\,[X_{2,1}]} \tag{8.14}$$

and we have achieved our main goal.

8.3 Determining Arbitrary δ-Coefficients

The methods for determining the concrete δ-coefficients, developed and applied in the last two sections, do not take advantage of the specific given setting in any way. As we will see in this section, they can be generalized to the point where every δ-coefficient $\delta^{\beta}_{k,n}$, defined via Equation (8.6), can be computed algorithmically. Even stronger, we no longer need to restrict ourselves to the first Pontryagin class, but can just as well compute the coefficients of any Pontryagin class with respect to the suitable basis of Poincaré duals. To explain how, let us first introduce some preliminary notation:

In the following, if not otherwise mentioned, we work with homology respectively cohomology with integer coefficients. Let $G := G_k(\mathbb{C}^n)$ be a fixed Grassmannian and assume $n > k$ [31]. A basis for $H^*(G)$ is given by $(PD(\sigma_\alpha))_{\alpha \in \mathcal{P}(n-k,k)}$, where α ranges over all integer sequences $\alpha = (\alpha_1, \ldots, \alpha_k)$ such that $n - k \geq \alpha_1 \geq \ldots \geq \alpha_k \geq 0$. A basis for $H^{2i}(G)$ is given by the subfamily $(PD(\sigma_\alpha))_{|\alpha|=i}$. By Milnor-Stasheff [MS74, Cor. 15.5] the i-th Pontryagin class of G can solely be computed by the Chern classes of G, more precisely

$$p_i(G) = p_i(TG) = c_i(G)^2 - 2c_{i-1}(G)c_{i+1}(G) + - \ldots \pm 2c_1(G)c_{2i-1}(G) \mp 2c_{2i}(G) \quad .$$

This however means that, to determine the coefficients $\delta^{\beta}_{k,n}$ in the linear combination

$$p_i(G) = \sum_{|\beta|=2i} \delta^{\beta}_{k,n}\, PD(\sigma_\beta) \ \in H^{4i}(G) \quad ,$$

it suffices to know the coefficients κ_α in the linear combination of the Chern class of G with respect to the Poincaré dual basis:

$$c(G) = \sum_{\alpha \in \mathcal{P}(n-k,k)} \kappa_\alpha\, PD(\sigma_\alpha) \ \in H^*(G) \quad . \tag{8.15}$$

The j-th Chern class of G is given by

$$c_j(G) = \sum_{|\alpha|=j} \kappa_\alpha\, PD(\sigma_\alpha) \ \in H^{2j}(G) \quad . \tag{8.16}$$

Expanding terms of the form $c_{i-l}(G)c_{i+l}(G)$, combined with the fact that PD is a ring isomorphism with respect to intersection and cup product, allows us to find the desired linear combination for $p_i(G)$ via Schubert calculus. Since Schubert calculus can be implemented as an algorithm, this computation for the δ-coefficients of the i-th Pontryagin class is algorithmic if we additionally

[31] If $n = k$, then G is a point and its total Pontryagin class is $p(G) = p(\text{pt.}) = 1$

assume that we can determine the coefficients κ_α from Equation (8.15) via an algorithm. Thus it remains to develop such an algorithm for the κ-coefficients from (8.15). This will be done in the remaining part of this section. Notice that expressing the Pontryagin class in terms of Chern classes and then determining their coefficients with respect to the Poincaré dual basis was exactly our idea before, e.g. just look at Equation (8.7).

8.3.1 Determining the κ-Coefficients

To determine the κ-coefficients we invoke Borel-Hirzebruch's formula (8.8), which can be translated to our setting as

$$c(G) \cdot \prod_{1\leq i<j\leq k} (1-(x_i-x_j)^2) \equiv \prod_{i=1}^{k} (1-x_i)^n \mod I$$

with $c(G)$ considered as a power series in the variables $x_1, \ldots, x_k$. This is of course not entirely correct, so let us explain this at full length without all the tacit identifications Borel-Hirzebruch work with: We deal with the ring of formal power series $\mathbb{Z}[[x_1, \ldots, x_n]]$ with integer coefficients and are particularly interested in the subrings $\mathbb{Z}[[x_1, \ldots, x_k]]$ and $S := S\{x_1, \ldots, x_k\} \otimes S\{x_{k+1}, \ldots, x_n\}$ [32], where $S\{x_1, \ldots, x_k\}$ is the subring of symmetric power series in the variables $x_1, \ldots, x_k$ and analogously for $S\{x_{k+1}, \ldots, x_n\}$. For $0 \leq r \leq k$, denote by e_r the r-th elementary symmetric polynomial in the variables $x_1, \ldots, x_k$, i.e.

$$e_r = \sum_{1\leq i_1<\ldots<i_r\leq k} x_{i_1} \cdot \ldots \cdot x_{i_r} \quad .$$

Let I be the ideal in S that is generated by the symmetric power series in the variables $x_1, \ldots, x_n$ without constant term. By Borel-Hirzebruch [BH58, Section 16.2] there is a canonical ring isomorphism

$$\Phi : H^*(G) \xrightarrow{\sim} S/I \ , \quad c_r(\gamma^k(\mathbb{C}^n)) \mapsto e_r + I \ , \quad r = 1, \ldots, k \, .$$

The following lemma makes use of this fact and will be of importance further below:

Lemma 48. *Let $s \in S$ be arbitrary and write s_i for the homogeneous degree-i-term of s. Then*

$$s_i = 0 \quad \forall\, i \leq \dim(G) \quad \Longrightarrow \quad s \in I \quad .$$

Proof. The proof is not difficult but for the sake of completeness is presented in Appendix A.3.3 nonetheless. □

In the following we will simply write γ^k for $\gamma^k(\mathbb{C}^n)$. Furthermore, we define

$$f := \prod_{1\leq i<j\leq k} (1-(x_i-x_j)^2)$$
$$g := \prod_{i=1}^{k} (1-x_i)^n$$

[32] This is the smallest subring of $\mathbb{Z}[[x_1, \ldots, x_k]]$ that contains both $S\{x_1, \ldots, x_k\}$ and $S\{x_{k+1}, \ldots, x_n\}$.

f and g are symmetric polynomials in the variables $x_1, \ldots, x_k$, i.e. they lie in $\mathbb{Z}[x_1, \ldots, x_k] \cap S\{x_1, \ldots, x_k\}$. Notice that by Lemma 23 [33] a power series in $\mathbb{Z}[[x_1, \ldots, x_n]]$ is invertible if and only if its constant term is ± 1. Furthermore, a power series P is symmetric in the $x_1, \ldots, x_k$ respectively $x_1, \ldots, x_n$ if and only if its degree-i-term P_i is a symmetric polynomial in the $x_1, \ldots, x_k$ respectively $x_1, \ldots, x_n$. This and the recursive formula (3.2) from Lemma 23, which holds by Remark 24, show that the inverses of symmetric power series with constant term ± 1 are again symmetric. Similarly, by considering the subring $\mathbb{Z}[[x_1, \ldots, x_k]]$, one sees that the inverses of power series with constant term ± 1, that are symmetric in the $x_1, \ldots, x_k$, are symmetric in the $x_1, \ldots, x_k$ too. This shows in particular that f and g are invertible in $S\{x_1, \ldots, x_k\}$, and thus in S. Hence their congruence classes $f + I$ and $g + I$ are invertible in S/I.

Having introduced all this notation, Borel-Hirzebruch's formula should strictly be written as

$$\Phi(c(G)) \cdot (f + I) = (g + I) \in S/I \quad .$$

Since f is invertible in $S\{x_1, \ldots, x_k\}$, we can consider the power series $f^{-1}g$ which is symmetric in the $x_1, \ldots, x_k$. Above equation shows that $f^{-1}g$ is a representative for $\Phi(c(G))$, i.e. $\Phi(c(G)) = (f^{-1}g) + I$. Therefore, we may also write (a little informally) $c(G)$ for the power series $f^{-1}g$, so

$$c(G) := f^{-1}g \in S\{x_1, \ldots, x_k\} \subseteq S \quad .$$

We will also write $c_i(G)$ for the degree-i-term of the power series $c(G)$. $c_i(G)$ is a symmetric polynomial in the $x_1, \ldots, x_k$. Above notation is motivated by the following observation: Since $c_i(G)$ is a symmetric polynomial, it can be written as polynomial in the elementary symmetric polynomials. More precisely, there is a unique polynomial $Q_i \in \mathbb{Z}[y_1, \ldots, y_k]$ such that $c_i(G) = Q_i(e_1, \ldots, e_k)$. Because $c_i(G)$ is of degree i, every monomial $y_1^{i_1} \cdot \ldots \cdot y_k^{i_k}$ with nonvanishing coefficient in Q_i fulfills $i = i_1 + 2\, i_2 + \ldots + k\, i_k = \sum_{l=1}^{k} l\, i_l$. In such a case we will also say that the polynomial has *weight* i. Now, by definition of Φ and since applying ring homomorphisms and evaluating polynomials (with integer coefficients) commutes, we have

$$\Phi^{-1}(c_i(G) + I) = \Phi^{-1}\left(Q_i(e_1 + I, \ldots, e_k + I)\right) = Q_i\left(c_1(\gamma^k), \ldots, c_k(\gamma^k)\right) \quad .$$

Q_i is of weight i, so $Q_i\left(c_1(\gamma^k), \ldots, c_k(\gamma^k)\right)$ lies in $H^{2i}(G)$. By the above Lemma 48, $c_i(G) + I = 0$ for all $i > \dim(G)$ as well as $\left(\sum_{i>\dim(G)} c_i(G)\right) + I = 0$, where the sum is to be understood as formal sum. Thus $c(G) + I = \sum_{i=0}^{\dim(G)} c_i(G) + I$, this time we mean an ordinary sum, and hence

$$\begin{aligned} H^*(G) \ni c(G) = \Phi^{-1}(c(G) + I) &= \sum_{i=0}^{\dim(G)} \Phi^{-1}(c_i(G) + I) \\ &= \sum_{i=0}^{\dim(G)} Q_i\left(c_1(\gamma^k), \ldots, c_k(\gamma^k)\right) = \sum_{i \geq 0} Q_i\left(c_1(\gamma^k), \ldots, c_k(\gamma^k)\right) \quad . \end{aligned}$$

Because $Q_i\left(c_1(\gamma^k), \ldots, c_k(\gamma^k)\right)$ lies in $H^{2i}(G)$, above equation implies that the i-th Chern class of G is exactly $Q_i\left(c_1(\gamma^k), \ldots, c_k(\gamma^k)\right)$. Thus $\Phi(c_i(G)) = c_i(G) + I$, where on the left-hand side we mean the usual i-th Chern class of G and on the right-hand side we mean the degree-i-term of the

[33] or rather Remark 24

power series $f^{-1}g$. This should explain why we denote by $c_i(G)$ the degree-i-term of the power series $c(G)$.

By definition of the power series $c(G)$, we now have

$$c(G) \cdot f = g \quad ,$$

which allows us to calculate the polynomials $c_0(G)\,,\, c_1(G)\,,\, c_2(G)\ldots$ recursively via

$$\begin{aligned} c_0(G) &= 1 \\ c_i(G) &= g_i - \sum_{j=1}^{i} f_j\, c_{i-j}(G) = g_i - \sum_{\substack{2\le j\le i \\ j \text{ even}}} f_j\, c_{i-j}(G) \end{aligned} \tag{8.17}$$

where f_j and g_j are the degree-j-terms of f respectively g. Now there are standard algorithms to express the symmetric polynomial $c_i(G)$ of degree i as a polynomial Q_i in the elementary symmetric polynomials $e_1, \ldots, e_k$. E.g. by comparison of coefficients, as it is done in Appendix A.3.2. Q_i is uniquely determined and must have weight i. In the above paragraph we concluded that $c_i(G) = Q_i(c_1(\gamma^k), \ldots, c_k(\gamma^k))$. We thus have derived the following algorithm:

Algorithm 49 (Expressing Chern class of Grassmannian as polynomial in Chern classes of the canonical bundle)**.**
Input: Numbers $k < n$ and $i \geq 0$
Output: Polynomial Q_i of weight i such that $c_i(G_k(\mathbb{C}^n)) = Q_i(c_1(\gamma^k), \ldots, c_k(\gamma^k))$
Instructions: Put $G := G_k(\mathbb{C}^n)$.

1. Compute the polynomials $c_0(G), \ldots, c_i(G)$ recursively via Equation (8.17).
2. Determine the unique polynomial Q_i such that $c_i(G) = Q_i(e_1, \ldots, e_k)$ via standard methods, e.g. via comparison of coefficients.

As a next step let us replace the Chern classes of the canonical bundle with Poincaré duals of Schubert classes:
Further down, we will present the Epsilon-algorithm (Algorithm 54), developed by myself, that calculates the values of $\varepsilon_1, \ldots, \varepsilon_i \in \{0,1\}$ for input i. Now by Corollary 47 we have $c_i(\gamma^k) = (-1)^{\varepsilon_i}[X_{\underbrace{1,\ldots,1}_{i \text{ times}},0,\ldots 0}]^\vee \in H^{2i}(G)$ for $1 \leq i \leq k$. By Corollary 30 we are able to algorithmically compute the change-of-basis matrix from $(PD(\sigma_\beta))_{|\beta|=i}$ to $(\sigma^\vee_\alpha)_{|\alpha|=\dim(G)-i}$ via Schubert calculus. Now invert this matrix via an algorithm from linear algebra to obtain the change-of-basis matrix from $(\sigma^\vee_\alpha)_{|\alpha|=\dim(G)-i}$ to $(PD(\sigma_\beta))_{|\beta|=i}$. Thus we are able to compute the coefficients $\eta_{i,\beta}$ in the following linear combination:

$$[X_{\underbrace{\underbrace{1,\ldots,1}_{i \text{ times}},0,\ldots 0}_{\text{length } k}}]^\vee = \sigma^\vee_{\underbrace{\underbrace{n-k,\ldots,n-k}_{(k-i)\text{ times}},n-k-1,\ldots n-k-1}_{\text{length } k}} = \sum_{|\beta|=i} \eta_{i,\beta}\, PD(\sigma_\beta) \quad .$$

Since $PD : H_*(G) \xrightarrow{\sim} H^*(G)$ is a ring isomorphism, we conclude

$$c_i(G) = PD\left(Q_i \left((-1)^{\varepsilon_1} \sum_{|\beta|=1} \eta_{1,\beta}\, \sigma_\beta\,,\, \ldots\,,\, (-1)^{\varepsilon_k} \sum_{|\beta|=k} \eta_{k,\beta}\, \sigma_\beta \right) \right) \quad . \tag{8.18}$$

The right-hand side can now be evaluated algorithmically via Schubert calculus. Then one obtains the linear combination of $c_i(G)$ with respect to the basis $(PD(\sigma_\alpha))_{|\alpha|=i}$. Now one can read off the coefficients κ_α for $|\alpha| = i$. These are exactly the coefficients in Equation (8.16). All in all, we have proven that the following algorithm is correct:

Algorithm 50 (Determining the κ-coefficients)**.**

Input: Numbers $k < n$ and $i \geq 0$

Output: All coefficients κ_α with $|\alpha| = i$, such that $c_i(G_k(\mathbb{C}^n)) = \sum_{|\alpha|=i} \kappa_\alpha \, PD(\sigma_\alpha)$, cf. Equation (8.16).

Instructions: Put $G := G_k(\mathbb{C}^n)$.

1. Apply Algorithm 49 with input (k, n, i) and obtain a polynomial Q_i.
2. Apply the Epsilon-algorithm (Algorithm 54) with input k and obtain the values of $\varepsilon_1, \ldots, \varepsilon_k$.
3. Compute the coefficients $\eta_{j,\beta}$ for $j = 1, \ldots, k$, $|\beta| = j,$, via Corollary 30, Schubert calculus and inverting the resulting matrix.
4. Evaluate the term on the right-hand side of Equation (8.18) via Schubert calculus. The final expression shall be of the form
$$\sum_{|\alpha|=i} \kappa_\alpha \, PD(\sigma_\alpha) \quad .$$
5. Read off the coefficients κ_α for $|\alpha| = i$.

8.3.2 The Epsilon-Algorithm

Now we present the already announced Epsilon-algorithm. It takes as input a number k and has the output $(\varepsilon_1, \ldots, \varepsilon_k)$. The Epsilon-algorithm will be of recursive nature. By Remark 46 we already know that $\varepsilon_1 = 1$. Now consider $k > 1$ fixed but arbitrary. We work inside the Grassmannian $G := G_k(\mathbb{C}^{k+1})$ which has complex dimension $\dim(G) = k$.

In the previous subsection we saw that there is a unique polynomial $Q_k \in \mathbb{Z}[y_1, \ldots, y_k]$ such that $c_k(G) = Q_k(e_1, \ldots, e_k)$, where on the left-hand side we mean the degree-k-term of the polynomial $c(G) = f^{-1}g$ and e_i denotes the i-th elementary symmetric polynomial. Q_k has weight k, so we may write

$$Q_k =: \sum_{i_1+2\,i_2+\ldots+k\,i_k=k} \lambda_{i_1,\ldots,i_k} \; y_1^{i_1} \cdot \ldots \cdot y_k^{i_k} = \sum_{\substack{I \\ \mathrm{weight}(I)=k}} \lambda_I \, y^I$$

where the right-hand side runs over all multi-indices $I = (i_1, \ldots, i_k)$ such that $\mathrm{weight}(I) := \sum_{l=1}^k l\, i_l$ is equal to k. Note that there is exactly one multi-index I with weight k such that $i_k > 0$, namely $I = (0, \ldots, 0, 1)$. Additionally, in the previous subsection we also derived that $Q_k(c_1(\gamma^k), \ldots, c_k(\gamma^k)) = c_k(G) \in H^{2k}(G)$, and hence

$$c_k(G) = \sum_{i_1+2\,i_2+\ldots+k\,i_k=k} \lambda_{i_1,\ldots,i_k} \; c_1(\gamma^k)^{i_1} \cdot \ldots \cdot c_k(\gamma^k)^{i_k} \quad . \tag{8.19}$$

By [MS74, Corollary 11.12] the Euler characteristic $\chi(G)$ can be computed as

$$\chi(G) = \langle\, e(TG)\,,\,[G]\,\rangle = \langle\, c_k(G)\,,\,[G]\,\rangle \quad ,$$

where we also used that the top Chern class of the tangent bundle TG equals its Euler class. On the other hand, since the Schubert cells give G the structure of a finite CW-complex with cells in only even dimension and highest cell dimension $2k$, we have

$$\begin{aligned}\chi(G) &= \sum_{l=0}^{2k} (-1)^l \,\#\{\text{Schubert cells in } G \text{ of dimension } l\} \\ &= \sum_{l=0}^{k} \#\{\text{Schubert cells in } G \text{ of dimension } 2\,l\} \\ &= \#\{\text{Schubert cells in } G\} \quad .\end{aligned}$$

The Schubert cells in G are $\mathring{X}_{0,\dots,0}$, $\mathring{X}_{1,0,\dots,0}$, $\mathring{X}_{1,1,0,\dots,0}$, $\dots$, $\mathring{X}_{1,\dots,1}$, so there are exactly $k+1$ of them. Thus $\chi(G) = k+1$ and we conclude

$$k+1 = \langle\, c_k(G)\,,\,[G]\,\rangle = \langle\, c_k(G)\,,\,\sigma_0\,\rangle \quad . \tag{8.20}$$

Lemma 51. *In $G = G_k(\mathbb{C}^{k+1})$ we have for arbitrary $0 \le p \le k$:*

$$[X_{\underbrace{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0}_{k \text{ times}}}]^\vee = PD(\sigma_{\underbrace{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0}_{k \text{ times}}}) \;\in H^{2p}(G) \quad .$$

Proof. Fix $0 \le p \le k$. Notice that $[X_{\underbrace{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0}_{k \text{ times}}}]^\vee = \sigma^\vee_{\underbrace{\underbrace{1,\dots,1}_{(k-p) \text{ times}},0,\dots,0}_{k \text{ times}}}$ and that the only admissible partitions are of the form $(\underbrace{1,\dots,1}_{i \text{ times}},0,\dots,0)$ for $0 \le i \le k$. Thus for each number $0 \le i \le k$ there is exactly one admissible partition of i. By Corollary 30 it thus suffices to show

$$\sigma_{\underbrace{1,\dots,1}_{(k-p) \text{ times}},0,\dots,0} \cdot \sigma_{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0} = 1 \quad .$$

Applying Pieri (Theorem 10) inductively we see that

$$\sigma_1^i = \sigma_{\underbrace{1,\dots,1}_{i \text{ times}},0,\dots,0} \quad \forall\; 0 \le i \le k\,,$$

where we omit trailing zeroes in the notation, i.e. σ_1 stands for $\sigma_{1,0,\dots,0}$. This shows

$$\sigma_{\underbrace{1,\dots,1}_{(k-p) \text{ times}},0,\dots,0} \cdot \sigma_{\underbrace{1,\dots,1}_{p \text{ times}},0,\dots,0} = \sigma_1^{k-p} \cdot \sigma_1^p = \sigma_1^k = \sigma_{1,\dots,1} = [X_{0,\dots,0}] = [\text{pt.}] = 1 \quad .$$

$\square$

Remark 52. The above proof showed

$$\sigma_1^i = \sigma_{\underbrace{1,\dots,1}_{i \text{ times}},0,\dots,0} \quad \forall\, 0 \le i \le k$$

in $G = G_k(\mathbb{C}^{k+1})$, where σ_1 stands for $\sigma_{1,0,\dots,0}$.

By Lemma 51 and Corollary 47 we deduce

$$c_p(\gamma^k) = (-1)^{\varepsilon_p}\, PD(\sigma_{\underbrace{\underbrace{1,\dots,1}_{p\text{ times}},0,\dots 0}_{k \text{ times}}}) = (-1)^{\varepsilon_p}\, PD(\sigma_1^p) \;\in H^{2p}(G) \quad \forall\, 1 \le p \le k \quad .$$

Inserting this into Equation (8.19) and exploiting that PD is a ring isomorphism yields

$$\begin{aligned} c_k(G) = &\sum_{i_1+2\,i_2+\dots+(k-1)\,i_{k-1}=k} (-1)^{\varepsilon_1\,i_1+\dots+\varepsilon_{k-1}\,i_{k-1}}\; \lambda_{i_1,\dots,i_{k-1},0}\; PD(\sigma_1^{i_1}\cdot\sigma_1^{2\,i_2}\cdot\ldots\cdot\sigma_1^{(k-1)\,i_{k-1}}) \\ &+ (-1)^{\varepsilon_k}\;\lambda_{0,\dots,0,1}\; PD(\sigma_1^k) \quad , \end{aligned}$$

where the last summand belongs to $I = (0,\dots,0,1)$. Combine this with Equation (8.20) to obtain

$$\begin{aligned} k+1 = &\sum_{i_1+2\,i_2+\dots+(k-1)\,i_{k-1}=k} (-1)^{\varepsilon_1\,i_1+\dots+\varepsilon_{k-1}\,i_{k-1}}\; \lambda_{i_1,\dots,i_{k-1},0}\; \langle\, PD(\sigma_1^k)\,,\,\sigma_0\,\rangle \\ &+ (-1)^{\varepsilon_k}\;\lambda_{0,\dots,0,1}\;\langle\, PD(\sigma_1^k)\,,\,\sigma_0\,\rangle \quad . \end{aligned}$$

Since σ_0 is the identity element with regard to the intersection product, Proposition 28 gives

$$\langle\, PD(\sigma_1^k)\,,\,\sigma_0\,\rangle = \sigma_1^k \overset{\text{Rem. 52}}{=} \sigma_{1,\dots,1} = [\text{pt.}] = 1 \quad ,$$

and thus we obtain

$$(-1)^{\varepsilon_k}\;\lambda_{0,\dots,0,1} = (k+1) - \sum_{i_1+2\,i_2+\dots+(k-1)\,i_{k-1}=k} (-1)^{\sum_{l=1}^{k-1}\varepsilon_l\,i_l}\;\lambda_{i_1,\dots,i_{k-1},0} \quad .$$

Our goal is to prove the following lemma:

Lemma 53. $\lambda_{0,\dots,0,1} \neq 0$.

Let us postpone the proof until the end of this section. Then we are allowed to divide by $\lambda_{0,\dots,0,1}$ and hence we have derived

$$(-1)^{\varepsilon_k} = \frac{1}{\lambda_{0,\dots,0,1}} \left((k+1) \;-\; \sum_{i_1+2\,i_2+\dots+(k-1)\,i_{k-1}=k} (-1)^{\sum_{l=1}^{k-1}\varepsilon_l\,i_l}\;\lambda_{i_1,\dots,i_{k-1},0} \right) \quad , \qquad (8.21)$$

which shows that the following recursive algorithm is indeed correct:

Algorithm 54 (Epsilon-algorithm)**.**
Input: Number $k \ge 1$
Output: Values of $\varepsilon_1,\dots,\varepsilon_k$
Instructions: Put $G := G_k(\mathbb{C}^{k+1})$.

1. **If** $k = 1$: **Return** $\varepsilon_1 = 1$ and **stop**.

2. **Else**:

   2.1. Run the Epsilon-algorithm for input $k-1$ and obtain the values of $\varepsilon_1, \ldots, \varepsilon_{k-1}$.

   2.2. Run Algorithm 49 with input $(k, n := k+1, i := k)$ to obtain the polynomial

$$Q_k =: \sum_{i_1 + 2\, i_2 + \ldots + k\, i_k = k} \lambda_{i_1, \ldots, i_k}\, y_1^{i_1} \cdot \ldots \cdot y_k^{i_k} \quad .$$

   2.3. Compute the right-hand side term in Equation (8.21).
   If its value is $+1$: **Return** $\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_k = 0$ and **stop**.
   If its value is -1: **Return** $\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_k = 1$ and **stop**.

The only thing that remains to be shown is Lemma 53, the proof of which will cover the rest of this section. Let us first point out some preliminary oberservations and introduce basic notation: It is well-known that the subring of symmetric polynomials in $\mathbb{Z}[x_1, \ldots, x_k]$ is generated by the elementary symmetric polynomials $e_1, \ldots, e_k$, i.e.

$$S\{x_1, \ldots, x_k\} \cap \mathbb{Z}[x_1, \ldots, x_k] = \mathbb{Z}[e_1, \ldots, e_k] \quad ,$$

and that $e_1, \ldots, e_k$ are algebraically independent over $\mathbb{Z}$. This means that the ring homomorphism

$$\mathbb{Z}[y_1, \ldots, y_k] \longrightarrow \mathbb{Z}[e_1, \ldots, e_k] \quad , \quad y_i \longmapsto e_i \quad \forall\ 1 \le i \le k$$

is an isomorphism[34]. For any symmetric polynomial $h \in \mathbb{Z}[x_1, \ldots, x_k]$ write $\widetilde{h}$ for the unique polynomial $\widetilde{h}$ such that $h = \widetilde{h}(e_1, \ldots, e_k)$, i.e. $\widetilde{h}$ is the preimage of h under above isomorphism. With this new notation we have

$$Q_k = \widetilde{c_k(G)} \quad .$$

It is standard knowledge that, adding some new variable t to $x_1, \ldots, x_k$, the following relation holds for the elementary symmetric polynomials, which sometimes functions as their definition too:

$$\prod_{i=1}^{k} (t - x_i) = \sum_{i=0}^{k} (-1)^i\, e_i\, t^{k-i} \quad .$$

Inserting 1 for t yields

$$\prod_{i=1}^{k} (1 - x_i) = \sum_{i=0}^{k} (-1)^i\, e_i \quad . \tag{8.22}$$

Besides the elementary symmetric polynomials one is also interested in the power sums: The i-th *power sum* p_i in variables $x_1, \ldots, x_k$ is defined as

$$p_i := \sum_{l=1}^{k} x_l^i = x_1^i + \ldots + x_k^i \quad .$$

[34] Of course $y_i = x_i$, but when we want to insert elementary symmetric polynomials into some given polynomial, we rather write y_i than x_i to point this out.

In the proof of Lemma 53 we will make use of the *Newton's identities*, which may be formulated as:

$$0 = \sum_{l=1}^{i} (-1)^{i-l}\, p_l\, e_{i-l} \;+\; (-1)^i\, i\, e_i \quad \forall\; 1 \le i \le k \quad . \tag{8.23}$$

Concretely, for $i = 1, 2, 3, \ldots$ we obtain

$$\begin{aligned} 0 &= p_1 - e_1 \\ 0 &= p_2 - p_1\, e_1 + 2\, e_2 \\ 0 &= p_3 - p_2\, e_1 + p_1\, e_2 - 3\, e_3 \\ &\cdots \end{aligned}$$

Also recall that for any polynomial $h \in \mathbb{Z}[x_1, \ldots, x_k]$ the degree-i-term is denoted by h_i, i.e. $h = \sum_{i \ge 0} h_i$ where all h_i are homogeneous of degree i. Given $h \in \mathbb{Z}[x_1, \ldots, x_k]$ and some multi-index $I = (i_1, \ldots, i_k)$, we denote by $h_{x_1^{i_1} \cdot \ldots \cdot x_k^{i_k}} \in \mathbb{Z}$ the $x_1^{i_1} \cdot \ldots \cdot x_k^{i_k}$-coefficient of h. Lastly, for arbitrary i and $h \in \mathbb{Z}[x_1, \ldots, x_k]$ we write $h_{\text{weight}=i}$ for the weight-i-term of h, i.e.

$$h_{\text{weight}=i} := \sum_{\substack{I=(i_1,\ldots,i_k) \\ \text{weight}(I)=i}} h_{x_1^{i_1} \cdot \ldots \cdot x_k^{i_k}}\; x_1^{i_1} \cdot \ldots \cdot x_k^{i_k} \quad .$$

We observe the following:

Lemma 55. *Let $h \in \mathbb{Z}[x_1, \ldots, x_k]$ be a symmetric polynomial and $i \ge 0$ arbitrary. Then*

$$\widetilde{(h_i)} = (\widetilde{h})_{\text{weight}=i} \quad .$$

Proof. First notice that h_i is symmetric, so it makes sense to speak of $\widetilde{(h_i)}$.
We have $\widetilde{h} = \sum_{j \ge 0} (\widetilde{h})_{\text{weight}=j}$ and therefore

$$h = \widetilde{h}(e_1, \ldots, e_k) = \sum_{j \ge 0} \Big((\widetilde{h})_{\text{weight}=j}\, (e_1, \ldots, e_k) \Big) \quad .$$

For j arbitrary, the polynomial $(\widetilde{h})_{\text{weight}=j}\, (e_1, \ldots, e_k)$ is homogeneous of degree j. Hence

$$h_i = (\widetilde{h})_{\text{weight}=i}\, (e_1, \ldots, e_k)$$

and thus, by definition of $\widetilde{(h_i)}$, it follows that $\widetilde{(h_i)} = (\widetilde{h})_{\text{weight}=i}$. □

Corollary 56. *If $h \in \mathbb{Z}[x_1, \ldots, x_k]$ is symmetric and homogeneous of degree i, then $\widetilde{h}$ is a polynomial of weight i, i.e. if $y_1^{i_1} \cdot \ldots \cdot y_k^{i_k}$ has nonvanishing coefficient in $\widetilde{h}$, then $\sum_{l=1}^{k} l\, i_l = i$.*

Proof.

$$\widetilde{h} = \widetilde{(h_i)} = (\widetilde{h})_{\text{weight}=i}$$

is of weight i. □

Proof of Lemma 53. First notice that $\lambda_{0,\ldots,0,1}$ is just the y_k-coefficient of Q_k, so

$$\lambda_{0,\ldots,0,1} = (Q_k)_{y_k} = (\widetilde{c_k(G)})_{y_k} \quad .$$

Since inserting elementary symmetric polynomials is a ring isomorphism, Equation (8.17) yields

$$\widetilde{g_k} = \widetilde{c_k(G)} + \sum_{i=1}^{k} \widetilde{f_i} \cdot \widetilde{c_{k-i}(G)}$$

and thus the y_k-coefficients on both sides coincide:

$$(\widetilde{g_k})_{y_k} = (\widetilde{c_k(G)})_{y_k} + \sum_{i=1}^{k} \left(\widetilde{f_i} \cdot \widetilde{c_{k-i}(G)}\right)_{y_k} \quad .$$

By Corollary 56 the polynomials $\widetilde{f_i}$ and $\widetilde{c_{k-i}(G)}$ are of weight i respectively $k-i$. If $1 \leq i \leq k-1$, this means that $(\widetilde{f_i})_{y_k} = 0$ and $(\widetilde{c_{k-i}(G)})_{y_k} = 0$ since y_k is of weight k. Thus $\left(\widetilde{f_i} \cdot \widetilde{c_{k-i}(G)}\right)_{y_k} = 0$ as well. For $i = k$ the polynomial $c_{k-i}(G)$ equals 1 and hence $\widetilde{c_{k-i}(G)} = 1$. Furthermore, by Lemma 55, we have the identities $(\widetilde{g_k})_{y_k} = (\widetilde{g})_{y_k}$ and $(\widetilde{f_k})_{y_k} = (\widetilde{f})_{y_k}$ since y_k is of weight k. Altogether we obtain

$$\lambda_{0,\ldots,0,1} = (\widetilde{c_k(G)})_{y_k} = (\widetilde{g})_{y_k} - (\widetilde{f})_{y_k} \quad .$$

Thus, to prove Lemma 53, we need to show $(\widetilde{g})_{y_k} \neq (\widetilde{f})_{y_k}$.

Let us first determine $(\widetilde{g})_{y_k}$:
By definition of g and Equation (8.22) we may write

$$g = \left(\prod_{i=1}^{k}(1-x_i)\right)^{k+1} = \left(\sum_{i=0}^{k}(-1)^i\, e_i\right)^{k+1} = \left(1 + \sum_{i=1}^{k}(-1)^i\, e_i\right)^{k+1}$$

and hence

$$\widetilde{g} = \left(1 + \sum_{i=1}^{k}(-1)^i\, y_i\right)^{k+1}$$

which already shows that $(\widetilde{g})_{y_k} = (-1)^k\,(k+1)$.

Now let us compute $(\widetilde{f})_{y_k}$:
Recall that f was defined as

$$f = \prod_{1 \leq i < j \leq k} (1 - (x_i - x_j)^2) \quad .$$

If k is odd, then clearly $f_k = 0$, so $(\widetilde{f})_{\text{weight}=k} = 0$ by Lemma 55. Thus $(\widetilde{f})_{y_k} = 0$, which clearly is not equal to $(\widetilde{g})_{y_k} = -(k+1)$.
Now consider the case k even with $k = 2m$:
For $1 \leq i < j \leq k$ let us introduce variables $z_{i,j}$. There are $\binom{k}{2} = \frac{k\,(k-1)}{2}$ of them in total. Consider the ring homomorphism

$$\varphi:\ \mathbb{Z}[z_{i,j}\,|\,1 \leq i < j \leq k] \longrightarrow \mathbb{Z}[x_1, \ldots, x_k] \quad , \quad z_{i,j} \longmapsto (x_i - x_j)^2 \quad \forall\ 1 \leq i < j \leq k \quad .$$

Then

$$f = \prod_{1\le i<j\le k} (1-\varphi(z_{i,j})) = \varphi\left(\prod_{1\le i<j\le k}(1-z_{i,j})\right) \quad .$$

Now let us write $e_l[\underline{z}]$ for the l-th elementary symmetric polynomial in the $z_{i,j}$'s and analogously $p_l[\underline{z}]$ for the l-th power sum in the $z_{i,j}$'s. Again applying Equation (8.22), we obtain

$$f = \varphi\left(\sum_{l=0}^{\binom{k}{2}} (-1)^l\, e_l[\underline{z}]\right) = \sum_{l=0}^{\binom{k}{2}} (-1)^l\, \varphi(e_l[\underline{z}]) \quad .$$

Since $e_l[\underline{z}]$ is of degree l, the polynomial $\varphi(e_l[\underline{z}]) = e_l[\underline{z}]\big((x_i-x_j)^2|1\le i<j\le k\big)$ is of degree $2l$. Hence

$$f_k = (-1)^m\, \varphi(e_m[\underline{z}])$$

and thus

$$(\widetilde{f})_{y_k} \overset{\text{Lemma 55}}{=} (\widetilde{f_k})_{y_k} = (-1)^m\, \left(\widetilde{\varphi(e_m[\underline{z}])}\right)_{y_k} \quad .$$

Before we continue with the computation of the e_k-coefficient of $\varphi(e_m[\underline{z}])$, let us make four short preliminary remarks:

Firstly, for the rest of the proof we may as well consider polynomials with rational coefficients instead of integer ones. This is no problem because $e_1,\dots,e_k$ are algebraically independent over $\mathbb{Q}$ as well and $\mathbb{Q}[e_1,\dots,e_k]$ is the ring of symmetric polynomials in rational coefficients, so inserting elementary symmetric polynomials again gives a ring isomorphism, and furthermore the following two diagrams obviously commute:

$$\begin{array}{ccccccc}
\mathbb{Z}[z_{i,j}\,|\,i<j] & \xrightarrow{\varphi} & \mathbb{Z}[x_1,\dots,x_k] & & \mathbb{Z}[y_1,\dots,y_k] & \xrightarrow{\cong} & \mathbb{Z}[e_1,\dots,e_k] \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathbb{Q}[z_{i,j}\,|\,i<j] & \xrightarrow{\varphi} & \mathbb{Q}[x_1,\dots,x_k] & & \mathbb{Q}[y_1,\dots,y_k] & \xrightarrow{\cong} & \mathbb{Q}[e_1,\dots,e_k]
\end{array}$$

where the horizontal arrows in the right diagram are inserting $e_1,\dots,e_k$ into a given polynomial.

Secondly, notice that recursive application of Newton's identities (8.23) allows us to express p_i as a polynomial in the $e_1,\dots,e_i$ and, vice versa, e_i as a polynomial (with rational coefficients) in the $p_1,\dots,p_i$. Taking a closer look at Newton's identities, one sees that:

$$\text{The } e_i\text{-coefficient in the polynomial expression for } p_i \text{ is } (-1)^{i-1}\, i \,. \tag{*}$$

$$\text{The } p_i\text{-coefficient in the polynomial expression for } e_i \text{ is } (-1)^{i-1}\, i^{-1} \,. \tag{**}$$

Thirdly, observe that for a symmetric polynomial P in the $z_{i,j}$, $i<j$, the polynomial $\varphi(P) = P((x_i-x_j)^2\,|\,i<j)$ is symmetric in the $x_1,\dots,x_k$. This is because every permutation of $1,\dots,k$ canonically induces a permutation of the 2-element subsets of $\{1,\dots,k\}$. In particular, the polynomials $p_l[\underline{z}]((x_i-x_j)^2\,|\,i<j)$ are symmetric in the $x_1,\dots,x_k$.

Fourthly, there is a standard trick to express $\varphi(p_s[\underline{z}])$ in terms of $p_1, \ldots, p_{2s}$[35]:

$$\begin{aligned}\varphi(p_s[\underline{z}]) &= \sum_{i<j} (x_i - x_j)^{2s} = \frac{1}{2} \sum_{i,j} (x_i - x_j)^{2s} = \frac{1}{2} \sum_{i,j} \sum_{l=0}^{2s} \binom{2s}{l} (-1)^l \, x_i^l \, x_j^{2s-l} \\ &= \frac{1}{2} \sum_{l=0}^{2s} (-1)^l \binom{2s}{l} p_l \, p_{2s-l} \quad .\end{aligned}$$

In particular, $\varphi(p_s[\underline{z}])$ is a polynomial in the $p_1, \ldots, p_{2s}$.

Now, let us continue with our original goal: By (**), we may write $e_m[\underline{z}]$ as

$$e_m[\underline{z}] = \frac{(-1)^{m-1}}{m} \, p_m[\underline{z}] + P(p_1[\underline{z}], \ldots, p_{m-1}[\underline{z}])$$

for some polynomial P. By the third remark, both $\varphi(p_m[\underline{z}])$ and $\varphi\left(P(p_1[\underline{z}], \ldots, p_{m-1}[\underline{z}])\right)$ are symmetric in the $x_1, \ldots, x_k$. Now

$$\varphi\left(P(p_1[\underline{z}], \ldots, p_{m-1}[\underline{z}])\right) = P\left(\varphi(p_1[\underline{z}]), \ldots, \varphi(p_{m-1}[\underline{z}])\right)$$

is a polynomial in the $p_1, \ldots, p_{2m-2}$ by our fourth preliminary remark and by the second one it thus is a polynomial in the $e_1, \ldots, e_{2m-2}$. In particular, no e_k-term appears in this polynomial representation. Thus, we deduce

$$\left(\widetilde{\varphi(e_m[\underline{z}])}\right)_{y_k} = \frac{(-1)^{m-1}}{m} \left(\widetilde{\varphi(p_m[\underline{z}])}\right)_{y_k} = \frac{(-1)^{m-1}}{m} \frac{1}{2} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left(\widetilde{p_l \, p_{k-l}}\right)_{y_k} \quad ,$$

where we also used the fourth preliminary remark for the last equation. By the second remark, we know that for $l = 1, \ldots, k-1$ the polynomial $p_l \, p_{k-l}$ can be written as polynomial in the $e_1, \ldots, e_{k-1}$ and hence $\left(\widetilde{p_l \, p_{k-l}}\right)_{y_k} = 0$. Additionally, by (*) we also know that the e_k-coefficient in the polynomial expression for p_k is $(-1)^{k-1} \, k$ and thus the e_k-coefficient in the polynomial expression for $p_0 \, p_k = k \, p_k$ is simply $(-1)^{k-1} \, k^2 = -k^2$. Altogether

$$\left(\widetilde{\varphi(e_m[\underline{z}])}\right)_{y_k} = \frac{(-1)^{m-1}}{m} (-k^2) = (-1)^m \, 2\, k$$

and hence

$$(\widetilde{f})_{y_k} = (-1)^m \left(\widetilde{\varphi(e_m[\underline{z}])}\right)_{y_k} = 2\, k \quad .$$

This shows that $(\widetilde{f})_{y_k} = 2\, k$ is not equal to $(\widetilde{g})_{y_k} = k + 1$ for k even. $\square$

Remark 57. The above proof even showed that

$$\lambda_{0,\ldots,0,1} = (\widetilde{g})_{y_k} - (\widetilde{f})_{y_k} = \begin{cases} -(k+1) & \text{if } k \text{ is odd} \\ 1-k & \text{if } k \text{ is even} \end{cases} = (-1)^k - k \quad .$$

[35] This is the main idea for the whole last part of the proof of Lemma 53, where we try to compute the y_k-coefficient of $\widetilde{f}$, and I should give credit to the *mathoverflow*-users in [htt], where I originally was introduced to this particular trick.

8.3.3 Addendum to the Epsilon-Algorithm

After I already had developed the Epsilon-algorithm, I combed through the literature and stumbled across the following - apparently classical - result in Griffiths-Harris [GH78], that we present here with regard to our own notation:

Theorem 58 (Gauss-Bonnet Theorem I; Chapt. 3.3, p. 410, in [GH78])**.**
In $G := G_k(\mathbb{C}^n)$ *we have*

$$c_r(\gamma^k(\mathbb{C}^n)) = (-1)^r \, PD(\sigma_{\underbrace{\underbrace{1,\dots,1}_{r \text{ times}},0\dots,0}_{\text{length } k}}) \in H^{2r}(G;\mathbb{Z}) \quad \forall\, r = 1,\dots,k\,.$$

At first glance, this statement does of course not look very similar to the usual Gauss-Bonnet theorem from differential geometry, but they seem to be strongly connected, as Griffiths-Harris decided to name the theorem this way. In addition to this fundamental theorem, we have the following generalization of Lemma 51, which we will prove further below:

Lemma 59. *In* $G := G_k(\mathbb{C}^n)$ *we have*

$$[X_{\underbrace{\underbrace{1,\dots,1}_{r \text{ times}},0,\dots,0}_{\text{length } k}}]^\vee = PD(\sigma_{\underbrace{\underbrace{1,\dots,1}_{r \text{ times}},0,\dots,0}_{\text{length } k}}) \in H^{2r}(G;\mathbb{Z}) \quad \forall\, r = 0,\dots,k\,.$$

Corollary 60 (Values of ε-coefficients)**.** $(-1)^{\varepsilon_k} = (-1)^k$ *for all* k*, thus*

$$\varepsilon_k = \begin{cases} 0 & , \ k \text{ even} \\ 1 & , \ k \text{ odd} \end{cases} \quad .$$

Proof. Choose some $n > k$ and work inside $G := G_k(\mathbb{C}^n)$. By Corollary 47, Gauss-Bonnet (Theorem 58) and Lemma 59 we have

$$(-1)^{\varepsilon_k} \, [X_{1,\dots,1}]^\vee = c_k(\gamma^k(\mathbb{C}^n)) = (-1)^k \, PD(\sigma_{1,\dots,1}) = (-1)^k \, [X_{1,\dots,1}]^\vee \quad .$$

Since $[X_{1,\dots,1}]^\vee \neq 0$ in the free abelian group $H^{2k}(G;\mathbb{Z})$, we conclude $(-1)^{\varepsilon_k} = (-1)^k$. □

Of course, and to my slight disappointment, this result renders the Epsilon-algorithm de facto obsolete. Let us now give a short proof of the above Lemma 59. We will need the following (weaker) version of a strong result from Banagl & Wrazidlo [BW22]:

Proposition 61 (Cor. 5.9 in [BW22])**.**
Recall that, given partitions $\alpha\,,\, \beta \in \mathcal{P}(n-k,k)$*,* β *is called complementary to* α *if* $\alpha_i + \beta_{k+1-i} = n-k$ *for all* $i = 1,\dots,k$*. Now work inside* $G := G_k(\mathbb{C}^n)$*. For arbitrary* $\alpha\,,\, \beta \in \mathcal{P}(n-k,k)$ *with* $|\alpha| + |\beta| = k\,(n-k) = \dim(G)$ *we have*

$$\sigma_\alpha \cdot \sigma_\beta = \begin{cases} 1 & , \ \text{if } \beta \text{ is complementary to } \alpha \\ 0 & , \ \text{else} \end{cases} \quad .$$

Corollary 62 (Enhanced version of Corollary 30)**.** *Given any partition $\beta \in \mathcal{P}(n-k,k)$. Let $\alpha \in \mathcal{P}(n-k,k)$ denote its complementary partition, i.e. $\alpha_i = (n-k) - \beta_{k+1-i}$ for all $i = 1, \ldots, k$. Then*

$$PD(\sigma_\beta) = \sigma_\alpha^\vee = [X_\beta]^\vee \in H^{2\,|\beta|}(G_k(\mathbb{C}^n);\mathbb{Z}) \quad .$$

Proof. The first equation follows directly from Corollary 30 and Proposition 61. The second equation is due to Remark 32. □

Proof of Lemma 59. This is a special case of Corollary 62. □

Remark 63. In view of Corollary 62, much of the effort we invested in Chapter 6 and this chapter to transform the Poincaré dual basis of Schubert classes into the respective linear dual basis of course seems superfluous. The main reason we did it anyway is because I was not aware of the vital Corollary 5.9 in [BW22] when I originally derived the values of λ and μ. Since Corollary 30, combined with Schubert calculus, suffices to compute any change-of-basis matrix algorithmically, I was content enough not to wonder whether one could simplify the formula in Corollary 30. However, an advantage of the way we did it now is, that it demonstrates that all of our previously achieved results do not depend on Corollary 5.9 in [BW22] but can be derived in a self-contained manner.

Chapter 9

Alternative Approach for Determining the δ-Coefficients

In this chapter we give an alternative approach for determining the δ-coefficients we are interested in. In contrast to the preceding chapter, which was in many respects self-contained or at least relied only on the highly classical works of Borel-Hirzebruch and Milnor-Stasheff, we now apply recent results, achieved by Aluffi & Mihalcea in [AM08].

In the following we tacitly work with homology and cohomology groups with integer coefficients. Let $G := G_k(\mathbb{C}^n)$ be a fixed Grassmannian. As already defined in Section 8.3, Equation (8.15), the κ-coefficients κ_α (for partitions α of length k such that $\alpha_1 \leq n-k$) are given by

$$c(G) = \sum_{\alpha} \kappa_\alpha \, PD(\sigma_\alpha)$$

and thus for the i-th Chern class we have

$$c_i(G) = \sum_{|\alpha|=i} \kappa_\alpha \, PD(\sigma_\alpha) \quad .$$

Admittedly, of course one should keep record of the parameters k and n in the notation of the κ-coefficients, i.e. rather write $\kappa^\alpha_{k,n}$ than κ_α (analogously to the notation for the δ-coefficients), but, other than for the δ-coefficients, κ-coefficients for different parameters $k\,,\, n$ will not appear simultaneously in any formula. Thus we informally omit these parameters in the notation.
As already described in detail in Section 8.3, we invoke [MS74, Cor. 15.5] to obtain[1]

$$p_1(G) = c_1(G)^2 - 2\, c_2(G) \quad ,$$

which enables us to express the δ-coefficients in terms of the κ-coefficients: As before, let us omit trailing zeroes in the notation[2]. For $k > 1\,,\; n-k \geq 2\,,$ Pieri yields $PD(\sigma_1)^2 = PD(\sigma_{1,1}) + PD(\sigma_2)$.

[1] cf. Equation (8.7)

[2] This is just informal notation. All sequences are meant to be of length k and, if the sequence is shorter than that, one should fill it up with the appropriate number of zeroes.

Hence, comparing coefficients with Equation (8.6), we see that

$$\delta^{1,1}_{k,n} = \kappa_1^2 - 2\,\kappa_{1,1}$$
$$\delta^{2}_{k,n} = \kappa_1^2 - 2\,\kappa_{2} \quad .$$

Analogously for $k = 1\,,\ n - k \geq 2\,$, applying Pieri gives $PD(\sigma_1)^2 = PD(\sigma_2)$ and thus

$$\delta^{2}_{1,n} = \kappa_1^2 - 2\,\kappa_{2} \quad .$$

All of what we did so far is in complete accordance with Section 8.3. The two methods only differ in the way how the κ-coefficients are determined:
While Algorithm 50 makes use of Borel-Hirzebruch's Formula (8.8), for the alternative approach we will first compute the so-called *Chern-Schwartz-MacPherson class* and then take its Poincaré dual. Certainly, the first way has the advantage that it only relies on long known results and all new considerations are proven in detail and at full length. Furthermore, this method is based only on classical homology and cohomology theory, whereas the alternative way requires basic underlying knowledge of homological characteristic classes for singular spaces[3]. However, the alternative way has the benefit that it is in some ways more direct because we directly obtain the desired linear combination with respect to the Poincaré dual basis, in contrast to our original approach where we had to transform certain bases (and generating systems) into one another[4]. This said, both methods are algorithms in the strict sense and lead to guaranteed success in the determination of any κ-coefficient in question, so from a theoretical point of view both ways solve the problem of finding the κ-coefficient for some given partition entirely satisfactorily.

9.1 Aluffi-Mihalcea's Notation

Let c_{SM} denote the *Chern-Schwartz-MacPherson*[5] class. This class lives in homology and has the property that for a nonsingular variety V it is the Poincaré dual of the total Chern class of V, i.e. $PD(c_{\mathrm{SM}}(V)) = c(V) \in H^*(V;\mathbb{Z})$ [6]. As before, for a partition $a = (a_1, \ldots, a_k)$ we denote by $\mathring{X}_a$ the Schubert cell for a, i.e. $\mathring{X}_a = X_a - \bigcup_{b<a} X_b$. Then

$$c_{\mathrm{SM}}(X_a) = \sum_{b \leq a} c_{\mathrm{SM}}(\mathring{X}_b)$$

and, with the notation of Aluffi-Mihalcea,

$$c_{\mathrm{SM}}(\mathring{X}_a) =: \sum_{b \leq a} \gamma_{a,b}\,[X_b]$$

[3]At least if one aims for some comprehension of the paper of Aluffi-Mihalcea [AM08].

[4]It took us a considerable amount of time to develop the theory for that, namely the whole Section 8.1 (for the proof of [MS74, Problem 14-D]) as well as Subsection 8.3.2 (for the novel Epsilon-algorithm). Even with this theory developed, it still is time-consuming and somewhat complex to deal with three given generating systems – the linear duals of Schubert classes, the Poincaré duals and the monomials in Chern classes of the canonical bundle – of the Grassmannian cohomology group of some fixed degree simultaneously.

[5]abbreviated: CSM

[6]$PD : H_*(V;\mathbb{Z}) \xrightarrow{\sim} H^*(V;\mathbb{Z})$ is defined as the sum of the Poincaré duality isomorphisms $PD_i : H_i(V;\mathbb{Z}) \xrightarrow{\sim} H^{\dim_{\mathbb{R}}(V)-i}(V;\mathbb{Z})$.

for uniquely determined integer coefficients $\gamma_{a,b}$ since the Schubert classes $[X_b]$, $b \leq a$, freely generate $H_*(X_a;\mathbb{Z})$. Aluffi-Mihalcea's main result [AM08, Theorem 1.1] allows for the computation of these coefficients $\gamma_{a,b}$. There is an even more explicit form presented by them directly beneath their main theorem. However, given our situation, we are only interested in partitions a of length 2. For those, Aluffi-Mihalcea find a nice way to reformulate their main theorem, namely in the form of their Theorem 4.5, which we now will explain[7]:

We are considering partitions $a = (a_1, a_2)$, $b = (b_1, b_2)$ of length 2 with $b \leq a$. A *lattice path* joining two lattice points A and B is a finite sequence $(p_n)_{0\leq n\leq n_0}$ of points in $\mathbb{Z}^2$ which starts in A and ends in B, such that each point is one step to the right or down from the preceding one, i.e. $p_{n+1} = p_n + (1,0)$ or $p_{n+1} = p_n - (0,1)$ for all $0 \leq n < n_0$. Two paths $(p_n)_{0\leq n\leq n_0}$ and $(q_n)_{0\leq n\leq n_1}$ *meet* if there are $0 \leq n \leq n_0$, $0 \leq k \leq n_1$ such that $p_n = q_k$. We are considering the following sets of $a_2 + 1$ points in the plane:

$$\begin{aligned} A_1 &:= \{A_1^l\}_{0\leq l\leq a_2} & \qquad A_2 &:= \{A_2^l\}_{0\leq l\leq a_2} \\ A_1^l &:= (l+2, a_1+2) & A_2^l &:= (1-l, a_2+1-l) \quad . \end{aligned}$$

Additionally, we fix two points on the diagonal:

$$B_1 := (b_1+2, b_1+2) \qquad\qquad B_2 := (b_2+1, b_2+1) \quad .$$

For $0 \leq l \leq a_2$ let $\Pi(l)$ be the set of pairs (π_1, π_2) such that π_i is a lattice path that joins A_i^l and B_i and π_1 and π_2 do not meet. Let $\#\Pi(l)$ denote the cardinality of $\Pi(l)$. Then the coefficient $\gamma_{a,b}$ can be computed via the following theorem:

Theorem 64 (Theorem 4.5 in [AM08])**.**

$$\gamma_{a,b} = \sum_{l=0}^{a_2} \#\Pi(l) \quad .$$

9.2 Applying Aluffi-Mihalcea

For $G = G_k(\mathbb{C}^n)$ we denote by $c_{\mathrm{SM}}(G)_i$ the homology component of $c_{\mathrm{SM}}(G)$ of degree $2i$. We have $c(G) = PD(c_{\mathrm{SM}}(G))$ and thus

$$c_i(G) = PD(c_{\mathrm{SM}}(G)_{\dim(G)-i}) = PD(c_{\mathrm{SM}}(G)_{k(n-k)-i}) \quad \forall\, i\ .$$

Determining $\delta_{2,5}^{1,1}$, $\delta_{2,5}^{2}$:
We first demonstrate this new technique by determining $\delta_{2,5}^{1,1}$, $\delta_{2,5}^{2,0} = \delta_{2,5}^{2}$, so we put $G := G_2(\mathbb{C}^5) =$

[7] Or simply look at [AM08, Section 4.4] directly since the following definitions are more or less copied and Aluffi-Mihalcea even provide enlightening illustrations.

$X_{3,3}$, $n := 5$, $k := 2$. Then

$$\begin{aligned}
c_1(G) &= PD(c_{\mathrm{SM}}(G)_{(2(5-2)-1)}) = PD(c_{\mathrm{SM}}(G)_5) \\
c_2(G) &= PD(c_{\mathrm{SM}}(G)_4) \\
c_{\mathrm{SM}}(G)_5 &= c_{\mathrm{SM}}(X_{3,3})_5 = c_{\mathrm{SM}}(\mathring{X}_{3,2})_5 + c_{\mathrm{SM}}(\mathring{X}_{3,3})_5 \\
&= \left(\gamma_{(3,2),(3,2)} + \gamma_{(3,3),(3,2)}\right)[X_{3,2}] \\
c_{\mathrm{SM}}(G)_4 &= c_{\mathrm{SM}}(X_{3,3})_4 = c_{\mathrm{SM}}(\mathring{X}_{2,2})_4 + c_{\mathrm{SM}}(\mathring{X}_{3,1})_4 + c_{\mathrm{SM}}(\mathring{X}_{3,2})_4 + c_{\mathrm{SM}}(\mathring{X}_{3,3})_4 \\
&= \left(\gamma_{(2,2),(2,2)} + \gamma_{(3,2),(2,2)} + \gamma_{(3,3),(2,2)}\right)[X_{2,2}] \\
&\quad + \left(\gamma_{(3,1),(3,1)} + \gamma_{(3,2),(3,1)} + \gamma_{(3,3),(3,1)}\right)[X_{3,1}] \quad .
\end{aligned}$$

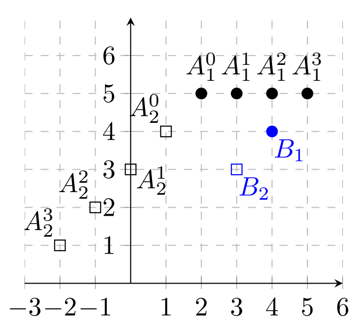
Hence

$$\begin{aligned}
c_1(G) &= \left(\gamma_{(3,2),(3,2)} + \gamma_{(3,3),(3,2)}\right) PD(\sigma_1) \\
c_2(G) &= \left(\gamma_{(2,2),(2,2)} + \gamma_{(3,2),(2,2)} + \gamma_{(3,3),(2,2)}\right) PD(\sigma_{1,1}) \\
&\quad + \left(\gamma_{(3,1),(3,1)} + \gamma_{(3,2),(3,1)} + \gamma_{(3,3),(3,1)}\right) PD(\sigma_2) \quad ,
\end{aligned}$$

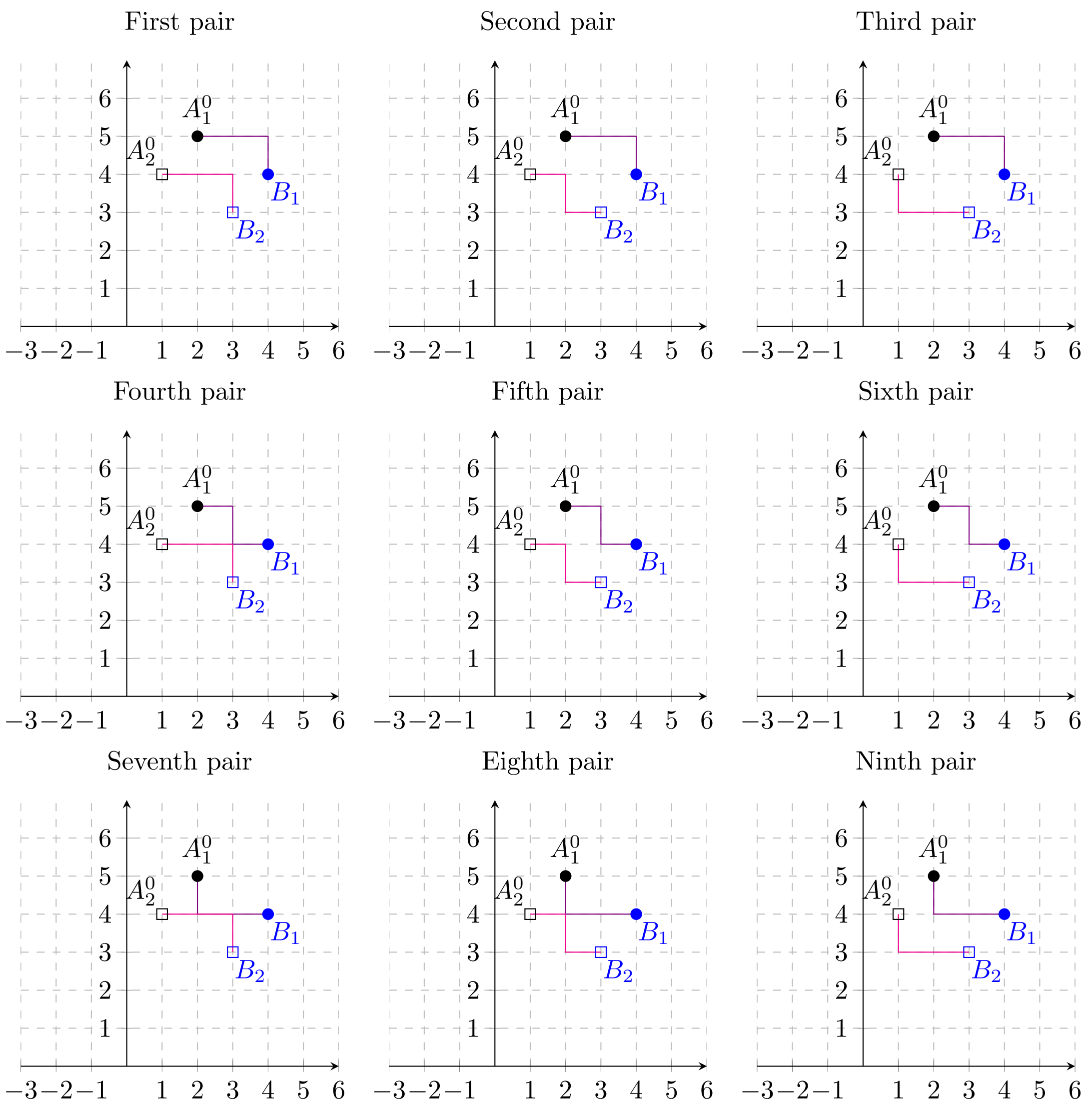
so by definition of the κ-coefficients:

$$\begin{aligned}
\kappa_1 &= \gamma_{(3,2),(3,2)} + \gamma_{(3,3),(3,2)} \\
\kappa_{1,1} &= \gamma_{(2,2),(2,2)} + \gamma_{(3,2),(2,2)} + \gamma_{(3,3),(2,2)} \\
\kappa_2 &= \gamma_{(3,1),(3,1)} + \gamma_{(3,2),(3,1)} + \gamma_{(3,3),(3,1)} \quad .
\end{aligned}$$

Next, we determine the γ-values by applying Aluffi-Mihalcea's theorem (Theorem 64). For instance let us deal with $\gamma_{(3,3),(2,2)}$: We set $a = (a_1, a_2) := (3,3)$ and $b = (b_1, b_2) := (2,2)$ and plot the correspondent lattice points A_j^l , $0 \le l \le 3$, $j = 1, 2$, as well as B_j , $j = 1, 2$:

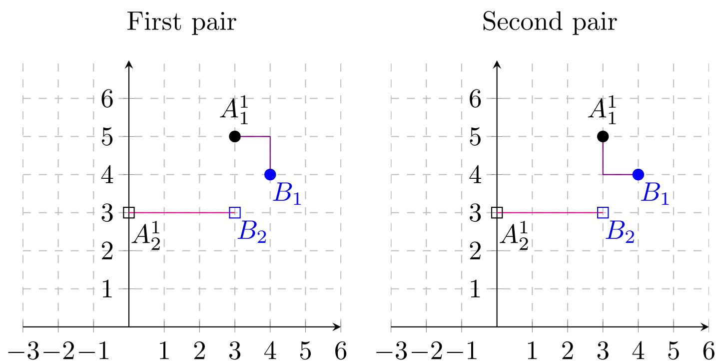


We have to determine the cardinality of the sets $\Pi(l)$, $0 \le l \le 3$. Let us first consider the case $l = 0$. Let us draw in all pairs of lattice paths connecting A_j^0 with B_j , $j = 1, 2$:



The fourth, seventh and eighth pair meet. Thus there are exactly 6 pairs of lattice paths that do not meet, in other words $\#\Pi(0) = 6$.

For $l = 1$ we have the following two pairs of lattice paths:



Obviously the paths in both these pairs do not meet each other, thus $\#\Pi(1) = 2$. Since there are no lattice paths connecting A_2^2 with B_2 respectively A_2^3 with B_2, the sets $\Pi(2)$ and $\Pi(3)$ are empty. All in all

$$\gamma_{(3,3),(2,2)} = \#\Pi(0) + \#\Pi(1) + \#\Pi(2) + \#\Pi(3) = 6 + 2 + 0 + 0 = 8 \quad .$$

Determining the other γ-values works analogously. For the partitions $\leq (3,3)$ of length 2 one obtains[8]:

Table 9.1: Values of $\gamma_{a,b}$ for partitions $b \leq a \leq (3,3)$

a, b	$\gamma_{a,b}$	a, b	$\gamma_{a,b}$	a, b	$\gamma_{a,b}$
$(0,0),(0,0)$	1	$(2,2),(0,0)$	1	$(3,2),(3,0)$	4
$(1,0),(0,0)$	1	$(2,2),(1,0)$	4	$(3,2),(2,2)$	3
$(1,0),(1,0)$	1	$(2,2),(1,1)$	4	$(3,2),(3,1)$	3
$(1,1),(0,0)$	1	$(2,2),(2,0)$	4	$(3,2),(3,2)$	1
$(1,1),(1,0)$	2	$(2,2),(2,1)$	3	$(3,3),(0,0)$	1
$(1,1),(1,1)$	1	$(2,2),(2,2)$	1	$(3,3),(1,0)$	6
$(2,0),(0,0)$	1	$(3,1),(0,0)$	1	$(3,3),(1,1)$	9
$(2,0),(1,0)$	2	$(3,1),(1,0)$	4	$(3,3),(2,0)$	12
$(2,0),(2,0)$	1	$(3,1),(1,1)$	3	$(3,3),(2,1)$	15
$(2,1),(0,0)$	1	$(3,1),(2,0)$	5	$(3,3),(3,0)$	8
$(2,1),(1,0)$	3	$(3,1),(2,1)$	3	$(3,3),(2,2)$	8
$(2,1),(1,1)$	2	$(3,1),(3,0)$	2	$(3,3),(3,1)$	7
$(2,1),(2,0)$	2	$(3,1),(3,1)$	1	$(3,3),(3,2)$	4
$(2,1),(2,1)$	1	$(3,2),(0,0)$	1	$(3,3),(3,3)$	1
$(3,0),(0,0)$	1	$(3,2),(1,0)$	5		
$(3,0),(1,0)$	3	$(3,2),(1,1)$	6		
$(3,0),(2,0)$	3	$(3,2),(2,0)$	8		
$(3,0),(3,0)$	1	$(3,2),(2,1)$	8		

[8] Actually the whole subsequent table can be found as Example 1.2 in [AM08], which furthermore definitely is more clearly arranged than here, so the interested reader may also take a look there.

Inserting these values in our above formulae yields

$$\kappa_1 = 1+4 = 5 \qquad \kappa_{1,1} = 1+3+8 = 12 \qquad \kappa_2 = 1+3+7 = 11$$

and hence

$$\delta^{1,1}_{2,5} = \kappa_1^2 - 2\,\kappa_{1,1} = 1 \qquad \delta^{2}_{2,5} = \kappa_1^2 - 2\,\kappa_2 = 3 \quad ,$$

in accordance with the proposed values in (8.5).

Determining $\delta^{1,1}_{2,4}$, $\delta^{2}_{2,4}$:

With the γ-values given in Table 9.1 it will not take long to determine $\delta^{1,1}_{2,4}$, $\delta^{2}_{2,4}$: Set $G := G_2(\mathbb{C}^4) = X_{2,2}$, $n = 4$, $k = 2$. Then $c_1(G) = PD(c_{\mathrm{SM}}(G)_3)$ and $c_2(G) = PD(c_{\mathrm{SM}}(G)_2)$. The only Schubert class of degree $2 \cdot 3$ is $[X_{2,1}] = \sigma_1$ and the Schubert classes of degree $2 \cdot 2$ are precisely $[X_{1,1}] = \sigma_{1,1}$ and $[X_2] = \sigma_2$. We derive

$$\begin{aligned}
\kappa_1 &= \gamma_{(2,1),(2,1)} + \gamma_{(2,2),(2,1)} \\
\kappa_{1,1} &= \gamma_{(1,1),(1,1)} + \gamma_{(2,1),(1,1)} + \gamma_{(2,2),(1,1)} \\
\kappa_2 &= \gamma_{(2,0),(2,0)} + \gamma_{(2,1),(2,0)} + \gamma_{(2,2),(2,0)}
\end{aligned}$$

and thus, reading off the relevant values in Table 9.1, we obtain

$$\kappa_1 = 1+3 = 4 \qquad \kappa_{1,1} = 1+2+4 = 7 \qquad \kappa_2 = 1+2+4 = 7 \quad ,$$

showing $\delta^{1,1}_{2,4} = \kappa_1^2 - 2\,\kappa_{1,1} = 2$ as well as $\delta^{2}_{2,4} = \kappa_1^2 - 2\,\kappa_2 = 2$ which verifies (8.4).

Determining $\delta^{2}_{1,4}$:

Before we deal with $\delta^{2}_{1,4}$ and thus complete this chapter, let us make a preliminary observation: Given an arbitrary Grassmannian $G := G_k(\mathbb{C}^n)$ and $0 \leq l \leq m$, we have the canonical embedding $\iota : G = G_k(\mathbb{C}^n) \hookrightarrow G' := G_{k+l}(\mathbb{C}^{n+m})$, given by $V \mapsto \mathbb{C}^l \oplus V$. In Lemma 15 we have proven that $\iota(X_a) = X_{a,\underline{0}}$ and $\iota_*\,[X_a]_G = [X_{a,\underline{0}}]_{G'}$ for all partitions $a \in \mathcal{P}(n-k,k)$. Now consider a fixed partition $a \in \mathcal{P}(n-k,k)$. The restriction $\iota| : X_a \xrightarrow{\sim} X_{a,\underline{0}}$ induces an isomorphism on homology $\iota|_* : H_*(X_a) \xrightarrow{\sim} H_*(X_{a,\underline{0}})$ with $\iota|_*\,[X_b]_{X_a} = [X_{b,\underline{0}}]_{X_{a,\underline{0}}}$ for all $b \in \mathcal{P}(n-k,k)$ with $b \leq a$ [9]. Furthermore, by definition of the CSM class it holds[10]

$$\iota|_*\, c_{\mathrm{SM}}(X_a) = c_{\mathrm{SM}}(X_{a,\underline{0}}) \quad , \tag{9.1}$$

which we verify in Appendix A.4. Writing $c_{\mathrm{SM}}(X_a) = \sum_{b \leq a} \rho_b\,[X_b]_{X_a}$ with coefficients $\{\rho_b\}_{b \leq a}$, it follows that

$$c_{\mathrm{SM}}(X_{a,\underline{0}}) = \sum_{b \leq a} \rho_b\,[X_{b,\underline{0}}]_{X_{a,\underline{0}}} \quad .$$

In other words, the $[X_b]_{X_a}$-coefficient ρ_b in the linear combination of $c_{\mathrm{SM}}(X_a)$ equals the $[X_{b,\underline{0}}]_{X_{a,\underline{0}}}$-coefficient in the linear combination of $c_{\mathrm{SM}}(X_{a,\underline{0}})$.

Now we can finally turn our attention to the remaining coefficient $\delta^{2}_{1,4}$: Put $G := G_1(\mathbb{C}^4) = X_3$, $n := 4$, $k = 1$. We would like to fall back on the now easily computable case of partitions of length 2. For this consider the canonical embedding $\iota : G \hookrightarrow G_2(\mathbb{C}^5)$, $V \mapsto \mathbb{C} \oplus V$, from Lemma 15.

[9] by functoriality of homology and because $H_*(X_a) \hookrightarrow H_*(G)$ maps $[X_b]_{X_a}$ to $[X_b]_G$ and $H_*(X_{a,\underline{0}}) \hookrightarrow H_*(G')$ is injective and maps $[X_{b,\underline{0}}]_{X_{a,\underline{0}}}$ to $[X_{b,\underline{0}}]_{G'}$

[10] cf. the third bullet point on p. 4 of [AM08]; it should be clear that this is what Aluffi-Mihalcea actually mean

By our preliminary observation we know that the $[X_b]$-coefficient of $c_{\mathrm{SM}}(G)$ equals the $[X_{b,0}]_{X_{3,0}}$-coefficient of $c_{\mathrm{SM}}(X_{3,0})$ for all partitions $b \in \mathcal{P}(3,1)$. The $[X_2]$-coefficient of $c_{\mathrm{SM}}(G)$ is just κ_1 and the $[X_1]$-coefficient of $c_{\mathrm{SM}}(G)$ is κ_2. This shows that

$$\begin{aligned}
\kappa_1 &= [X_{2,0}]\text{-coefficient of } c_{\mathrm{SM}}(X_{3,0}) = \gamma_{(2,0),(2,0)} + \gamma_{(3,0),(2,0)} \\
\kappa_2 &= [X_{1,0}]\text{-coefficient of } c_{\mathrm{SM}}(X_{3,0}) = \gamma_{(1,0),(1,0)} + \gamma_{(2,0),(1,0)} + \gamma_{(3,0),(1,0)} \quad .
\end{aligned}$$

Reading off the relevant values from Table 9.1, the κ-values in question are $\kappa_1 = 1 + 3 = 4$ and $\kappa_2 = 1 + 2 + 3 = 6$. We conclude $\delta^2_{1,4} = \kappa_1^2 - 2\,\kappa_2 = 4$, in accordance with (8.3).

Chapter 10

Outlook

While I originally computed the concrete values of λ and μ from Banagl's paper [Ban20], Banagl & Wrazidlo further extended the techniques for determining the coefficients in normally nonsingular expansions of Goresky-MacPherson L-classes of Schubert varieties in their joint work [BW22], which currently is available as preprint. A vital result in this paper is their Theorem 7.1, which allows for a recursive calculation of the coefficients in normally nonsingular expansions of inclusions of closed irreducible algebraic subvarieties into Grassmannians with regard to Gysin coherent characteristic classes[1], as Banagl-Wrazidlo note directly beneath said Theorem 7.1. Since the Goresky-MacPherson L-class indeed induces a Gysin coherent characteristic class[2], this in particular applies to normally nonsingular expansions of L-classes of Schubert varieties in arbitrary real codimension, whereas the methods in [Ban20] were restricted to real codimension 4 [3]. Some of the terms in the resulting recursive formula are of similar form to the ones for λ and μ, given in Equations (1.3) and (1.4). Thus the question arises whether we are able to compute them with the techniques developed so far in this thesis. In this chapter we try to argue why this is not the case - or at least is not directly evident - in general. The following is intended only for the ones who have fully read and understood Banagl-Wrazidlo's paper [BW22], in particular we require the reader to be familiar with the notation in [BW22], as we will completely adopt it and not give the definitions a second time. Furthermore, any subsequent references given to theorems etc. that are not within this thesis are tacitly assumed to target [BW22], unless explicitly stated otherwise.

Staring at Equation (10) in Theorem 7.1 in [BW22], we see that the recursive calculation of the coefficients in the normally nonsingular expansion, given in Equation (9), is based upon computing the normally nonsingular integration, introduced in Section 7.1, and the genera of characteristic subvarieties, defined in Section 7.2. Determining the genera of the characteristic subvarieties depends strongly on the underlying Gysin coherent characteristic class, e.g. they are Witt-signatures of certain intersections if the underlying Gysin coherent characteristic class is the Goresky-MacPherson L-class. This is a purely topological task, which was not the focus of this thesis. So we will concentrate on the normally nonsingular integration and demonstrate that the general setting even yields enough new problems that computing those numbers seems not in reach with the simple tools and tricks developed and applied in this thesis. The way to calculate the normally nonsingular integra-

[1] see [BW22, Def. 6.2]
[2] see [BW22, Thm. 9.2]
[3] cf. [BW22, p. 34]

tion $\langle c\ell^* \rangle(b', a'')$ for arbitrary partitions $b' \in \mathcal{P}(m', k')$, $a'' \in \mathcal{P}(m'', k'')$ is via the formula given in Remark 7.9, i.e.

$$\langle c\ell^* \rangle(b', a'') = \langle\, c\ell^*(f)^{-1} \,,\, [R_{abc}]_M \,\rangle \tag{10.1}$$

with $R_{abc} := X_a(D_*) \cap X_b(E_*) \cap X_c(F_*)$, Z the singular set of $X_c(F_*)$, $M = X_c(F_*) - Z$ the nonsingular part of $X_c(F_*)$ and $f : M \hookrightarrow G - Z = W$ the inclusion. Looking at the proof of Theorem 7.8, one sees that $X_a(D_*)$ and $X_b(E_*)$ as well as $R_{ab} := X_a(D_*) \cap X_b(E_*)$ and $X_c(F_*)$ are generically transverse in $G = G_k(\mathbb{C}^n)$. So, by Corollary 3.4, we can determine the codimension of R_{abc} in G if we additionally assume that $R_{abc} \neq \varnothing$ [4]:

$$\operatorname{codim}_G(R_{abc}) = \operatorname{codim}_G(X_a(D_*)) + \operatorname{codim}_G(X_b(E_*)) + \operatorname{codim}_G(X_c(F_*)) \quad .$$

Thus the pure-dimensional closed subvariety R_{abc} is of dimension

$$\dim(R_{abc}) = \dim(G) - (3\,\dim(G) - |a| - |b| - |c|) = |a| + |b| + |c| - 2\,\dim(G)$$

and, since $a = a'' \sqcup c'$, $b = b' \sqcup c''$, $c = c' \sqcup c''$, $k = k' + k''$, $m = m' + m''$, $n = m + k$ in the situation of the proof of Theorem 7.8, we furthermore have

$$\begin{aligned} |a| &= |a''| + |c'| + m''k' \ , & |b| &= |b'| + |c''| + m'k'' \ , \\ |c| &= m'k' + m''k'' + m'k'' \ , & \dim(G) &= mk = m'k' + m''k' + m'k'' + m''k'' \ . \end{aligned}$$

All in all, the complex dimension of R_{abc}, if it is not empty, is given by

$$R_{abc} \neq \varnothing \implies \dim(R_{abc}) = |a''| + |b'| - m''k' \quad . \tag{10.2}$$

So, in view of Equation (10.1), we are interested in the cohomology degree-2 $(|a''|+|b'|-m''k')$-term of $c\ell^*(f)^{-1}$. Further below we will show that $H^{2*+1}(M) = 0$ and thus that $H^*(M)$ is commutative. Furthermore, since M is a manifold, its cohomology groups vanish in almost every degree, so the ring $A = H^*(M)$ fulfills the requirements of Lemma 23. Therefore, as $c\ell^0(f) = 1$, Formula (3.2) allows for the recursive computation of $c\ell^*(f)^{-1}$ in terms of $c\ell^*(f)$.

Remark 65. Throughout this thesis, when we applied Lemma 23, we consistently checked that the respective ring in question indeed fulfills (ordinary) commutativity. As the cohomology ring $H^*(-)$ is graded commutative, we therefore often restricted ourselves to the commutative subring $H^{2*}(-)$. However, this actually is not necessary, as the statement of Lemma 23 also holds for all graded rings $A = \bigoplus_{n\geq 0} A_n$ with $A_n = 0$ for almost all n such that A is graded commutative. The proof of this well-known fact is via induction and a little more involved than that of Lemma 23, so, as for all our cases the commutative version suffices, we will refrain from presenting it here.

10.1 The Trivial Case

Before we expound why it seems so difficult to compute the normally nonsingular integration in (10.1), let us actually solve a trivial case:
Assume $X_c(F_*)$ is nonsingular. This is equivalent to $c = c' \sqcup c''$ being a rectangular partition which in turn is equivalent to one of the numbers k', k'', m', m'' being zero. For instance, a natural case where this occurs is when, in the setting of Theorem 7.1, a' is a rectangular partition, because then

[4] It makes sense to speak of the codimension of R_{abc} since R_{abc} is pure-dimensional by Corollary 3.4.

$k'' = 0$ by definition. Nonetheless, this case is more of a toy example to demonstrate what tricks we would like to apply in general.

Now, if $X_c(F_*)$ is nonsingular, then $Z = \varnothing$ and $M = X_c(F_*)$. Let $h : M = X_c(F_*) \hookrightarrow G$ denote the inclusion[5], as in the proof of Theorem 7.8. Then $h^* : H^*(G) \twoheadrightarrow H^*(M)$ is surjective by Lemma 25 and since the linear duals of Schubert classes induce bases on the cohomology groups respectively. Now let $\{\mu_f^d\}_{d\in\mathcal{P}(m,k)\,,\,d\leq c}$ be the unique family of rational coefficients such that

$$c\ell^*(f) = \sum_{\substack{d\in\mathcal{P}(m,k)\\ d\leq c}} \mu_f^d\,[X_d(F_*)]_M^\vee \in H^*(M) = H^*(M;\mathbb{Q}) \quad , \tag{10.3}$$

where we try to imitate the notation of Banagl-Wrazidlo, cf. [BW22, Equation (9)]. Then, by Lemma 25, the element $\widetilde{c\ell^*(f)}$ defined via

$$\widetilde{c\ell^*(f)} := \sum_{\substack{d\in\mathcal{P}(m,k)\\ d\leq c}} \mu_f^d\,[X_d]_G^\vee \in H^*(G)$$

is mapped onto $c\ell^*(f)$ via h^*. As the degree-0-term of $\widetilde{c\ell^*(f)}$ is 1 [6], it is invertible in $H^*(G)$ via Lemma 23. Furthermore, by Proposition 3.6 in [BW22], it holds

$$[R_{abc}]_G = [X_a(D_*)]_G \cdot [X_b(E_*)]_G \cdot [X_c(F_*)]_G = [X_a]\cdot[X_b]\cdot[X_c] \quad ,$$

cf. Equation (28) in [BW22]. Hence, for the normally nonsingular integration we have

$$\langle c\ell^*\rangle(b',a'') \overset{(10.1)}{=} \langle\, c\ell^*(f)^{-1}\,,\,[R_{abc}]_M\,\rangle = \langle\, h^*(\widetilde{c\ell^*(f)}^{-1})\,,\,[R_{abc}]_M\,\rangle = \langle\,\widetilde{c\ell^*(f)}^{-1}\,,\,[X_a]\cdot[X_b]\cdot[X_c]\,\rangle \quad .$$

By Corollary 62, i.e. the enhanced version of Corollary 30, we know that $[X_d]_G^\vee = PD(\sigma_d)$, so

$$\widetilde{c\ell^*(f)} = \sum_{\substack{d\in\mathcal{P}(m,k)\\ d\leq c}} \mu_f^d\,PD(\sigma_d) \quad .$$

By the Inversion Lemma 23 one can recursively express the coefficients $\{\eta_f^d\}_{d\in\mathcal{P}(m,k)}$ in

$$(\widetilde{c\ell^*(f)}^{-1})^{2r} = \sum_{\substack{d\in\mathcal{P}(m,k)\\ |d|=r}} \eta_f^d\,PD(\sigma_d) \in H^{2r}(G)$$

in terms of the $\{\mu_f^d\}_d$ via Schubert calculus. Finally, Proposition 28 yields

$$\langle c\ell^*\rangle(b',a'') = \sum_{\substack{d\in\mathcal{P}(m,k)\\ |d|=\dim(R_{abc})}} \eta_f^d\,(\,\sigma_d\cdot[X_a]\cdot[X_b]\cdot[X_c]\,) \in \mathbb{Q} \quad ,$$

[5] In this trivial case we have $W = G$ and thus $h = f$.

[6] since $[X_{0,\ldots,0}(F_*)]_M^\vee = 1 \in H^0(M)$ for the empty partition, so $\mu_f^{0,\ldots,0} = 1$, and $[X_{0,\ldots,0}]_G^\vee = 1 \in H^0(G)$

which can be evaluated via Schubert calculus[7]. Thus, once the coefficients $\{\mu_f^d\}_d$ are known, one can algorithmically compute the normally nonsingular integration $\langle c\ell^* \rangle (b', a'')$. These coefficients $\{\mu_f^d\}_d$ of course depend on the Gysin coherent characteristic class $c\ell$ and it remains a task to determine them if a concrete characteristic class is given. E.g. in the case of the Gysin coherent characteristic class $\mathcal{L} = (\mathcal{L}^*, \mathcal{L}_*)$ of Theorem 9.2 in [BW22], i.e. the one corresponding to the Goresky-MacPherson L-class, we developed all tools necessary for that in this thesis, mainly in Chapter 3, cf. Equation (3.5), Chapter 4, cf. Lemma 25, and Section 8.3. Notice that the tricks applied for this trivial case are practically the same as the ones we used in Chapters 4 and 6. The general idea can be briefly summarized as follows: We tried to "push" everything into the cohomology respectively homology of the ambient Grassmannian and then apply the rules of Schubert calculus. This principle will fail if $X_c(F_*)$ is singular, which we will detail in the next two sections.

10.2 Dimension of the Cohomology Groups of M

Unless otherwise stated, we consider homology and cohomology with rational coefficients, i.e. $H_*(X) = H_*(X; \mathbb{Q})$ for any topological space X and analogously for cohomology.

First notice that, if $X_c(F_*)$ is nonsingular, then $M = X_c(F_*)$ is a Schubert variety and we completely understand the homology and cohomology groups of M, cf. Section 1.2. In particular $H^{2*+1}(M) = 0$ holds, which is what we claimed before.

Now assume that $X_c(F_*)$ is singular, i.e. c is not a rectangular partition. Then, by Lakshmibai-Weyman (Theorem 4.3 in [BW22]), the singular set $Z = X_{\widetilde{c}}(F_*) \neq \varnothing$ of $X_c(F_*)$ is itself a Schubert variety with

$$\widetilde{c} = [(m' - 1) \times (k' + 1)] \sqcup [(m'' + 1) \times (k'' - 1)]$$

(cf. Figure 3(b) in [BW22]). Clearly $\widetilde{c} < c$ and thus $\varnothing \subsetneq M \subsetneq X_c(F_*)$. Since Z is compact, it is closed in $X_c(F_*)$ and M is open in $X_c(F_*)$. If M was compact, then it would also be closed in $X_c(F_*)$ and hence $X_c(F_*)$ would be disconnected, which clearly is a contradiction[8]. Hence, M is noncompact and so is not homeomorphic to any Schubert variety.

Now let us deal with the cohomology groups of M: Since M is an oriented manifold of real dimension $2|c|$ [9], generalized Poincaré duality yields an isomorphism

$$H^i(M) \xrightarrow{\sim} H^{\mathrm{BM}}_{2|c|-i}(M)$$

from singular cohomology to Borel-Moore homology for all i. Now it is a well-known fact[10] that for a locally compact space X with some compactification $\overline{X}$ it holds

$$H_j^{\mathrm{BM}}(X; \mathbb{Z}) \cong H_j(\overline{X}, \overline{X} - X; \mathbb{Z})$$

for arbitrary j and analogously for rational coefficients. Clearly $X_c(F_*)$ is a compactification of M, as $X_c(F_*)$ is compact as Schubert variety and $\overline{M} = X_c(F_*)$ because $\mathring{X}_c(F_*) \subseteq M \subseteq X_c(F_*)$ and

[7] Here we tacitly assumed that R_{abc} is non-empty, so that we can speak of the dimension of R_{abc}. Let us now state the precise algorithm: First compute $[X_a] \cdot [X_b] \cdot [X_c]$ via Schubert calculus. If this is zero, then one can return the value 0 for $\langle c\ell^* \rangle (b', a'')$. If the intersection product is not zero, then $[R_{abc}]_G \neq 0$ and thus R_{abc} is non-empty. In this case, compute the dimension of R_{abc} via Equation (10.2) and proceed as described above.

[8] $X_c(F_*)$ is the closure of the topological cell $\mathring{X}_c(F_*)$. Thus it is the closure of a connected subspace and therefore also connected.

[9] as it is the open subset of nonsingular points in $X_c(F_*)$

[10] e.g. see [Max19, Rem. 2.2.5]

$X_c(F_*)$ is the closure of $\mathring{X}_c(F_*)$. Thus we have an isomorphism

$$H^{\mathrm{BM}}_{2|c|-i}(M) \cong H_{2|c|-i}(X_c(F_*)\,,\, Z) \cong H_{2|c|-i}(X_c\,,\, X_{\widetilde{c}}) \quad ,$$

where the second isomorphism comes from Lemma 17. Now consider the long exact homology sequence of the pair $(X_c\,,\, X_{\widetilde{c}})$:

$$\ldots \longrightarrow H_{j+1}(X_c\,,\, X_{\widetilde{c}}) \longrightarrow H_j(X_{\widetilde{c}}) \longrightarrow H_j(X_c) \longrightarrow H_j(X_c\,,\, X_{\widetilde{c}}) \longrightarrow H_{j-1}(X_{\widetilde{c}}) \longrightarrow \ldots$$

Since the inclusion of Schubert varieties $X_{\widetilde{c}} \hookrightarrow X_c$ induces an injection $H_*(X_{\widetilde{c}}) \hookrightarrow H_*(X_c)$, cf. Section 1.2, we obtain short exact sequences of the form

$$0 \longrightarrow H_j(X_{\widetilde{c}}) \longrightarrow H_j(X_c) \longrightarrow H_j(X_c\,,\, X_{\widetilde{c}}) \longrightarrow 0$$

and thus

$$H_j(X_c\,,\, X_{\widetilde{c}}) \cong H_j(X_c) \,/\, H_j(X_{\widetilde{c}})$$

for all j. All in all, we have shown that the dimension of the $\mathbb{Q}$-vector space $H^i(M)$ is given by

$$\dim_{\mathbb{Q}}\, H^i(M) = \dim_{\mathbb{Q}}\, H_{2|c|-i}(X_c) \,/\, H_{2|c|-i}(X_{\widetilde{c}}) = \dim_{\mathbb{Q}}\, H_{2|c|-i}(X_c) - \dim_{\mathbb{Q}}\, H_{2|c|-i}(X_{\widetilde{c}}) \quad .$$

In particular, as the homology of Schubert varieties in odd degrees vanishes, $H^{2*+1}(M) = 0$, which we claimed further above.

Remark 66. If one formally repeats all of the previous arguments for integer instead of rational coefficients, then this proves

$$H^i(M\,;\mathbb{Z}) \cong H_{2|c|-i}(X_c\,;\mathbb{Z}) \,/\, H_{2|c|-i}(X_{\widetilde{c}}\,;\mathbb{Z}) \quad .$$

In particular, $H^i(M;\mathbb{Z})$ is finitely generated for arbitrary i. Thus, also all homology groups of M with respect to integer coefficients are finitely generated[11]. By the Universal Coefficient Theorem for Homology[12] it follows that all homology groups of M with respect to rational coefficients are finite-dimensional vector spaces and hence $H^i(M) \cong H_i(M)^\vee \cong H_i(M)$ via the Universal Coefficient Theorem. In particular, all odd dimensional homology groups of M vanish as well.

Because $([X_d])_{d\le e\,,\,|d|=i}$ is a basis of the homology group $H_{2i}(X_e)$, for $i \geq 0$ and $e \in \mathcal{P}(m,k)$ arbitrary, the dimension of $H_{2i}(X_e)$ is given by

$$\dim_{\mathbb{Q}}\, H_{2i}(X_e) = \#\big\{d \in \mathcal{P}(m,k) \,\big|\, d \le e\,,\, |d| = i\big\} \quad .$$

Therefore, the dimension of $H^{2i}(M)$ is given by

$$\begin{aligned} \dim_{\mathbb{Q}}\, H^{2i}(M) &= \#\big\{d \;\big|\; d \le c\,,\, |d| = |c| - i\big\} - \#\big\{d \;\big|\; d \le \widetilde{c},\, |d| = |c| - i\big\} \\ &= \#\big\{d \in \mathcal{P}(m,k) \,\big|\, d \le c\,,\, d \not\le \widetilde{c},\, |d| = |c| - i\big\} \quad , \end{aligned} \tag{10.4}$$

where the last equation holds because the second set in the first line is a subset of the first one. Now there is a canonical bijection (see Figure 10.1)

$$\begin{aligned} \Phi : \mathcal{P}(m',k') \times \mathcal{P}(m'',k'') &\xrightarrow{\sim} \big\{d \in \mathcal{P}(m,k) \,\big|\, d \le c\,,\, d \not\le \widetilde{c}\big\} \\ (d',d'') &\mapsto d := d' \sqcup d'' \end{aligned} \quad .$$

[11] e.g. by [Hat01, Prop. 3F.12]

[12] see [Bre93, Chapt. V, Cor. 7.5]

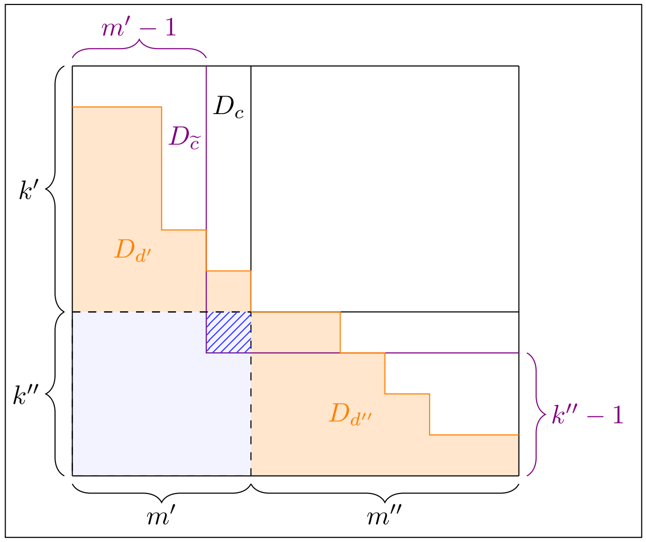


Figure 10.1: Definition of Φ. The Young diagram of a partition α is denoted by D_α.

Remark 67. Let us briefly argue why this is well-defined and a bijection: For arbitrary $d \in \mathcal{P}(m,k)$ it holds

$$\begin{aligned} d \leq c &\iff d_{k''+1} \leq m' \\ d \not\leq \widetilde{c} &\iff d_{k''} \geq m' \quad . \end{aligned}$$

If $d_{k''+1} \leq m'$, then also $d \leq c$ because d is nonincreasing. The converse direction is trivial. If $d \not\leq \widetilde{c}$, then there exists an $i \geq k''$ with $d_i > m'-1$. Since d is nonincreasing, this implies $d_{k''} \geq m'$. The converse direction again is trivial. Pictorially, $d_{k''} \geq m'$ is equivalent to the Young diagram D_d of d containing the hatched box in Figure 10.1. This shows that $d := d' \sqcup d''$ indeed fulfills $d \leq c$, $d \not\leq \widetilde{c}$ and thus well-definedness.
The inverse of Φ is given by

$$\begin{aligned} &d \longmapsto (d', d'') \\ &d'_i := d_{k''+i} \quad \forall\, i = 1, \ldots, k' \\ &d''_i := d_i - m' \quad \forall\, i = 1, \ldots, k''\,. \end{aligned}$$

In view of the bijection Φ, the equation for the dimension of the $2i$-th cohomology group of M can further be simplified to

$$\begin{aligned} \dim_{\mathbb{Q}} H^{2i}(M) &\overset{(10.4)}{=} \#\big\{d \in \mathcal{P}(m,k) \,\big|\, d \leq c\,,\ d \not\leq \widetilde{c}\,,\ |d| = |c| - i\big\} \\ &= \#\,\Phi^{-1}\left(\big\{d \in \mathcal{P}(m,k) \,\big|\, d \leq c\,,\ d \not\leq \widetilde{c}\,,\ |d| = |c| - i\big\}\right) \\ &= \#\big\{(d', d'') \in \mathcal{P}(m', k') \times \mathcal{P}(m'', k'') \,\big|\, |d' \sqcup d''| = |c| - i\big\} \\ &= \#\big\{(d', d'') \in \mathcal{P}(m', k') \times \mathcal{P}(m'', k'') \,\big|\, |d'| + |d''| + m'k'' = |c| - i\big\} \\ &= \#\big\{(d', d'') \in \mathcal{P}(m', k') \times \mathcal{P}(m'', k'') \,\big|\, |d'| + |d''| = m'k' + m''k'' - i\big\} \quad . \end{aligned} \tag{10.5}$$

10.3 The Singular Case

In this section we assume that $X_c(F_*)$ is singular. As announced in Section 10.1, we will now explain why we cannot simply adopt the tricks given there in the present, singular case. It was indispensable in Section 10.1 that we could define $\widetilde{c\ell^*(f)}$ in $H^*(G)$ and then "push" all cohomology classes from $H^*(M)$ to $H^*(G)$: Outside a Grassmannian we cannot apply the standard algorithms from Schubert calculus; furthermore we we want to make use of the generic transversality, so that the homology class of R_{abc} can be computed as intersection product of the homology classes of $X_a(D_*)$, $X_b(E_*)$ and $X_c(F_*)$, but (it seems that) this can only be achieved in $G = G_k(\mathbb{C}^n)$. However, we will show that, if $X_c(F_*)$ is singular, then it is possible that the dimension of $H^{2i}(M)$ exceeds the one of $H^{2i}(G)$ and of $H^{2i}(X_c)$ for $i = \dim(R_{abc})$, so it cannot exist an epimorphism $H^{2i}(G) \twoheadrightarrow H^{2i}(M)$ or an epimorphism $H^{2i}(X_c(F_*)) \twoheadrightarrow H^{2i}(M)$. In particular, the map $h^* : H^{2i}(G) \to H^{2i}(M)$, induced by the inclusion $h : M \hookrightarrow G$, is not surjective. This of course makes it highly unlikely that we always can lift $c\ell^*(f)$ to $H^*(G)$ or even $(c\ell^*(f)^{-1})^{2\dim(R_{abc})}$ to $H^{2\dim(R_{abc})}(G)$ [13]. Hence, (it appears that) the tricks from Section 10.1 cannot be applied in the (more important) singular case.

Concretely, set $i := 1$. Then both $H^{2i}(G) = H^{2i}(X_{[m\times k]})$ and $H^{2i}(X_c)$ have dimension 1 because there clearly is exactly one partition of 1 of length k, namely $(1, 0, \ldots, 0)$, and it is contained in c and $[m \times k]$. On the other hand, it is easy to see that

$$\big\{(d', d'') \in \mathcal{P}(m', k') \times \mathcal{P}(m'', k'') \,\big|\, |d'| + |d''| = m'k' + m''k'' - 1\big\}$$

is equal to the disjoint union

$$\big\{\, (d', d'') \,\big|\, d' = [m' \times k']\,,\, |d''| = m''k'' - 1 \,\big\} \sqcup \big\{\, (d', d'') \,\big|\, |d'| = m'k' - 1\,,\, d'' = [m'' \times k''] \,\big\} \quad .$$

Thus, by Equation (10.5), we obtain

$$\begin{aligned}
\dim_{\mathbb{Q}} H^2(M) &= \#\big\{(d', d'') \in \mathcal{P}(m', k') \times \mathcal{P}(m'', k'') \,\big|\, |d'| + |d''| = m'k' + m''k'' - 1\big\} \\
&= \#\big\{\, (d', d'') \,\big|\, d' = [m' \times k']\,,\, |d''| = m''k'' - 1 \,\big\} \\
&\quad + \#\big\{\, (d', d'') \,\big|\, |d'| = m'k' - 1\,,\, d'' = [m'' \times k''] \,\big\} \\
&= \#\big\{\, d'' \in \mathcal{P}(m'', k'') \,\big|\, |d''| = m''k'' - 1 \,\big\} + \#\big\{\, d' \in \mathcal{P}(m', k') \,\big|\, |d'| = m'k' - 1 \,\big\} \\
&\overset{(*)}{=} 1 + 1 = 2 \quad ,
\end{aligned}$$

where $(*)$ holds because for arbitrary $\widetilde{m}\,,\, \widetilde{k} > 0$ the only element in

$$\big\{\, d \in \mathcal{P}(\widetilde{m}, \widetilde{k}) \,\big|\, |d| = \widetilde{m}\widetilde{k} - 1 \,\big\}$$

is $(\underbrace{\widetilde{m}, \ldots, \widetilde{m}}_{(\widetilde{k}-1) \text{ times}}, \widetilde{m} - 1)$. Altogether, we have derived

$$\dim_{\mathbb{Q}} H^2(M) = 2 > 1 = \dim_{\mathbb{Q}} H^2(G) = \dim_{\mathbb{Q}} H^2(X_c) \quad .$$

It remains to give an explicit example where $1 = \dim(R_{abc})$. By (10.2) we have to find numbers k', k'', m', $m'' > 0$ (so that c is not rectangular) and partitions $b' \in \mathcal{P}(m', k')$, $a'' \in \mathcal{P}(m'', k'')$

[13] Of course it remains to prove that there really exists a Gysin coherent characteristic class $c\ell$, numbers k', m', k'', m'' and partitions a'', b' so that one cannot lift $c\ell^*(f)$ to $H^*(G)$ (for any admissible choice of the flag F_*).

such that $|b'| + |a''| = m''k' + 1$ and $R_{abc} \neq \varnothing$, where $a := a'' \sqcup c'$, $b := b' \sqcup c''$ and R_{abc} is the triple intersection from above[14]. We will present an even stronger example, where we show that such partitions will arise in the formula of Theorem 7.1 (Formula (10)) in Banagl Wrazidlo [BW22]. This demonstrates that our problems in computing the normally nonsingular integration actually affect the recursive computation of the coefficients in the normally nonsingular expansion. For that, however, it is best to alter notation and now write b'' instead of a'', as in the setting of Theorem 7.1 the variable b'' plays the role of the variable a'' from Remark 7.9. Thus, we aim at finding numbers k' , $m' > 0$, an inclusion $i' : X' \hookrightarrow G' = G_{k'}(\mathbb{C}^{m'})$ and a partition $a' \in \mathcal{P}(m', k')$ with $|a'| \leq \dim(X')$ (as in Theorem 7.1), such that k'' , $m'' > 0$ [15] and there exists a tuple (r, r', r'', b', b'') with $0 \leq r = r' + r'' \leq l$, $0 \leq r' < l$, $0 \leq r'' \leq l$, $b' \in \mathcal{P}(m', k')$, $|b'| = d' - r'$, $b'' \in \mathcal{P}(m'', k'')$, $|b''| = |a''| - r''$ and

$$|b'| + |b''| = m''k' + 1 \ , \quad R_{b''\sqcup c', b'\sqcup c'', c} \neq \varnothing \ ,$$

where the last line assures that the triple intersection indeed is 1-dimensional.

For that let $m' := 4$, $k' := 2$ and take X' to be the Schubert variety $X_{4,3}$ in the Grassmannian $G' = G_{k'}(\mathbb{C}^{k'+m'})$. Then the dimension d' of X' is $d' = 7$. Furthermore, put $a' := (3, 2) \in \mathcal{P}(m', k)$, so $l = d' - |a'| = 2$. By definition, $m'' = 3$, $k'' = 1$, $m = m' + m'' = 7$, $k = k' + k'' = 3$ and $a'' = (1) \in \mathcal{P}(m'', k'')$. We consider the tuple $(r, r', r'', b', b'') := (1, 0, 1, b', b'')$ with $b' := (4, 3) \in \mathcal{P}(m', k')$ and $b'' := (0) \in \mathcal{P}(m'', k'')$. Apart from the triple intersection being non-empty, all of the above requirements are obviously fulfilled. We claim that

$$[X_{b''\sqcup c'}]_G \cdot [X_{b'\sqcup c''}]_G \cdot [X_c]_G \neq 0 \quad .$$

Then the fundamental class of $R_{b''\sqcup c', b'\sqcup c'', c}$ in the homology of the Grassmannian $G = G_k(\mathbb{C}^{k+m})$ is not zero (by Equation (28) in [BW22]), so $R_{b''\sqcup c', b'\sqcup c'', c}$ is non-empty. By Schubert calculus

$$[X_{b''\sqcup c'}]_G \cdot [X_{b'\sqcup c''}]_G \cdot [X_c]_G = \sigma_{7,0,0} \cdot \sigma_{4,3,0} \cdot \sigma_{3,3,0} = \sigma_{7,7,6} = [X_{1,0,0}]_G \neq 0 \quad .$$

This finishes our example which hopefully conveys to the reader how much considerably harder the singular case is than the nonsingular one – or that it requires outright different techniques to tackle this problem.

[14]Of course we also have to find admissible flags D_* , E_* , F_* , as they appear in the definition of R_{abc}. However, in our example all admissible flags will work.

[15]with k'' , m'' , l , a'' being defined as in Theorem 7.1

Appendix A

Appendix

A.1 Regarding Chapter 6

A.1.1 Schubert Calculus for Determining μ_c in Section 6.1

Here we present the computations in Schubert calculus regarding the determination of μ_c in Section 6.1. As always when explicitly computing expressions in Schubert calculus, one only applies Pieri's and Giambelli's formulae (Theorems 10 and 11 respectively).

$$\sigma_{1,1} \overset{\text{Giamb}}{=} \det \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_0 & \sigma_1 \end{pmatrix} = \sigma_1^2 - \sigma_0\,\sigma_2 = \sigma_1^2 - \sigma_2$$
$$\sigma_{3,1} \cdot \sigma_{1,1} = \sigma_{3,1} \cdot (\sigma_1^2 - \sigma_2) \overset{\text{Pieri}}{=} (\sigma_{3,1}\,\sigma_1)\,\sigma_1 - \sigma_{3,3} \overset{\text{Pieri}}{=} \sigma_{3,2}\,\sigma_1 - \sigma_{3,3} \overset{\text{Pieri}}{=} 0$$

Here we used multiple times that $\sigma_{\alpha_1,\alpha_2} = 0$ if $\alpha_1 > n - k = 5 - 2 = 3$. We also used $\sigma_0 = 1$ in the first line.

$$\sigma_{3,1} \cdot \sigma_2 \overset{\text{Pieri}}{=} \sigma_{3,3} = 1$$

Recall that we (informally) omit the augmentation map ε_* in the notation, cf. the paragraph before Corollary 30 in Chapter 5. Strictly speaking, we should of course write $\varepsilon_*\,\sigma_{3,3} = \varepsilon_*\,[\text{pt.}] = 1$.

$$\sigma_{2,2} \cdot \sigma_2 \overset{\text{Pieri}}{=} \sigma_{4,2} = 0$$
$$\sigma_{2,2} \cdot \sigma_{1,1} = \sigma_{2,2}\,(\sigma_1^2 - \sigma_2) = \sigma_{2,2}\,\sigma_1^2 \overset{\text{Pieri}}{=} \sigma_{3,2}\,\sigma_1 \overset{\text{Pieri}}{=} \sigma_{3,3} = 1$$

Remark 68. Notice that there is a nice way for a sanity check here which is quite helpful since it is easy to miscalculate in Schubert calculus: By Remark 33 the change-of-basis matrix with respect to the bases of PD of Schubert classes and duals of Schubert classes is invertible as matrix over $\mathbb{Z}$. This is of course equivalent to being a matrix with integer entries and determinant ± 1. Indeed, this is the case in our present setting, the change-of-basis matrix being

$$\begin{pmatrix} \sigma_{3,1} \cdot \sigma_{1,1} & \sigma_{3,1} \cdot \sigma_2 \\ \sigma_{2,2} \cdot \sigma_{1,1} & \sigma_{2,2} \cdot \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad .$$

A.1.2 Schubert Calculus for Section 6.2

The ambient Grassmannian is $G = G_2(\mathbb{C}^4)$.

$$\begin{aligned}
& \sigma_{1,1} \cdot \sigma_{1,1} = (\sigma_1^2 - \sigma_2)^2 = \sigma_1^4 - 2\,\sigma_1^2\,\sigma_2 + \sigma_2^2 && \text{by Appendix ??} \\
& \sigma_1^2 = \sigma_2 + \sigma_{1,1} && \text{by Pieri} \\
\Longrightarrow\ & \sigma_1^3 = \sigma_3 + \sigma_{2,1} + \sigma_{2,1} = 2\,\sigma_{2,1} && \text{by Pieri and since } 3 > 4-2 \\
\Longrightarrow\ & \sigma_1^4 = 2\,(\sigma_{3,1} + \sigma_{2,2}) = 2\,\sigma_{2,2} = 2 && \text{by Pieri and since } 3 > 4-2 \\
& \sigma_2^2 = \sigma_4 + \sigma_{3,1} + \sigma_{2,2} = \sigma_{2,2} = 1 && \text{by Pieri and since } 4,3 > 4-2 \qquad \text{(A.1)} \\
\Longrightarrow\ & \sigma_1^2\,\sigma_2 = (\sigma_2 + \sigma_{1,1})\,\sigma_2 = \sigma_2^2 + \sigma_{3,1} = 1 && \text{by Pieri and since } 3 > 4-2 \\
\Longrightarrow\ & \sigma_{1,1}^2 = 2 - 2 + 1 = 1 && \text{(A.2)} \\
& \sigma_{1,1} \cdot \sigma_2 = \sigma_{3,1} = 0 && \text{by Pieri and since } 3 > 4-2 \qquad \text{(A.3)}
\end{aligned}$$

A.2 Proofs regarding Chapter 7

In this section we prove Facts 34 and 35 as well as Proposition 38. Actually, the arguments for Facts 34 and 35 are standard verifications and furthermore highly similar. Thus, we will only present a proof for Fact 35 and leave out the even easier one for Fact 34.

Proof of Fact 35. By abuse of notation we not only write can_*, can^* for the (co-)homological canonical maps but also for the respective chain maps $\mathrm{can}_* : C_*(X) \to C_*(X;\mathbb{Q})$, $\mathrm{can}^* : C^*(X) \to C^*(X;\mathbb{Q})$. Obviously, it suffices to prove the statement on the (co-)chain level, i.e. we want to show

$$\forall\, \alpha \in C^i(X)\,,\ b \in C_k(X):\ \mathrm{can}_{k-i}(\alpha \frown_{\mathbb{Z}} b) = \mathrm{can}^i(\alpha) \frown_{\mathbb{Q}} \mathrm{can}_k(b)\ \in C_{k-i}(X;\mathbb{Q}) \quad .$$

By $\mathbb{Z}$-(bi-)linearity, we can assume that b is a singular k-simplex, i.e. there is a $\sigma : \Delta^k \to X$ with $b = 1 \cdot \sigma$. Furthermore we may assume $k \geq i$. Then, by definition of the cap product given in Hatcher [Hat01, p. 239]:

$$\begin{aligned}
\mathrm{can}_{k-i}(\alpha \frown_{\mathbb{Z}} b) &= \mathrm{can}_{k-i}\left(\alpha(1 \cdot \sigma|_{[v_0,\dots,v_i]}) \cdot \sigma|_{[v_i,\dots,v_k]}\right) \\
&= \alpha(1 \cdot \sigma|_{[v_0,\dots,v_i]})\ \mathrm{can}_{k-i}\left(1 \cdot \sigma|_{[v_i,\dots,v_k]}\right) \\
\mathrm{can}_{k-i}\left(1 \cdot \sigma|_{[v_i,\dots,v_k]}\right) &= 1 \cdot \sigma|_{[v_i,\dots,v_k]}
\end{aligned}$$

$$\begin{aligned}
& \mathrm{can}_k(b) = 1 \cdot \sigma \in C_k(X;\mathbb{Q}) \\
\Longrightarrow\ & \mathrm{can}^i(\alpha) \frown_{\mathbb{Q}} \mathrm{can}_k(b) = \mathrm{can}^i(\alpha)(1 \cdot \sigma|_{[v_0,\dots,v_i]}) \cdot \sigma|_{[v_i,\dots,v_k]}
\end{aligned}$$

Now, by definition of can^*, we have

$$\begin{aligned}
\mathrm{can}^i(\alpha)(1 \cdot \sigma|_{[v_0,\dots,v_i]}) &= \mathrm{tensor} - \mathrm{hom}((\mathbb{Z} \hookrightarrow \mathbb{Q}) \circ \alpha)\ ((1 \cdot \sigma|_{[v_0,\dots,v_i]}) \otimes 1) \\
&= \alpha(1 \cdot \sigma|_{[v_0,\dots,v_i]})
\end{aligned}$$

All in all

$$\mathrm{can}_{k-i}(\alpha \frown_{\mathbb{Z}} b) = \alpha(1 \cdot \sigma|_{[v_0,\dots,v_i]}) \cdot \sigma|_{[v_i,\dots,v_k]} = \mathrm{can}^i(\alpha) \frown_{\mathbb{Q}} \mathrm{can}_k(b) \quad .$$

□

Remark 69. After finishing most of this thesis, I realized a subtle problem regarding the sign convention, although this has factually no consequences whatsoever. We used the identity $(\alpha \smile \beta) \frown c = \alpha \frown (\beta \frown c)$, relating cup and cap product, throughout this thesis, e.g. in the proof of Proposition 28. However, with the definitions in Hatcher [Hat01] this only is correct up to sign. The identity holds if one works with the definitions of cup and cap product given in Bredon [Bre93, pp. 328, 335], see Theorem 5.2. in [Bre93, p. 336]. The proof of Fact 35 obviously still works with Bredon's sign convention, mutatis mutandis. The reason that I did not change over to Bredon's definition is that with Bredon's convention the coboundary map in general is not the dual of the boundary map anymore but has a sign attached to it, cf. [Bre93, p. 321]. Altering this in the notation could be a source of several typing errors, so for Chapter 7 I sticked to my original choice of Hatcher's definition.

Proof of Proposition 38. Given any integer cocycle $\varphi \in Z^i(X;\mathbb{Z})$, we want to show that $(\mathrm{nat}\,\mathrm{Kron})\,[\varphi] = (\mathrm{Kron}\,\mathrm{can}^i)\,[\varphi]$. For that take any rational cycle $c \in Z_i(X;\mathbb{Q})$. We need to prove $(\mathrm{nat}\,\mathrm{Kron}\,[\varphi])[c] = (\mathrm{Kron}\,\mathrm{can}^i\,[\varphi])[c]$. Without loss of generality suppose c has only integer coefficients[1], i.e. c lies in the image of $C_i(X;\mathbb{Z}) \to C_i(X;\mathbb{Q})$. We see that $[c] \otimes 1$ [2] is mapped to $[c] \in H_i(X;\mathbb{Q})$ under the natural map $H_i(X) \otimes \mathbb{Q} \xrightarrow{\sim} H_i(X;\mathbb{Q})$ given by (7.1). Now, by definition of nat [3], we observe $(\mathrm{nat}\,\mathrm{Kron}\,[\varphi])[c] = (\mathrm{Kron}\,[\varphi])\,[c] = \varphi(c)$.

On the other hand we have $\mathrm{can}^i\,[\varphi] = [\mathrm{can}^i(\varphi)]$, where by definition $\mathrm{can}^i(\varphi)$ is the unique $\mathbb{Q}$-linear map $C_i(X;\mathbb{Q}) \to \mathbb{Q}$ such that $\mathrm{can}^i(\varphi) \circ (C_i(X) \otimes \mathbb{Q} \xrightarrow{\sim} C_i(X;\mathbb{Q}))$ corresponds to $(\mathbb{Z} \hookrightarrow \mathbb{Q}) \circ \varphi$ via tensor-hom adjunction. By definition of the Kronecker map we have $(\mathrm{Kron}\,\mathrm{can}^i\,[\varphi])[c] = (\mathrm{can}^i(\varphi))(c)$. Now $(\mathrm{can}^i(\varphi))(c) = \mathrm{can}^i(\varphi) \circ (C_i(X) \otimes \mathbb{Q} \xrightarrow{\sim} C_i(X;\mathbb{Q}))\ (c \otimes 1)$. By definition of the concrete isomorphism in the tensor-hom adjunction this is equal to $((\mathbb{Z} \hookrightarrow \mathbb{Q}) \circ \varphi)(c)$. Hence $(\mathrm{can}^i(\varphi))(c) = \varphi(c)$.

Altogether we have proven $(\mathrm{nat}\,\mathrm{Kron}\,[\varphi])[c] = \varphi(c) = (\mathrm{Kron}\,\mathrm{can}^i\,[\varphi])[c]$. □

A.3 Regarding Chapter 8

A.3.1 Schubert Calculus

Our aim is to prove the equalities in (8.11). Recall that we are working inside $G_2(\mathbb{C}^5)$, so $\sigma_\alpha = 0$ if $\alpha_1 > 5 - 2 = 3$. Applying Pieri multiple times yields

$$\begin{aligned}
\sigma_1^2 &= \sigma_{1,1} + \sigma_2 \\
\sigma_1^3 &= \sigma_{2,1} + (\sigma_3 + \sigma_{2,1}) = 2\,\sigma_{2,1} + \sigma_3 \\
\sigma_1^4 &= 2\,(\sigma_{3,1} + \sigma_{2,2}) + \sigma_{3,1} = 3\,\sigma_{3,1} + 2\,\sigma_{2,2} \\
\sigma_1^5 &= 3\,\sigma_{3,2} + 2\,\sigma_{3,2} = 5\,\sigma_{3,2} \\
\sigma_1^6 &= 5\ \ \sigma_{3,3} = 5
\end{aligned}$$

[1] Since c is nonzero for only finitely many singular simplices σ, we may write $c = \frac{1}{q}\tilde{c}$ with $q \in \mathbb{Z}$ and $\tilde{c} \in C_i(X;\mathbb{Q})$ having only integer coefficients. $\tilde{c}$ is also a cycle since $Z_i(X;\mathbb{Q})$ is a $\mathbb{Q}$-linear subspace of $C_i(X;\mathbb{Q})$. It suffices to prove $(\mathrm{nat}\,\mathrm{Kron}\,[\varphi])[\tilde{c}] = (\mathrm{Kron}\,\mathrm{can}^i\,[\varphi])[\tilde{c}]$. Then also $(\mathrm{nat}\,\mathrm{Kron}\,[\varphi])[c] = (\mathrm{Kron}\,\mathrm{can}^i\,[\varphi])[c]$, since both $(\mathrm{nat}\,\mathrm{Kron}\,[\varphi])$ and $(\mathrm{Kron}\,\mathrm{can}^i\,[\varphi])$ are $\mathbb{Q}$-linear.

[2] We informally write c when, strictly speaking, we mean the preimage of c in $Z_i(X;\mathbb{Z})$ under the map $C_i(X;\mathbb{Z}) \hookrightarrow C_i(X;\mathbb{Q})$.

[3] cf. diagram (7.3)

Combining the results from the first and third row also gives

$$\begin{aligned}\sigma_1^4 \cdot \sigma_{1,1} &= (3\,\sigma_{3,1} + 2\,\sigma_{2,2}) \cdot (\sigma_1^2 - \sigma_2)\\ &= (3\,\sigma_{3,1} + 2\,\sigma_{2,2}) \cdot \sigma_1^2 - (3\,\sigma_{3,1} + 2\,\sigma_{2,2}) \cdot \sigma_2\\ &= 3\,\sigma_{3,3} + 2\,\sigma_{3,3} - 3\,\sigma_{3,3} - 2 \cdot 0\\ &= 2\end{aligned}$$

Furthermore, we compute:

$$\begin{aligned}\sigma_{1,1}^2 &= \sigma_{1,1} \cdot (\sigma_1^2 - \sigma_2) = (\sigma_{3,1} + \sigma_{2,2}) - \sigma_{3,1} = \sigma_{2,2}\\ \sigma_1^2 \cdot \sigma_{1,1}^2 &= \sigma_1^2 \cdot \sigma_{2,2} = \sigma_1 \cdot \sigma_{3,2} = 1\\ \sigma_{1,1}^3 &= \sigma_{2,2} \cdot (\sigma_1^2 - \sigma_2) = \sigma_{3,2} \cdot \sigma_1 - 0 = 1\end{aligned}$$

A.3.2 Expressing the Polynomial as Polynomial in Elementary Symmetric Polynomials

Recall that $\sigma_1 = x_1 + x_2$, $\sigma_2 = x_1 x_2$ are the elementary symmetric polynomials in the variables x_1, x_2. We are given the polynomial $P \in \mathbb{Z}[x_1, x_2]$:

$$P := 16\,x_1^6 + 24\,x_1^5 x_2 + 40\,x_1^4 x_2^2 + 50\,x_1^3 x_2^3 + 40\,x_1^2 x_2^4 + 24\,x_1 x_2^5 + 16\,x_2^6 \quad .$$

This is a symmetric polynomial, thus there exists a unique polynomial $Q \in \mathbb{Z}[y_1, y_2]$ such that $P = Q(\sigma_1, \sigma_2)$. Since P is homogeneous of degree 6, it follows that Q is of the form

$$Q = \alpha_1\, y_1^6 + \alpha_2\, y_1^4 y_2 + \alpha_3\, y_1^2 y_2^2 + \alpha_4\, y_2^3$$

for uniquely determined $\alpha_1, \ldots, \alpha_4 \in \mathbb{Z}$. Clearly $\sigma_2^n = x_1^n x_2^n$ and by the binomial theorem $\sigma_1^n = \sum_{k=0}^n \binom{n}{k} x_1^{n-k} x_2^k$ for all $n \geq 1$. Hence

$$\begin{aligned}Q(\sigma_1, \sigma_2) = {}&\alpha_1\, x_1^6 + \left(\alpha_1 \binom{6}{1} + \alpha_2\right) x_1^5 x_2 + \left(\alpha_1 \binom{6}{2} + \alpha_2 \binom{4}{1} + \alpha_3\right) x_1^4 x_2^2\\ &+ \left(\alpha_1 \binom{6}{3} + \alpha_2 \binom{4}{2} + \alpha_3 \binom{2}{1} + \alpha_4\right) x_1^3 x_2^3\\ &+ \left(\alpha_1 \binom{6}{4} + \alpha_2 \binom{4}{3} + \alpha_3\right) x_1^2 x_2^4 + \left(\alpha_1 \binom{6}{5} + \alpha_2\right) x_1 x_2^5 + \alpha_1\, x_2^6\end{aligned}$$

Comparing coefficients yields

$$\begin{aligned}16 &= \alpha_1\\ 24 &= \left(\alpha_1 \binom{6}{1} + \alpha_2\right) = 6\,\alpha_1 + \alpha_2\\ 40 &= \left(\alpha_1 \binom{6}{2} + \alpha_2 \binom{4}{1} + \alpha_3\right) = 15\,\alpha_1 + 4\,\alpha_2 + \alpha_3\\ 50 &= \left(\alpha_1 \binom{6}{3} + \alpha_2 \binom{4}{2} + \alpha_3 \binom{2}{1} + \alpha_4\right) = 20\,\alpha_1 + 6\,\alpha_2 + 2\,\alpha_3 + \alpha_4\end{aligned}$$

Solving this system of linear equations (in lower triangular form) we obtain

$$\alpha_1 = 16 \qquad \alpha_2 = 24 - 6\,\alpha_1 = -72$$
$$\alpha_3 = 40 - 15\,\alpha_1 - 4\,\alpha_2 = 88 \qquad \alpha_4 = 50 - 20\,\alpha_1 - 6\,\alpha_2 - 2\,\alpha_3 = -14$$

which proves our claim in Section 8.2.

A.3.3 Proof of Lemma 48

Here we present a proof for Lemma 48 in Subsection 8.3.1.

Let $S(x_1,\ldots,x_k) := S\{x_1,\ldots,x_k\} \cap \mathbb{Z}[x_1,\ldots,x_k]$ denote the subring of symmetric polynomials in the $x_1,\ldots,x_k$ and $S(x_1,\ldots,x_k)_i$ the subgroup of homogeneous symmetric polynomials in the $x_1,\ldots,x_k$ of degree i. Put $c_i := c_i(\gamma^k(\mathbb{C}^n))$. By Fact 41, $H^{2i}(G)$ is generated by monomials of the form $c_1^{i_1} \cdot \ldots \cdot c_k^{i_k}$ with $i_1 + 2\,i_2 + \ldots + k\,i_k = i$ and thus, by definition of Φ, Φ maps $H^{2i}(G)$ to $(S(x_1,\ldots,x_k)_i + I)\,/I$. Hence we obtain a commutative diagram

$$\begin{tikzcd} H^*(G) \arrow[rr, "\cong", "\Phi"'] \arrow[d, equal] & & S/I \\ \bigoplus_{i=0}^{\dim(G)} H^{2i}(G) \arrow[rr, "\bigoplus_i \Phi"] & & \bigoplus_{i=0}^{\dim(G)} (S(x_1,\ldots,x_k)_i + I)\,/I \arrow[u] \end{tikzcd}$$

and, since Φ is an isomorphism, the vertical arrow is surjective[4]. Now, given $s \in S$ with $s_i = 0$ for all $i \le \dim(G)$, we write

$$s + I = \sum_{j=0}^{\dim(G)} (t_j + I)$$

with $t_j \in S(x_1,\ldots,x_k)_j$. Hence $\sum_j t_j - s \in I$ and $t_i = \left(\sum_j t_j - s\right)_i$ for all $i \le \dim(G)$. We now have the following lemma, which will be proved below:

Lemma 70. *For arbitrary $t \in S$: $t \in I \implies t_j \in I \quad \forall\, j \ge 0\,.$*

By the lemma we conclude that $t_i \in I$ for $i \le \dim(G)$ because $\sum_j t_j - s$ lies in I. Thus $s + I = \sum_j (t_j + I) = 0$ and $s \in I$.

Proof of Lemma 70. $I \subseteq S$ is the ideal generated by the symmetric power series in the $x_1,\ldots,x_n$ without constant term. Thus, for $t \in I$ we may write

$$t = \sum_{\substack{i \\ \text{finite}}} h_i\, r_i$$

with $h_i \in S$ and symmetric power series r_i in the variables $x_1,\ldots,x_n$ without constant term. As r_i is a symmetric power series, for arbitrary l its degree-l-term $(r_i)_l$ is symmetric as well and hence $(r_i)_l \in I$ for all l. Now the degree-j-term of t is given by

$$t_j = \sum_{\substack{i \\ \text{finite}}} (h_i\, r_i)_j = \sum_{\substack{i \\ \text{finite}}} \sum_{l=0}^{j} \underbrace{(h_i)_l}_{\in S}\, \underbrace{(r_i)_{j-l}}_{\in I} \in I \quad .$$

[4]Actually it is an isomorphism as well since the $\Phi : H^{2i}(G) \to (S(x_1,\ldots,x_k)_i + I)\,/I$ all are isomorphisms, as $S(x_1,\ldots,x_k)_i$ is generated by monomials of the form $e_1^{i_1} \cdot \ldots \cdot e_k^{i_k}$ with $i_1 + 2\,i_2 + \ldots + k\,i_k = i$.

$\square$

A.4 Verification of the CSM Class Formula in Section 9.2

Here we prove the CSM class formula (9.1) we claimed in Section 9.2. The situation is as follows: $\iota : G = G_k(\mathbb{C}^n) \hookrightarrow G' = G_{k+l}(\mathbb{C}^{n+m})$ is the canonical embedding from Lemma 15 and $a \in \mathcal{P}(n-k,k)$ is an arbitrary fixed partition. The homeomorphism $\iota| : X_a \xrightarrow{\sim} X_{a,\underline{0}}$ induces an isomorphism $\iota|_* : H_*(X_a\,;\mathbb{Z}) \xrightarrow{\sim} H_*(X_{a,\underline{0}}\,;\mathbb{Z})$ on homology. The formula we need to prove is

$$\iota|_*\, c_{\mathrm{SM}}(X_a) = c_{\mathrm{SM}}(X_{a,\underline{0}}) \quad .$$

In the following we only work with the definition of the CSM class respectively its defining properties, presented in the original paper by MacPherson [Mac74] [5]. Let us briefly recall them. For that we adopt MacPherson's notation.

The covariant functor $\mathbf{F}$ from compact complex algebraic varieties to abelian groups assigns to such a variety V the group of constructible functions $V \to \mathbb{Z}$ [6]. $\mathbf{F}$ maps a morphism $f : V \to V'$ to $\mathbf{F}(f) = f_*$, defined by

$$f_*(1_W)(p) = \chi(f^{-1}(p) \cap W)$$

for $W \subseteq V$ a subvariety and $p \in V'$. Here 1_W denotes the constructible function that is 1 on W and 0 elsewhere and χ is the usual Euler characteristic. The CSM class c_{SM} is the unique natural transformation $c_* : \mathbf{F} \to H_*(-\,;\mathbb{Z})$ with the property that for every nonsingular variety V the image of 1_V under c_* [7] is Poincaré dual to the total Chern class of V, i.e. $c_*(1_V) = c(V) \frown [V]$. Then the CSM class $c_{\mathrm{SM}}(V)$ of any compact complex algebraic variety V is defined to be $c_{\mathrm{SM}}(1_V)$.

Now let us return to our setting. By definition of the CSM class of a variety and since c_{SM} is a natural transformation, we have

$$\iota|_*\, c_{\mathrm{SM}}(X_a) = \iota|_*\, c_{\mathrm{SM}}(1_{X_a}) = c_{\mathrm{SM}}\, \iota|_*(1_{X_a}) \in H_*(X_{a,\underline{0}}\,;\mathbb{Z}) \quad .$$

To prove our claim, it thus suffices to show that $\iota|_*(1_{X_a}) = 1_{X_{a,\underline{0}}}$ as constructible functions. Let $p \in X_{a,\underline{0}}$ be arbitrary. By definition it holds

$$\iota|_*(1_{X_a})(p) = \chi(\iota|^{-1}(p) \cap X_a) = \chi(\mathrm{pt.}) = 1 = 1_{X_{a,\underline{0}}}(p) \quad ,$$

which finishes the argument. As announced in Section 9.2, our claim is merely a direct consequence of the definition of the CSM class. Instead of working in our concrete setting, we also could have shown the more general statement that for any bijective morphism of varieties $f : V \to V'$ the induced map on homology f_* sends $c_{\mathrm{SM}}(V)$ to $c_{\mathrm{SM}}(V')$. Obviously, the proof is completely analogous.

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[5] All the necessary definitions and statements are given on just the first two pages of [Mac74], most of them in Proposition 1 and Theorem 1.

[6] A *constructible set* in V is one obtained from the subvarieties by finitely many of the usual set-theoretic operations. A *constructible function* is one for which there is a finite partition of the variety V into constructible subsets such that the function is constant on each of those sets.

[7] more precisely: $(c_*)_V$

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Eidesstattliche Erklärung

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