A polyfold perspective on APS operator families and topology-changing domains

Tobias Witt

Master's Thesis in Mathematics
under the supervision of Prof. Dr. Peter Albers

November 2022

Ruprecht Karl University of Heidelberg
Faculty of Mathematics and Computer Science
Mathematical Institute

Abstract

While proving the sc-Fredholm property of APS-type operators $D_A = \frac{\mathrm{d}}{\mathrm{d}t} - A(t)$ on both unweighted and weighted Floer path spaces $\mathcal{W}_n = \bigcap_{k+r=n} W^{r,2}(\mathbb{R}, W_k)$ and $(\mathcal{W}^{\delta_n}_{n+k})_{n\geq 0}$, we argue that the latter case requires a bound on the weight sequence $0 = \delta_0 < \delta_1 < \ldots$ whose value δ_∞ can be calculated in terms of the operator family $A(t)$. Moreover, in an attempt to replace the classical Floer cylinder $\Sigma = \mathbb{R} \times S^1$ by a pair-of-pants worldsheet with topology-changing level sets, we prove the sc-smoothness of a retraction $r_\Sigma : (\text{-}\epsilon, \epsilon) \oplus W_n^{\oplus 2} \longrightarrow W_n^{\oplus 2}$ that interpolates between topologically distinct fibres $r_{t<0}(W_n^{\oplus 2}) \cong W^{n+1,2}(S^1)$ and $r_{t>0}(W_n^{\oplus 2}) \cong W^{n+1,2}(S^1) \oplus W^{n+1,2}(S^1)$, leaving it for future investigation to interpret the Cauchy-Riemann operator $\partial_{\bar{z}} \sim \partial_t + i\partial_\omega$ as calculating the flow of a sc-smooth vector field $A(t) \sim i\partial_\omega$ on the M-polyfold $\mathrm{im}(r_\Sigma)$.

Kurzzusammenfassung

In dieser Arbeit beweisen wir die sc-Fredholm-Eigenschaft von APS-Operatoren $D_A = \frac{\mathrm{d}}{\mathrm{d}t} - A(t)$ auf ungewichteten und gewichteten Pfadräumen $\mathcal{W}_n = \bigcap_{k+r=n} W^{r,2}(\mathbb{R}, W_k)$ und $(\mathcal{W}^{\delta_n}_{n+k})_{n\geq 0}$. In letzterem Fall muss die Gewichtsfolge $0 = \delta_0 < \delta_1 < \ldots$ eine obere Schranke besitzen, deren Wert wir in Abhängigkeit von der Operatorfamilie $A(t)$ eingrenzen können. Des Weiteren unternehmen wir den Versuch den klassischen Floer-Zylinder $\Sigma = \mathbb{R} \times S^1$ durch eine 'pair-of-pants'-Weltfläche zu ersetzen und konstruieren dazu eine sc-glatte 'retraction' $r_\Sigma : (\text{-}\epsilon, \epsilon) \oplus W_n^{\oplus 2} \longrightarrow W_n^{\oplus 2}$, die zwischen topologisch distinkten Fasern $r_{t<0}(W_n^{\oplus 2}) \cong W^{n+1,2}(S^1)$ und $r_{t>0}(W_n^{\oplus 2}) \cong W^{n+1,2}(S^1) \oplus W^{n+1,2}(S^1)$ interpoliert. Wir überlassen es zukünftigen Untersuchungen zu klären, inwieweit sich der Cauchy-Riemann-Operator $\partial_{\bar{z}} \sim \partial_t + i\partial_\omega$ durch den Fluss eines sc-glatten Vektorfelds $A(t) \sim i\partial_\omega$ auf der M-Polyfold $\mathrm{im}(r_\Sigma)$ beschreiben lässt.

Contents

Chapter 1

Introduction

1.1 Motivation

Before describing our precise results in sections 1.2 and 1.3, let us outline the basic dictionary that defines the philosophy behind our approach.

- In studying maps $u : \Sigma \longrightarrow \mathbb{H}$ from a 2-dimensional worldsheet Σ to an in our case linear target space $\mathbb{H}$, we will fix a Morse function ν on Σ and regard the time slices
$$u(t) = u|_{\nu^{-1}(t)} : \Sigma_t \longrightarrow \mathbb{H}$$
as points in an M-polyfold $\mathcal{M}_\Sigma$.

- For the classical Floer cylinder $\Sigma = \mathbb{R}\times S^1$ this M-polyfold will be given as $\mathcal{M}_\Sigma = \mathbb{R} \oplus W$, with a constant fibre
$$W = \left[L^2(S^1, \mathbb{H}) \supset W^{1,2}(S^1, \mathbb{H}) \supset \dots\right]$$
hosting maps $S^1 \longrightarrow \mathbb{H}$ of all regularities.

- Hence, we may reinterpret a given map $u : \mathbb{R}\times S^1 \longrightarrow \mathbb{H}$ as a path $\mathbb{R} \longrightarrow W$ through the sc-Banach space $W = \left[W^{k,2}(S^1, \mathbb{H})\right]_{k\geq 0}$, the 'regularity' of such a path being defined by its place in the filtration $\mathcal{H} \supset \mathcal{W}_1 \supset \dots$ of nested Sobolev spaces
$$\mathcal{W}_n = \bigcap_{k+r=n} W^{r,2}(\mathbb{R}, W_k)$$

- Operator families $A(t) : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$, on the other hand, can be understood as time-dependent vector fields on W, whose flow lines are the solutions to
$$[D_A u](t) = \left[\frac{\mathrm{d}}{\mathrm{d}t} - A(t)\right] u(t) = 0$$
Since $D_A : L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, W_0) \longrightarrow L^2(\mathbb{R}, W_0)$ extends to a regularizing sc-operator on the filtration $(\mathcal{W}_n)_{n\geq 0}$, only flow lines of the highest regularity $\mathcal{W}_\infty \subset \bigcap_{k\geq 0} C^\infty(\mathbb{R}, W_k)$ will be allowed.

- In the case of a pair-of-pants worldsheet $\Sigma = \mathbb{C}P^1 \setminus \{\pm 1, \infty\}$ with Morse function ν, the topology-changing level sets $\nu^{-1}(t)$ will be represented by the topologically distinct fibres
$$r_{t>0}(M_n) \cong W^{n+1,2}(S^1) \oplus W^{n+1,2}(S^1) \quad \text{and} \quad r_{t<0}(M_n) \cong W^{n+1,2}(S^1)$$
of a sc-smooth splicing $r : \mathbb{R} \oplus M \longrightarrow M$. Thus, it should be possible to study the solutions to a PDE on Σ as those flow lines $u : \mathbb{R} \longrightarrow M$ through the ambient fibre M which stay within a time-dependent "constraint surface" $r_t(M) \subset M$.

1.2 Summary of results (Part I)

In the paper [RS], Robbin and Salamon prove the classical Fredholm property of operators

$$D_A = \tfrac{\mathrm{d}}{\mathrm{d}t} - A : \; L^2(\mathbb{R}, W) \cap W^{1,2}(\mathbb{R}, H) \longrightarrow L^2(\mathbb{R}, H)$$

where, among other things, $A(t) : W \longrightarrow H$ is a family of self-adjoint operators on a Hilbert space H. Part I of this thesis studies the implications of replacing the pair $W \subset H$ by a sc-Banach space

$$\ldots \subset W_k \subset \ldots \subset W_1 \subset W_0 = H$$

in the sense of [HWZ21], where as before the norm $\|\cdot\|_{W_0}$ is assumed to arise from a Hilbert space structure on H (in which case we also call the filtration $(W_k)_{k\geq 0}$ a "sc-Hilbert space").

Our findings, as well as their relation to pre-existing work, can be summarized as follows:

$\boxed{\text{Chapter 2}}$

Motivated by Floer theory, we assume our operator family to decompose as

$$A(t) = A_0 + B(t)$$

where, in addition to $B(t) \in \mathcal{L}(H)$ and $A_0 : W_1 \longrightarrow H$ being self-adjoint, $A_0 : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ is a regularizing sc-operator on the Banach scale.

Prop. 2.18

Now that a "baseline operator" A_0 has been singled out, we can regard not only W_0 but also the higher levels W_k as Hilbert spaces with inner product

$$\langle v, w\rangle_{W_k} := \langle v, w\rangle_{W_{k-1}} + \langle A_0 v, A_0 w\rangle_{W_{k-1}} ,$$

thereby maintaining the property that $A_0 : W_{k+1} \longrightarrow W_k$ is self-adjoint as an unbounded operator on $\big(W_k, \langle\cdot,\cdot\rangle_{W_k}\big)$.

Lem. 2.20

Hence, at every $k \geq 0$ we can import the "vertical regularization property" $D_{A_0}^{-1}\big(W^{r,2}(\mathbb{R}, W_k)\big) \subset W^{r,2}(\mathbb{R}, W_{k+1}) \cap W^{r+1,2}(\mathbb{R}, W_k)$ from [RS] Thm. 3.13.

Lem. 2.19

Moreover, as a built-in feature of our baseline operator A_0, we have an additional "horizontal regularization property"

$$A_0^{-1}\big(L^2(\mathbb{R}, W_k)\big) = L^2(\mathbb{R}, W_{k+1}).$$

Note, however, that we cannot get along without a perturbation $B(t) \in \mathcal{L}(H)$ as the classical Fredholm property of $D_A : L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H) \longrightarrow L^2(\mathbb{R}, H)$ requires our family $A(t)$ to approach invertible endpoints $A_\pm \in \mathcal{L}(W_1, H)$ at $t = \pm\infty$.

Def. 2.21
+2.23

In our setting, we will have to impose the additional property of $B(t) \in \mathcal{L}(H)$ being a "moderate family of sc-operators" on $(W_k)_{k\geq 0}$ to ensure that, while operating on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}, H)$,

Cor. 2.29

B preserves the bifiltration $W_k^r := W^{r,2}(\mathbb{R}, W_k)$ in the sense that

$$B(W_k^r) \subset W_k^r \quad \text{for all } r, k \geq 0.$$

Cor. 2.34

This leads to $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ being a sc-operator on the filtration given by diagonals $\mathcal{W}_n := \bigcap_{k+r=n} W_k^r$, with the lowest orders reading

$$\mathcal{H} = L^2(\mathbb{R}, H), \quad \mathcal{W}_1 = L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H), \quad \mathcal{W}_2 = L^2(\mathbb{R}, W_2) \cap W^{1,2}(\mathbb{R}, W_1) \cap W^{2,2}(\mathbb{R}, H).$$

Prop. 2.35
+ 2.36

As illustrated in Fig. 2.1, the vertical and horizontal regularization properties can be combined to prove that first $D_{A_0} : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ and with B being a moderate perturbation also $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ is a regularizing sc-operator in the sense that $D_A^{-1}(\mathcal{W}_n) = \mathcal{W}_{n+1}$.

This seemingly innocent observation in fact opens the floodgates with most of our subsequent findings building on this result.

Before further elaborating on this, let us remark that all assumptions (A_0 being a baseline operator, B being a good perturbation) are invariant under the replacement $A \to -A$, so everything that has been proven about D_A will be valid for $D_{\text{-}A}$ as well.

Although this has not been emphasized by Robbin and Salamon, [RS] Thm. 3.10 ('Elliptic Regularity') can be reinterpreted

Lem. 2.37 as saying that $\text{-}D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ and $D_{\text{-}A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$ are mutually adjoint as unbounded operators on $\mathcal{H} = L^2(\mathbb{R}, H)$.

Thm. 2.38 + 2.40 Given such a pair of mutually adjoint Fredholm operators $\varphi_\pm = \mp D_{\pm A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$, now with the bonus property that $\varphi_\pm : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ are regularizing sc-operators, we will arrive at our first main result that the operator

$$-\varphi_+ = D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$$

(and therefore similarly $\varphi_- = D_{\text{-}A}$) is sc-Fredholm.

The more general fact that for a regularizing sc-operator to be sc-Fredholm it is enough to be classically Fredholm at the lowest level, has already been pointed out by Wehrheim (see [We] Def. 3.1 and [We] Lem. 3.6). However, whereas the proof given in [We] traces back to abstract Hahn-Banach-style arguments about the existence of complementary subspaces, our more restrictive setup allows completely explicit decompositions of the spaces $\mathcal{W}_n$ as summarized by the "double helix"

$$\begin{tikzcd}
\ker D_A \oplus D_{\text{-}A}(\mathcal{W}_1) \arrow[r, phantom, "="] & \mathcal{H} \arrow[r, phantom, "="] & D_A(\mathcal{W}_1) \oplus \ker D_{\text{-}A} \\
\ker D_A \oplus D_{\text{-}A}(\mathcal{W}_2) \arrow[urr, "D_A"'] \arrow[r, phantom, "="] & \mathcal{W}_1 \arrow[r, phantom, "="] & D_A(\mathcal{W}_2) \oplus \ker D_{\text{-}A} \arrow[ull, red, "D_{\text{-}A}"] \\
\ker D_A \oplus D_{\text{-}A}(\mathcal{W}_3) \arrow[urr, "D_A"'] \arrow[r, phantom, "="] & \mathcal{W}_2 \arrow[r, phantom, "="] & D_A(\mathcal{W}_3) \oplus \ker D_{\text{-}A} \arrow[ull, red, "D_{\text{-}A}"] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

It is this very picture that will come to our rescue in the proof of Theorem 4.15.

Cor. 2.41 As a by-product, the sc-Fredholm property guarantees that $D_{\pm A} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ are classical Fredholm operators at every level $n \geq 0$.

Chapter 3 This makes the composition

$$S := D_A D_{\text{-}A} : \mathcal{W}_2 \longrightarrow \mathcal{H},$$

Thm. 3.4 which is constructed to be of type "$-T^*T$", a non-positive, self-adjoint Fredholm operator.

Lem. 3.6, Cor. 3.7 The spectrum of such operators admits a gap $\epsilon > 0$ so that $\sigma(S) \subset (\text{-}\infty, \text{-}\epsilon] \cup \{0\}$.

As a result, the resolvent map $R_\bullet(S) : \rho(S) \longrightarrow \mathcal{L}(\mathcal{H}, \mathcal{W}_2)$ features a Laurent expansion

Lem. 3.14+ Prop. 3.15

$$R_\lambda(S) = -\tfrac{P}{\lambda} + Q[\mathrm{id}_\mathcal{H} - \lambda Q]^{-1}, \quad \lambda \in B_\epsilon(0) \setminus \{0\}$$

whose coefficients

$$P = -\frac{1}{2\pi i} \int_\circlearrowleft \mathrm{d}\lambda\, [S - \lambda]^{-1} \quad \text{and} \quad Q = \frac{1}{2\pi i} \int_\circlearrowleft \frac{\mathrm{d}\lambda}{\lambda} [S - \lambda]^{-1}$$

apart from being represented as $\mathcal{L}(\mathcal{H}, \mathcal{W}_2)$-valued contour integrals around the origin, admit the following interpretation:

- $P\colon \mathcal{H} = D_A(\mathcal{W}_1) \oplus \ker D_{\text{-}A} \longrightarrow \ker D_{\text{-}A}$ is the orthogonal projector onto $\ker D_{\text{-}A}$
- Q is a parametrix (quasi-inverse) for S in the sense that
$$SQ = \mathrm{id}_{\mathcal{H}} - P \quad \text{and} \quad QS = \mathrm{id}_{\mathcal{W}_2} - P$$

This interpretation relies on the absence of higher poles "λ^{-k}"
which is guaranteed by the " $\leq$ "-direction of the formula

Prop. 3.11
$$\big\| R_\lambda(S) \big\|_{\mathcal{L}(\mathcal{H})} = \mathrm{dist}\big(\lambda, \sigma(S)\big)^{-1}$$
being valid for self-adjoint operators S.

Now that we have studied the properties of D_A as an operator on the unweighted filtration $\mathcal{W}_n = L^2(\mathbb{R}, W_n) \cap W^{1,2}(\mathbb{R}, W_{n-1}) \cap \ldots \cap W^{n,2}(\mathbb{R}, H)$,
note that $(\mathcal{W}_n)_{n\geq 0}$ fails to be an honest sc-Banach space,
in the sense required by polyfold theory [HWZ21],
since a simple "bump escape argument" (see Lemma 2.44) shows that
unboundedness of the domain $\mathbb{R}$ prevents $L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H) \hookrightarrow L^2(\mathbb{R}, H)$
from being a compact inclusion.

Chapter 4

As explained for example in the paper [FW] by Frauenfelder and Weber,
this issue can be remedied by introducing a weight sequence $0 = \delta_0 < \delta_1 < \ldots$

In a dual approach to [FW], we consider the inverted weight factors $\gamma_{\text{-}\delta_i} = e^{-\delta_i \eta(t)}$
(with $\eta \in C^\infty(\mathbb{R})$ satisfying $\eta(t) = |t|$ for $|t| \geq 1$)
as a sequence of injective sc-operators $\gamma_{-\delta_i} : (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$
and use the identification
$$\gamma_{\text{-}\delta} : \mathcal{W}_n \xrightarrow{\sim} \gamma_{\text{-}\delta}(\mathcal{W}_n) \subset \mathcal{W}_n$$
to make $\mathcal{W}_n^\delta := \gamma_{\text{-}\delta}(\mathcal{W}_n)$ a Banach space with norm $\big\|\gamma_{+\delta}(\cdot)\big\|_{\mathcal{W}_n}$.

Lem. 4.1 + 4.3

Since one can rephrase results from the paper [FW] as saying that
$\gamma_{\text{-}\Delta\delta_i}\colon \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ with $\Delta\delta_i = \delta_{i+1} - \delta_i > 0$ is a compact operator,
the commutative triangle
$$\begin{tikzcd}
\mathcal{W}_{n+1} \arrow[rr, "\gamma_{\text{-}\Delta\delta_i}"] \arrow[dr, "\gamma_{\text{-}\delta_{i+1}}"'] & & \mathcal{W}_n \arrow[dl, "\gamma_{\text{-}\delta_i}"] \\
& \mathcal{H} &
\end{tikzcd}$$

Rem. 4.7

translates into a compact inclusion $\mathcal{W}_{n+1}^{\delta_{i+1}} \hookrightarrow \mathcal{W}_n^{\delta_i}$.

Thus, our "twisting sequence" $\gamma_{\text{-}\delta_i}\colon (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ produces a bifiltration
$$\begin{tikzcd}
\cdots\ \mathcal{W}_3 \arrow[r, phantom, "\subset"] & \mathcal{W}_2 \arrow[r, phantom, "\subset"] & \mathcal{W}_1 \arrow[r, phantom, "\subset"] & \mathcal{H} \\
\cdots\ \mathcal{W}_3^{\delta_1} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, "\text{compact}"] & \mathcal{W}_2^{\delta_1} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, "\text{compact}"] & \mathcal{W}_1^{\delta_1} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, "\text{compact}"] & \mathcal{H}^{\delta_1} \arrow[u, phantom, "\cup"] \\
\cdots\ \mathcal{W}_3^{\delta_2} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, "\text{compact}"] & \mathcal{W}_2^{\delta_2} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, "\text{compact}"] & \mathcal{W}_1^{\delta_2} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] \arrow[ur, "\text{compact}"] & \mathcal{H}^{\delta_2} \arrow[u, phantom, "\cup"] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$
with compact diagonals indicated in red, leaving us with

Lem. 4.8

a k-family of honest sc-Banach spaces $(\mathcal{W}_{n+k}^{\delta_n})_{n\geq 0}$.

Rem. 4.5 + **Prop. 2.42**

The density property of these sc-Banach spaces is guaranteed by
the set $\mathcal{S} := C_0^\infty(\mathbb{R}) W_\infty$ being dense in every $\mathcal{W}_n$
and at the same time invariant under $\gamma_{\text{-}\delta}$ in the sense that $\gamma_{\text{-}\delta}(\mathcal{S}) = \mathcal{S}$.

Now we are ready to state our main result:

Once our weight sequence $0 = \delta_0 < \delta_1 < \ldots$ is bounded by a suitable δ_∞, whose value depends on the invertible endpoints $A_\pm \in \mathcal{L}(W_1, H)$ of our family $A(t)$ as well as the spectral gap of $S = D_A D_{\text{-}A} : \mathcal{W}_2 \longrightarrow \mathcal{H}$,

Thm. 4.17

$$D_A : (\mathcal{W}_{n+1}^{\delta_n})_{n\geq 0} \longrightarrow (\mathcal{W}_n^{\delta_n})_{n\geq 0}$$

will be a sc-Fredholm operator between honest sc-Banach spaces.

As before, the key ingredient consists in

Prop. 4.13 $D_A\colon (\mathcal{W}_{n+1}^{\delta_n})_{n\geq 0} \longrightarrow (\mathcal{W}_n^{\delta_n})_{n\geq 0}$ being a regularizing sc-operator, which itself is the combination of two separate properties:

Lem. 4.14 + 4.11 First, one easily verifies that D_A is "strongly twistable", which guarantees that

$$D_A : (\mathcal{W}_{n+1}^{\delta})_{n\geq 0} \longrightarrow (\mathcal{W}_n^{\delta})_{n\geq 0}$$

is a regularizing sc-operator for arbitrary but fixed $\delta > 0$.

Note, however, that working with an increasing weight sequence $0 = \delta_0 < \delta_1 < \ldots$ forces us to vary the size of δ as we proceed to higher levels of regularity.

In order to adapt to these changes of δ, D_A has to be "twist-regularizing", which means that at least for $0 < \delta < \delta_\infty$ with a suitable upper bound δ_∞ the unweighted operator $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ needs to satisfy

Thm. 4.15

$$D_A^{-1}(\mathcal{H}^\delta) \subset \mathcal{W}_1^\delta$$

According to the decomposition $\mathcal{W}_1 = D_{\text{-}A}(\mathcal{W}_2) \oplus \ker D_A$, this property requires fundamentally different proofs in the cases $w \in D_{\text{-}A}(\mathcal{W}_2)$ and $v \in \ker D_A$, making Theorem 4.15 perhaps the most interesting result of this thesis:

- Building on the techniques from Chapter 3, Step 1 uses spectral perturbation theory to characterize the restricted parametrix $Q|_{\mathcal{H}^\delta}$
- Step 2, on the other hand, is a spiced-up version of ideas from [Sa] Lem. 2.11 and [RS] Prop. 3.14 that when combined can be used to prove exponential decay of solutions to $D_A v = 0$, now however with some extra complexity coming from the fact that working in $\mathcal{W}_1$ requires us to control not only $v \in L^2(\mathbb{R}, H)$ but in fact $v \in L^2(\mathbb{R}, W_1)$ as well as its derivative $\dot{v} \in L^2(\mathbb{R}, H)$.
  Our calculation leads to an inequality involving $\|\dot{B}(t)\|_{\mathcal{L}(H)}$ and $\|\ddot{B}(t)\|_{\mathcal{L}(H)}$ so for Theorem 4.15 we have to work with "localized" (or "asymptotically constant") perturbations that satisfy
  $$\lim_{t\to\pm\infty} \|B^{(l)}(t)\|_{\mathcal{L}(H)} = 0 \quad \text{for } l = 1, 2.$$

$\boxed{\text{Chapter 5}}$ The final chapter of Part I is devoted to criteria for restrictions of "$J_0\partial_s : W^{1,2}(I, \mathbb{H}) \longrightarrow L^2(I, \mathbb{H})$" to define an actual baseline operator in the sense used before. More specifically, we take $I = (0, 1)$ to be the standard interval and assume that $\mathbb{H}$ is a (possibly infinite-dimensional) real Hilbert space with complex structure $J_0 \in \mathcal{L}(\mathbb{H})$.
Then $\omega = \langle J_0 \cdot, \cdot \rangle_{\mathbb{H}}$ and $\Omega = (-\omega) \oplus \omega$ serve as symplectic forms on $\mathbb{H}$ and $\mathbb{H} \oplus \mathbb{H}$, respectively.

	In writing $u \in W^{1,2}_\Lambda(I,\mathbb{H})$, we require the endpoints $\big(u(0),u(1)\big)$ to be contained in a prescribed subspace $\Lambda \subset \mathbb{H}\oplus\mathbb{H}$ and it turns out that taking the adjoint of
Prop. 5.3	$$J_0\partial_s : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$$ amounts to replacing Λ by its Ω-orthogonal complement Λ^Ω.
Cor. 5.5	As a result, the self-adjoint restrictions of $J_0\partial_s : W^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ correspond to Lagrangian subspaces $\Lambda = \Lambda^\Omega$.
	With this understood, note that there is a unique way to define the higher levels $W^{n+1,2}_\Lambda(I,\mathbb{H})$ once we require $J_0\partial_s : \big(W^{n+1,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0} \longrightarrow \big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$
Lem. 5.6	to be a regularizing sc-operator.
	Interestingly, this generalizes the Lagrangian boundary conditions considered in [FW] section 7.
	With $I=(0,1)$ being a bounded interval,
Prop. 5.8	the density property of our Banach scale $\big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$ seems far from obvious, but is in fact guaranteed by the mere presence of a baseline operator.
	This suggests the conclusion that a baseline operator generates (rather than just lives on) the Banach scale.
	Now that we have described an explicit baseline operator, it remains to explain which maps $\Gamma : \mathbb{R}_t \times I \longrightarrow \mathcal{L}(\mathbb{H})$ represent good perturbations $B(t) \in \mathcal{L}(H)$.
	For moderate perturbations this boils down to the requirement
Prop. 5.20	$$\sup_{(t,x)\in\mathbb{R}\times I} \big\Vert \partial_t^l \partial_x^k \Gamma \big\Vert_{\mathcal{L}(\mathbb{H})} < \infty \qquad \text{for all } l,k \geq 0$$ which in "sigma model situations" $\Gamma : \mathbb{R}\times I \xrightarrow{\Phi} \mathbb{H} \xrightarrow{F} \mathcal{L}(\mathbb{H})$ reduces to a condition
Prop. 5.23	$$\sup_{(t,x)\in\mathbb{R}\times I} \big\Vert \partial_t^l \partial_x^k \Phi \big\Vert_{\mathbb{H}} < \infty$$ on the base curve $\Phi : \mathbb{R}\times I \longrightarrow \mathbb{H}$ that we study perturbations $D_A\delta\Phi = 0$ around.
	We expect this condition to be guaranteed by a-priori estimates similar to [Sa] Prop. 1.21.

1.3 Summary of results (Part II)

Considering maps $\Sigma \longrightarrow X$ from a cylinder worldsheet $\Sigma = \mathbb{R} \times S^1$ to a linear target space $X = \mathbb{H}$ meant that in Part I we were dealing with constant time slices S^1.
A map $u_t : S^1 \longrightarrow \mathbb{H}$ was taken to be a point in $L^2(S^1, \mathbb{H})$, with information about the regularity of u_t determining its place in the filtration $L^2(S^1, \mathbb{H}) \supset W^{1,2}(S^1; \mathbb{H}) \supset ....$
As a result, maps $u : \mathbb{R} \times S^1 \longrightarrow \mathbb{H}$ were interpreted as paths $u : \mathbb{R} \longrightarrow (W_k)_{k \geq 0}$ through an sc-Banach space $W_k = W^{k,2}(S^1, \mathbb{H})$, with the regularity of such a path being determined by its place in the filtration $\mathcal{H} \supset \mathcal{W}_1 \supset .....$ where

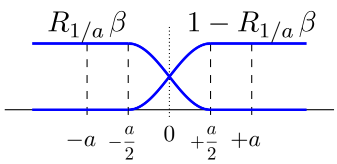
$$\mathcal{W}_n = \bigcap_{k+r=n} W^{r,2}\big(\mathbb{R}, W^{k,2}(S^1, \mathbb{H})\big)$$

In the more experimental Part II, we explore the possibility of replacing $\Sigma = \mathbb{R} \times S^1$ by a Riemann surface with topology-changing level sets. Our strategy is to fix a Morse function ν on Σ and identify its value with the gluing parameter $a \in B$ of a sc-smooth splicing $r : B \oplus M \longrightarrow M$. Levelwise maps $u_a : \nu^{-1}(a) \longrightarrow \mathbb{H}$ will now be points $u_a \in r_a(M)$ and maps $u : \Sigma \longrightarrow \mathbb{H}$ can be interpreted as paths $u : \mathbb{R} \longrightarrow \mathrm{im}(r) \subset M$ through an M-polyfold. Note that just as in the topology-preserving case, the ambient sc-Banach space M serves as a constant target through which such a path can travel, with information about the worldsheet topology now repackaged into a constraint "$u_a \in r_a(M)\ \forall a$".

Working with the example of a pair-of-pants worldsheet $\Sigma = S^2 \setminus \{\pm 1, \infty\}$, we manage to construct such a sc-smooth splicing $r : (\text{-}\epsilon, \epsilon) \oplus W_n^{\oplus 2} \longrightarrow W_n^{\oplus 2}$ that interpolates between fibres $r_{a>0}(W_n^{\oplus 2}) \cong W^{n,2}(S^1, \mathbb{H}) \oplus W^{n,2}(S^1, \mathbb{H})$ and $r_{a<0}(W_n^{\oplus 2}) \cong W^{n,2}(S^1)$.[1]

Let us explain the idea behind this construction: Whereas the polyfold construction of Morse theory (see e.g. [FFGW]) is concerned with gluing copies of $\mathbb{R}$, our basic building block consists in the gluing of two adjacent intervals (-1, 0) and (0, 1). The level sets of our Morse function will be patched together from two copies of (-1, 0) and two copies of (0, 1), with a breaking process at $a = 0$ allowing for a change of gluing partners to model the topology-changing level sets at a Morse critical point. In addition to this a-dependent, "dynamical" gluing, we implement "sheaf-like" or "static" gluing to connect neighbouring intervals at safe distance from the Morse critical point where the effects of our dynamical gluing process are invisible. However, in the following we will mostly focus on the dynamical gluing.

Depending on the value of our parameter a, our intervals will be glued on an overlap of size $2a$, by using cut-off functions $R_{1/a}\beta$ and $1 - R_{1/a}\beta$ as shown below to interpolate between Sobolev functions that were originally defined on (-1, 0) and (0, 1).



It will be useful to replace β and $1 - \beta$ by their normalized look-alikes

$$\alpha := \frac{\beta}{\sqrt{\beta^2 + (1-\beta)^2}} \quad \text{and} \quad \gamma := \frac{1-\beta}{\sqrt{\beta^2 + (1-\beta)^2}}$$

for in this case the matrix $\begin{pmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{pmatrix}$ belongs to $SO\big(2, C^\infty(\mathbb{R})\big)$.

Moreover, we will use the notation $R_\lambda f(\cdot) = f(\lambda \cdot)$ and $\tau_b f(\cdot) = f(\cdot + b)$ to denote the rescaling and shift maps, respectively.

[1] Caution: Here W_n will be a different space than the W_n from Part I

With the mapping spaces $F_n(-1,0)$, $E_n(0,1)$ and $\mathcal{N}_n(\text{-}a,a)$ yet to be determined, we define our gluing map as the upper row in

$$\begin{array}{r c} & \begin{array}{cc} F_n(-1,0) & E_n(0,1) \end{array} \\ \begin{array}{r} W^{n,2}(\text{-}1+a,1\text{-}a) \\ \mathcal{N}_n(-a,a) \end{array} & \begin{bmatrix} R_{1/a}\alpha\cdot\tau_{-a} & R_{1/a}\gamma\cdot\tau_{+a} \\ -R_{1/a}\gamma\cdot\tau_{-a} & R_{1/a}\alpha\cdot\tau_{+a} \end{bmatrix} \end{array} = \begin{bmatrix} R_{1/a}\alpha & R_{1/a}\gamma \\ -R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}\cdot\begin{bmatrix} \tau_{-a} & \\ & \tau_{+a} \end{bmatrix}$$

and take inspiration in [FFGW] Example 5.9 to consider the retraction

$$\begin{bmatrix} \tau_{+a} & \\ & \tau_{-a} \end{bmatrix} \underbrace{\begin{bmatrix} R_{1/a}\alpha & -R_{1/a}\gamma \\ R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}}_{|\Psi\rangle\langle\Psi|} \underbrace{\begin{bmatrix} R_{1/a}\alpha & R_{1/a}\gamma \\ -R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}}_{B^T} \begin{bmatrix} \tau_{-a} & \\ & \tau_{+a} \end{bmatrix}$$

$$= \begin{bmatrix} \tau_{+a} & \\ & \tau_{-a} \end{bmatrix} \underbrace{\begin{bmatrix} R_{1/a}\alpha^2 & R_{1/a}\alpha\gamma \\ R_{1/a}\alpha\gamma & R_{1/a}\gamma^2 \end{bmatrix}}_{|B\Psi\rangle\langle B\Psi|} \begin{bmatrix} \tau_{-a} & \\ & \tau_{+a} \end{bmatrix} = \begin{array}{r c} & \begin{array}{cc} F_n(-1,0) & E_n(0,1) \end{array} \\ \begin{array}{r} F_n(-1,0) \\ E_n(0,1) \end{array} & \begin{bmatrix} R_{1/a}\tau_{+1}\alpha^2\cdot\mathrm{id} & R_{1/a}\tau_{+1}\alpha\gamma\cdot\tau_{+2a} \\ R_{1/a}\tau_{-1}\alpha\gamma\cdot\tau_{-2a} & R_{1/a}\tau_{-1}\gamma^2\cdot\mathrm{id} \end{bmatrix} \end{array}$$

Note that the naive choice "$F_n = W^{n,2}(\text{-}1,0)$", "$E_n = W^{n,2}(0,1)$" leads to a breakdown of sc^0-continuity because the derivatives

$$\partial_x^k\big[R_{1/a}\tau_{+1}\alpha^2\big] = \frac{1}{a^k}R_{1/a}\big[\partial_x^k\tau_{+1}\alpha^2\big] \quad\text{and}\quad \partial_x^k\big[R_{1/a}\tau_{+1}\alpha\gamma\big] = \frac{1}{a^k}R_{1/a}\big[\partial_x^k\tau_{+1}\alpha\gamma\big]$$

introduce arbitrarily high pole divergences at $a=0$. We circumvent this issue by defining our Sobolev spaces w.r.t. vector fields $V_\pm = \pm x\partial_x$ on $(-1,0)$ and $(0,+1)$, respectively. These simply commute with the rescaling map, in the sense that for $f\in C^\infty(\mathbb{R})$ we have

$$V_\pm\big[R_{1/a}f\big] = R_{1/a}\big[V_\pm f\big]$$

After this heuristic introduction, let us continue with a more precise summary of our results:

Rem. 6.1	Our 'contravariant Sobolev spaces' $W^{n,2}_{V,g}(\Omega)$ are defined on open subsets $\Omega\subset\mathbb{R}^m$, thereby depending on the datum of a metric g and distinguished vector field V. Elements of $W^{n,2}_{V,g}(\Omega)$ are tuples $(u_0,...,u_n)\in L^2_g(\Omega)^{\oplus n+1}$ which obey a suitable realization of the constraint "$u_{k+1}=V[u_k]$".
Prop. 6.2	The spaces $W^{n,2}_{V,g}$ exhibit a strikingly simple transformation behaviour in the sense that each diffeomorphism $\Phi:\Omega'\longrightarrow\Omega$ induces an isomorphism of Banach spaces $$W^{n,2}_{V,g}(\Omega)\xrightarrow{\sim}W^{n,2}_{\Phi^*V,\Phi^*g}(\Omega'),\qquad (u_0,...,u_n)\longmapsto(u_0\circ\Phi,...,u_n\circ\Phi)$$ that while acting by componentwise pullback of the tuple $(u_k)_{k=0,...,n}$ packages all non-trivial information into the "smooth data" (V,g). This data is needed to recover the interpretation of the higher entries u_k.
	While the Sobolev spaces $W^{n,2}_{V,g}(\Omega)$ can be studied over any open subset $\Omega\subset\mathbb{R}^m$, we will focus exclusively on the case $m=1$ with $\Omega=I$ an open interval. Regarding the notation, it is sometimes useful to write $W^{n,2}_{V;\rho}(I):=W^{n,2}_{V,\rho^4}(I)$ or drop the subscript g from $W^{n,2}_{V,g}(I)$ once a particular metric has been singled out.
Rem. 6.3 **Lem. 6.4**	As an application of our transformation rule, we use the vehicle of a canonical 'straightening diffeomorphism' $\Phi:I_x\longrightarrow I$ to trivialise a given (non-vanishing) vector field as $\Phi^*V=\partial_x$, making it possible to identify $W^{n,2}_{V,g}(I)$ with an (almost) standard Sobolev space $W^{n,2}_{\partial_x,\Phi^*g}(I_x)$ and to analyse (non-)compactness of the inclusion $W^{1,2}_{V,g}(I)\hookrightarrow L^2_g(I)$, $(u_0,u_1)\longmapsto u_0$.

Moreover, in an algebraic interlude we develop calculation rules to exchange the distinguished vector field, while keeping the interval I and metric g fixed.
Rem. 6.5 Specifically, we introduce the subring $\mathcal{R}(I,W) \subset C^\infty(I)$ of 'smooth functions with bounded W-derivatives' and exhibit $f \in \mathcal{R}(I,W)$ as a sufficient condition for $W^{n,2}_V(I)$
Lem. 6.13 to embed into the Sobolev space $W^{n,2}_W(I)$ with rescaled vector field $W = f \cdot V$.
Returning to our vector field $V_+ = x\partial_x$, note that this calculation rule yields a bounded linear inclusion $W^{n,2}_{\partial_x}(0,1) \hookrightarrow W^{n,2}_{V_+}(0,1)$ whereas the reverse inclusion does not exist.

Having introduced vector-field dependent Sobolev spaces $W^{n,2}_{V,g}$ to circumvent the sc^0-catastrophe mentioned above, our main effort consists in proving that the off-diagonal part of the retraction defines a sc^∞-map

$$(a,u) \longmapsto \begin{cases} R_{1/a}(\tau_{+1}\alpha\gamma)\tau_{2a}u & \text{for } a>0 \\ 0 & \text{for } a \leq 0 \end{cases}$$

Note that $R_{1/a}(\tau_{+1}\alpha\gamma) \in C^\infty_0(\mathbb{R})$ is a bump supported in $\left(-\frac{3}{2}\,a, -\frac{a}{2}\right)$, making it possible to evaluate $\tau_{2a}u$ in the space $W^{n,2}_{V_-}(-1,0)$ even though it originates from $W^{n,2}_{V_+}(0,1)$.
Figure 8.1 is an illustration of this transfer process and the technique of 'comoving intervals' that is required for differentiation w.r.t. to a.

In differentiating the expression $a \longrightarrow R_{1/a}f \cdot \tau_{2a}w \in W^{n,2}_{V_-}(-1,0)$ with $w \in W^{n,2}_{V_+}(0,1)$ and a bump $f \in C^\infty_0\left(-\frac{3}{2}, -\frac{1}{2}\right)$, one has to distinguish between 'longitudinal derivatives' in a-direction and 'transversal derivatives' along the level sets.

First of all, the transversal derivative $V^k_-\big[R_{1/a}f\tau_{2a}w\big]$ is just a sum of similar terms

Prop. 8.22
$$R_{1/a}M_{k,l}[f] \cdot \tau_{2a}V^l_+ w$$

with $V^l_+ w \in W^{n-l,2}_{V_+}(0,1)$ and $M_{k,l}[f] \in C^\infty_0(\mathbb{R})$ a bump supported in $\left(-\frac{3}{2}, -\frac{1}{2}\right)$.

On the other hand, derivatives in a require more care:
As we will see below, it is only necessary to calculate the a-derivatives of $a \longmapsto R_{1/a}(f)\tau_{2a}u \in W^{1,2}_{V_-}(-1,0)$, with the n-th a-derivative forcing us to focus on $u \in W^{n+1,2}_{V_+}(0,1)$.
With $\chi_{n,l}[f] \in C^\infty_0(\mathbb{R})$ yet another kind of bump supported in $\left(-\frac{3}{2}, -\frac{1}{2}\right)$, one finds

Prop. 8.19
$$\left(\frac{\partial}{\partial a}\right)^n R_{1/a}f \cdot \tau_{2a}u \;=\; \frac{1}{a^n}\sum_{l=0}^{n} R_{1/a}\,\chi_{n,l}[f] \cdot \tau_{2a}V^l_+ u \;\in W^{1,2}_{V_-}(-1,0)$$

so the n-th a-derivative translates into a pole of order n.

Prop. 8.25 To prove sc-smoothness of a fibre-linear sc^0-map $\partial^0\pi : B \oplus E \longrightarrow F$ where the base is an open subset $B \subset \mathbb{R}$ and the fibres are sc-Banach spaces E and F, it is enough to find a sequence of sc^0-maps $\partial^n\pi : B \oplus E^n \longrightarrow F$, $n \geq 0$ such that at *fixed* $e \in E_{n+1}$ the map $b \longmapsto \partial^{n+1}\pi_b(e) \in F_0$ is the derivative of its predecessor $b \longmapsto \partial^n\pi_b(e) \in F_0$.

Prop. 8.19 suggests that in our case this sequence will be given by

$$\partial^n\pi_a(u) \;=\; \frac{1}{a^n}\sum_{l=0}^{n} R_{1/a}\,\chi_{n,l}[f] \cdot \tau_{2a}V^l_+ u$$

with transversal derivatives

$$V^m_-[\partial^n\pi_a u] = \frac{1}{a^n}\sum_{s=0}^{m+n} R_{1/a}L^{m,n}_s \cdot \tau_{2a}V^s_+ u, \qquad L^{m,n}_s \in C^\infty_0(-\tfrac{3}{2}, -\tfrac{1}{2})$$

entering the proof of sc^0-continuity.

Thm. 8.26 Crucially, our sc^∞-criterion can be extended to the case of removable singularities $0 \in (\text{-}\epsilon, \epsilon)$, the main condition being that the left and right limits

$$\lim_{b \nearrow 0} \partial^n \pi_b(e), \lim_{b \searrow 0} \partial^n \pi_b(e) \in F_k$$

have to exist and agree for every fixed $e \in E_{n+k}$.

In the case of the map

$$(a, u) \longmapsto \begin{cases} R_{1/a}(\tau_{+1}\alpha\gamma)\tau_{2a}u & \text{for } a \in (0, \epsilon) \\ 0 & \text{for } a \in (\text{-}\epsilon, 0) \end{cases}$$

this will be achieved by choosing

$$E_n = W^{n+1,2}_{V_+;\rho^n}(0,1) \qquad F_n = W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0)$$

with a fine-tuned weight factor $\rho = \frac{1}{|x|}$.

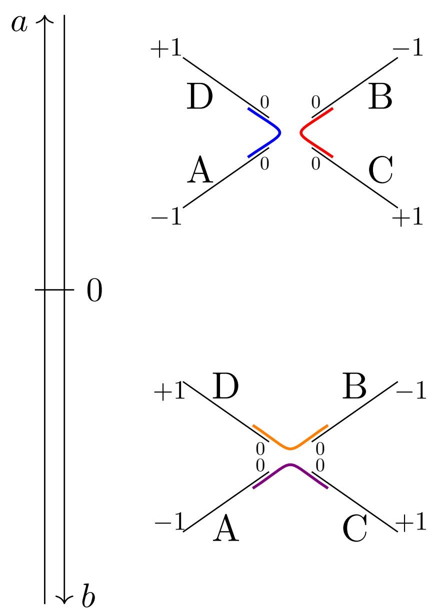
Prop. 8.27 Apart from providing compact inclusions (which itself would not require any fine-tuning), our weight sequence $\rho^n = |x|^{-n}$ can be used to cancel the pole $\frac{1}{a^n}$ from $V^m_-[\partial^n \pi_a u]$. Indeed, the map $\partial^n \pi : (0, \epsilon) \oplus E_{n+m} \longrightarrow F_m$ involves a "weight difference" between $W^{n+m+1,2}_{V_+;\rho^{n+m}}$ and $W^{m+1,2}_{V_-;\rho^m}$, allowing us to substitute $\rho^m = |x|^n \rho^{n+m}$ and use the support of $R_{1/a}L^{m,n}_s$ to compensate $\frac{1}{a^n}$ by $|x|^n < (2a)^n$.

Thm. 7.4 The transition at a Morse critical point can now be modelled through a sc-smooth 'crossover splicing'

$$r_{\text{Cross}} : \underset{a}{\underset{\cup}{(\text{-}\epsilon, \epsilon)}} \oplus \underset{A,B}{\left[W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0)\right]^{\oplus 2}} \oplus \underset{C,D}{\left[W^{n+1,2}_{V_+;\rho^n}(0,1)\right]^{\oplus 2}} \longrightarrow \underset{A,B}{\left[W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0)\right]^{\oplus 2}} \oplus \underset{C,D}{\left[W^{n+1,2}_{V_+;\rho^n}(0,1)\right]^{\oplus 2}}$$

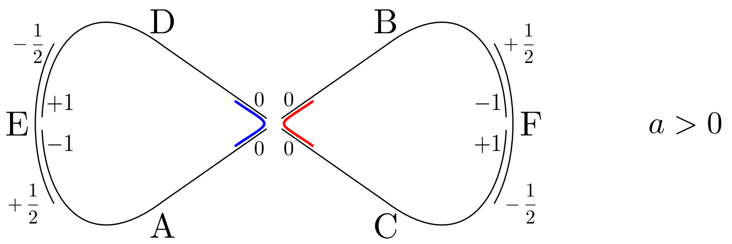
that is explicitly given by

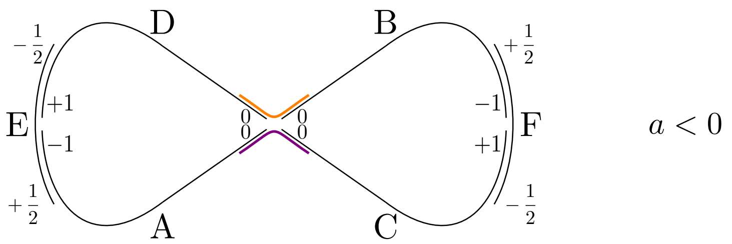
$$\begin{array}{c|cccc} & \text{A} & \text{B} & \text{C} & \text{D} \\ \hline \text{A} & R_{\frac{1}{a}}(\tau_{+1}\alpha^2)\,\text{id} & & & R_{\frac{1}{a}}(\tau_{+1}\alpha\gamma)\,\tau_{+2a} \\ \text{B} & & R_{\frac{1}{a}}(\tau_{+1}\alpha^2)\,\text{id} & R_{\frac{1}{a}}(\tau_{+1}\alpha\gamma)\,\tau_{+2a} & \\ \text{C} & & R_{\frac{1}{a}}(\tau_{-1}\alpha\gamma)\,\tau_{-2a} & R_{\frac{1}{a}}(\tau_{-1}\gamma^2)\,\text{id} & \\ \text{D} & R_{\frac{1}{a}}(\tau_{-1}\alpha\gamma)\,\tau_{-2a} & & & R_{\frac{1}{a}}(\tau_{-1}\gamma^2)\,\text{id} \end{array}$$



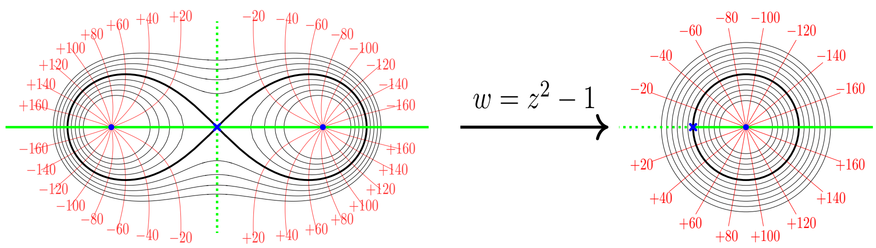
$$\begin{array}{c|cccc} & \text{A} & \text{B} & \text{C} & \text{D} \\ \hline \text{A} & R_{\frac{1}{b}}(\tau_{+1}\alpha^2)\,\text{id} & & R_{\frac{1}{b}}(\tau_{+1}\alpha\gamma)\,\tau_{+2b} & \\ \text{B} & & R_{\frac{1}{b}}(\tau_{+1}\alpha^2)\,\text{id} & & R_{\frac{1}{b}}(\tau_{+1}\alpha\gamma)\,\tau_{+2b} \\ \text{C} & R_{\frac{1}{b}}(\tau_{-1}\alpha\gamma)\,\tau_{-2b} & & R_{\frac{1}{b}}(\tau_{-1}\gamma^2)\,\text{id} & \\ \text{D} & & R_{\frac{1}{b}}(\tau_{-1}\alpha\gamma)\,\tau_{-2b} & & R_{\frac{1}{b}}(\tau_{-1}\gamma^2)\,\text{id} \end{array}$$

By restricting r_{Cross} to suitable closed subspaces
$$W_n \subset W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0) \oplus W^{n+1,2}_{V_+;\rho^n}(0,1)$$
Thm. 7.9 we arrive at a sc-smooth splicing $r_\Sigma : (\text{-}\epsilon,\epsilon) \oplus W_n^{\oplus 2} \longrightarrow W_n^{\oplus 2}$
that interpolates between fibres
Prop. 7.11 $r_{a>0}\big(W_n^{\oplus 2}\big) \cong \big[W^{n+1,2}(S^1)\big]^{\oplus 2}$ and $r_{a<0}\big(W_n^{\oplus 2}\big) \cong W^{n+1,2}(S^1)$
corresponding to distinct topologies $S^1 \sqcup S^1$ and S^1:





It remains to describe a way by which
geometrical data defined on the worldsheet Σ
can be transferred to our M-polyfold $\mathrm{im}(r_\Sigma)$.
In particular, we need a collective parametrization for all level sets Σ_a
so that each one of them can be identified with the ensemble of
unit intervals used in the construction of r_Σ.
Prop. 7.1 Luckily, this can be achieved without integrating any gradient flow:
Our pair-of-pants worldsheet $\Sigma = \mathbb{C} \setminus \{\pm 1\}$
can be interpreted as a ramified cover of the cylinder $C = \mathbb{C} \setminus \{0\}$.
This allows us to equip Σ with a distinguished Morse function ν
coming from the radial coordinate on C.
Meanwhile, the angular coordinate ω provides
the desired collective parametrization of all level sets:



pair of pants $\mathbb{C} \setminus \{\pm 1\}$ cylinder $C = \mathbb{C} \setminus \{0\}$

Away from the critical point at $z = 0$
we can implement a holomorphic change of coordinates $z \longmapsto \nu + i\omega$,
allowing us to split the operator

$$\partial_{\bar{z}} \sim \partial_\nu + i\partial_\omega$$

into a longitudinal part ∂_ν and a transversal part $i\partial_\omega$.

Once we identify our Morse function ν with the gluing parameter a, the gluing map from Proposition 7.3 should make it possible to study a combination of various copies of

$$A(a) := \begin{bmatrix} \tau_{+a} & \\ & \tau_{-a} \end{bmatrix} \begin{bmatrix} R_{1/a}\alpha & -R_{1/a}\gamma \\ R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix} \begin{bmatrix} i\partial_\omega & \\ & \textbf{?} \end{bmatrix} \begin{bmatrix} R_{1/a}\alpha & R_{1/a}\gamma \\ -R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix} \begin{bmatrix} \tau_{-a} & \\ & \tau_{+a} \end{bmatrix}$$

as an operator family on the ambient space of our retraction r_Σ, similar to the operator families encountered in Part I.

We leave the complete formulation as well as the properties of this operator family to future investigation and conclude our discussion with a series of heuristic remarks:

- The matrix $\begin{bmatrix} i\partial_\omega & \\ & \textbf{?} \end{bmatrix}$ being diagonal ensures that $A(a)$ commutes with the retraction, in the sense that $r_a \circ A(a) = A(a) \circ r_a$ at every value of the gluing parameter a.
As a result, $A(a)$ will preserve $\mathrm{im}(r_a)$ as a closed subspace of the ambient fibre.

- The term "?" is invisible to vectors from the subspace $\mathrm{im}(r_a)$.
However, one might consider choosing an injective operator

$$\textbf{?} : W^{n+1,2}_{V^a_{int};\rho^a_{int}}(\text{-}a, a) \longrightarrow W^{n,2}_{V^a_{int};\rho^a_{int}}(\text{-}a, a)$$

to prevent $A(a)$ from having an "unphysical" kernel on the ambient fibre.

- Note that the term $\partial[R_{1/a}\alpha] = \frac{1}{a}R_{1/a}[\partial\alpha]$ introduces a pole at $a = 0$.
However, at the same time $\partial_\omega : W^{n+1,2} \longrightarrow W^{n,2}$ leads a decrease in regularity. This decrease in regularity translates to a weight difference between $W^{n+1,2}_{V;\rho^n}$ and $W^{n,2}_{V;\rho^{n-1}}$ by which we can absorb the pole in way similar to Proposition 8.27.
Thus, we expect our operator family to extend continuously beyond $a = 0$.

- Even more, it should be possible to apply our methods from section 8.2 to prove sc-smoothness of the map $(a, u) \longmapsto A(a)u$, in essentially the same way as we proved sc-smoothness of the retraction $(a, u) \longmapsto r_a(u)$.

These remarks suggest the interpretation of A as a vector field with sc-smooth time-dependence, whose flow arises as the kernel of our APS-operator

$$\partial_{\bar{z}} \sim \frac{\mathrm{d}}{\mathrm{d}a} - A(a)$$

Part I

The sc-Fredholm property of APS-type operators on weighted Floer path spaces

Chapter 2

APS operators on unweighted Floer path spaces

2.1 Basic definitions about almost and honest sc-Banach spaces

In this section we introduce some very basic notions encountered in (linear) polyfold theory [HWZ07], while disentangling them from the requirement of 'compact inclusions'. First of all, remark that the notion of a 'regularizing sc-operator' makes sense on any kind of filtration without any assumption on how the levels are related:

Definition 2.1 (Filtration-compatible maps)
Let $U_0 \supset U_1 \supset \ldots \supset U_k \supset \ldots$ and $V_0 \supset V_1 \supset \ldots \supset V_k \supset \ldots$ be filtrations of vector spaces.

1) A map $\phi : U_0 \longrightarrow V_0$ is called *regularizing* if for every $k \geq 0$ we have $\phi^{-1}(V_k) \subset U_k$
2) Assume each U_k carries a norm $\|\cdot\|_{U_k}$ and each V_k carries a norm $\|\cdot\|_{V_k}$.
   A linear map $\phi : U_0 \longrightarrow V_0$ is called *sc-operator* if for every $k \geq 0$ we have $\phi(U_k) \subset V_k$ and $\phi \in \mathcal{L}(U_k, V_k)$

Remark. A sc-operator $\phi : U_0 \longrightarrow V_0$ is regularizing if and only if $\phi^{-1}(V_k) = U_k$

For our purposes, it will be useful to study 'almost sc-Banach spaces' as a precursor to the usual sc-Banach spaces of [HWZ07]:

Definition 2.2 (Banach scales)
Given a filtration of vector spaces $V_0 \supset V_1 \supset \ldots \supset V_k \supset \ldots$ each carrying a norm such that $\left(V_k, \|\cdot\|_{V_k}\right)$ is a Banach space and the inclusions are continuous we say that

- $(V_k)_{k\in\mathbb{N}}$ is an *almost sc-Banach space* if $V_\infty := \bigcap_{n\in\mathbb{N}} V_n$ is dense in V_k for every $k \in \mathbb{N}$
- $(V_k)_{k\in\mathbb{N}}$ is a *sc-Banach space* if in addition to this all inclusions $V_{k+1} \hookrightarrow V_k$ are compact operators

Notation. Given a filtration $V = (V_k)_{k\in\mathbb{N}}$ we denote by V^1 the truncated filtration $(V^1)_k = V_{k+1}$
If V is an (almost) sc-Banach space, then so is V^1.

A sc-Banach space will sometimes be called *honest* sc-Banach space to highlight the difference. Similarly, we will introduce a distinction between almost and honest sc-subspaces:

Definition 2.3 (almost/honest sc-subspace)
Assume that along with an almost sc-Banach space $(U_k)_{k\in\mathbb{N}}$ we are given a sequence of subspaces $V_0 \subset U_0$, $V_{k+1} \subset V_k \cap U_{k+1}$ such that, when V_k is equipped with the norm coming from U_k, the intersection $\bigcap_{m\geq 0} V_m$ is dense in every V_k.

In this case we say that $V \subset U$ is an *almost sc-subspace.*

We call $V \subset U$ an *(honest) sc-subspace* if in addition all our subspaces $V_k \subset U_k$ are closed.

Remark. Every almost sc-Banach space is an honest sc-subspace of itself.

Note that, as we have defined it, an honest sc-subspace $V \subset U$ does not necessarily have to be an honest sc-Banach space. However, it will automatically be an honest sc-Banach space, once U is:

Lemma 2.4 *Let $V \subset U$ be a sc-subspace. If U is an honest sc-Banach space, then so is V.*

Proof. To verify compactness of the inclusion $V_{k+1} \hookrightarrow V_k$ consider any bounded sequence $x_n \in V_{k+1}$. Since $V_{k+1} \subset U_{k+1} \hookrightarrow U_k$ is compact, we can find a subsequence that converges in U_k. Its limit in contained in V_k because $V_k \subset U_k$ is a closed subspace. $\square$

Almost sc-subspaces, on the other hand, exhibit better stability properties:

Lemma 2.5 (The image of a sc-operator is an almost sc-subspace)
*Let $\phi : U \longrightarrow V$ be a sc-operator between almost sc-Banach spaces U and V.
If X is an almost sc-subspace of U, then $\phi(X) = \big[\phi(X_k)\big]_{k\geq 0}$ is an almost sc-subspace of V.*

Proof. The property $\phi(X_{k+1}) \subset \phi(X_k) \cap V_{k+1}$ is immediate from $X_{k+1} \subset X_k$ and our assumption that ϕ is a sc-operator. It remains to show that $\phi(X)_\infty$ is dense in every $\phi(X_k)$. In doing so the smaller set $\phi(X_\infty) \subset \phi(X)_\infty$ will be sufficient. Indeed, $X_\infty \subset X_k$ is a dense subset w.r.t the norm $\|\cdot\|_{U_k}$ so taking into account that $\phi \in \mathcal{L}(U_k, V_k)$ is continuous, every $\phi(x) \in \phi(V_k)$ can be approximated by a sequence $\phi(x_n)$ with $x_n \in X_\infty$ $\square$

The analogous result for honest sc-subspaces requires some extra conditions:

Corollary 2.6 (Case where the image is an honest sc-subspace)
Let $\phi : U \longrightarrow V$ be a regularizing sc-operator and assume that $\phi(U_0) \subset V_0$ is closed subspace. Then $\phi(U) = \big[\phi(U_k)\big]_{k\geq 0}$ is an honest sc-subspace of V.

Proof. By Lemma 2.5 we already know that $\phi(U) = \big[\phi(U_k)\big]_{k\geq 0}$ is an almost sc-subspace.
That it is indeed an honest sc-subspace can be seen as follows: For a regularizing sc-operator we have $\phi^{-1}(V_k) = U_k$, by which we can rewrite $\phi(U_k) = \phi(U_0) \cap V_k$. Moreover, continuity of the inclusion $V_k \hookrightarrow V_0$ guarantees that $\phi(U_0) \cap V_k$ is a closed subspace of V_k. $\square$

The reader may observe that arguments similar to Lemmas 2.4, 2.5 and Corollary 2.6 are involved in the proof of [We] Lem. 3.6. However, we prefer to highlight them as individual properties.

Now we are ready to state the main property that we are after, namely that of a linear operator being sc-Fredholm. We phrase our definition in such a way that it emphasizes the concrete input needed for an operator to be sc-Fredholm, while making sense not only on honest but also on almost sc-Banach spaces:

Definition 2.7 (sc-Fredholm operator)
Let $U = (U_k)_{k\in\mathbb{N}}$ and $V = (V_k)_{k\in\mathbb{N}}$ be almost sc-Banach spaces. Assume that along with finite-dimensional subspaces $K \subset U_\infty$ and $C \subset V_\infty$ we can find decompositions $U_k = K \oplus X_k$ and $V_k = Y_k \oplus C$ where the $X_k \subset U_k$ and $Y_k \subset V_k$ are closed subspaces that organise into sc-subspaces of U and V, respectively. Assume further that $\varphi : U_0 \longrightarrow V_0$ is a sc-operator with $\ker \varphi = K$ and $\varphi(U_k) = Y_k$. In this situation φ will be called *sc-Fredholm.*

Remark 2.8 (Classical Fredholm property at every level)
With the notation of Definition 2.7 it is immediately clear that if $\varphi : (U_k)_{k\geq 0} \longrightarrow (V_k)_{k\geq 0}$ is sc-Fredholm, the operator $\varphi : U_k = K \oplus X_k \longrightarrow Y_k \oplus C = V_k$ will be classically Fredholm at every level $k \geq 0$.

It is straightforward, be it a bit tedious, to verify that in the case of honest sc-Banach spaces our definition agrees with the standard one from [HWZ07], the moral reason for why this works being Auxiliary Lemma 2.9 and Lemma 2.10 below.

Auxiliary Lemma 2.9 (Finite-dimensional subspaces are closed)
*Let U be a normed space over a complete field (in our case $\mathbb{R}$ or $\mathbb{C}$).
Then every finite-dimensional subspace $C \subset U$ is closed.*

Proof. Given a basis $\{e_i\}_{i=1,\ldots,n}$ of C the norm $\left\|\sum_i \lambda^i e_i\right\|_C = \sum_i \left|\lambda^i\right|$ is complete. Since all norms on a finite-dimensional vector space are equivalent, we conclude that $\left(C, \|\cdot\|_U\right)$ is a Banach space. Hence Cauchy sequences in C cannot have their limit outside of C. □

Lemma 2.10 (Topological decomposition of Banach spaces)
*Let $U = X \oplus C$ be a Banach space, decomposed into closed subspaces X and C.
Then $\|\cdot\|_U$ is equivalent to the canonical norm of $X \oplus C$*

Proof. The canonical norm on $X \oplus C$ is given by $\|(x,c)\| = \|x\|_U + \|c\|_U$ so the triangle inequality of $\|\cdot\|_U$ shows that $X \oplus C \longrightarrow U$, $(x,c) \longmapsto x + c$ is a bounded linear isomorphism, which by the Inverse Mapping Theorem has a bounded inverse. The condition that X and C are closed subspaces is to ensure that $X \oplus C$ is a Banach space, as required for the Inverse Mapping Theorem. □

To conclude this warm-up section, let us comment on a repeating pattern that we will encounter in regularization proofs over and over again:

Definition 2.11 (Escalator)
Given a filtration of sets $V_0 \supset V_1 \supset \ldots \supset V_k \supset \ldots$ we say that a map $\phi : V_1 \longrightarrow V_0$ is an *escalator* for $(V_k)_{k\in\mathbb{N}}$ if at every $k \geq 1$ we have

$$x, \phi(x) \in V_k \implies x \in V_{k+1} \tag{2.1}$$

or in other words $V_k \cap \phi^{-1}(V_k) \subset V_{k+1}$

Auxiliary Lemma 2.12 (Escalators are regularizing)
Let $\phi : V_1 \longrightarrow V_0$ be an escalator for $(V_k)_{k\in\mathbb{N}}$. Then $\phi^{-1}(V_k) \subset V_{k+1}$

Proof. At $k = 0$ we trivially have $\phi^{-1}(V_0) \subset V_1$. Assume by induction that $\phi^{-1}(V_k) \subset V_{k+1}$ holds for a given $k \geq 0$. Then any $x \in V_1$ with $\phi(x) \in V_{k+1} \subset V_k$ will belong to V_{k+1} so property (2.1) implies $x \in V_{k+2}$ and therefore $\phi^{-1}(V_{k+1}) \subset V_{k+2}$ □

Auxiliary Lemma 2.13 (Stability of the preimage)
Assume that on a filtration of vector spaces $(V_k)_{k\in\mathbb{N}}$ we are given operators $A_0 : V_1 \longrightarrow V_0$ and $B : V_0 \longrightarrow V_0$ with $A_0^{-1}(V_k) \subset V_{k+1}$ and $B(V_k) \subset V_k$. Then $A_0 + B : V_1 \longrightarrow V_0$ satisfies $(A_0 + B)^{-1}(V_k) \subset V_{k+1}$ as well.

Proof. By Auxiliary Lemma 2.12 it suffices to verify that $A_0 + B : V_1 \longrightarrow V_0$ is an escalator for $(V_k)_{k\in\mathbb{N}}$. Indeed, consider $x \in V_k$ such that $(A_0 + B)x \in V_k$. Then $A_0 x = (A_0 + B)x - Bx \in V_k$ implies $x \in V_{k+1}$ □

2.2 Baseline Operators

In this section we explain how a Hilbert space structure on the lowest level of an almost sc-Banach space $W_0 \supset W_1 \supset \dots$ can be transferred to all higher levels W_k, making it possible to apply the techniques of [RS] there as well. The key ingredient will be that of a distinguished 'baseline operator'.

Definition 2.14 (Hilbert scales)
An (almost) sc-Banach space $(W_k)_{k\in\mathbb{N}}$ will be called *(almost) sc-Hilbert space* if the norm $\|\cdot\|_{W_0}$ arises from a Hilbert space structure on $H := W_0$

Definition 2.15 (Baseline operator)
Let $H \supset W_1 \supset \dots \supset W_k \supset \dots$ be an almost sc-Hilbert space.
An operator $A_0 : W_1 \longrightarrow H$ will be called *sc-self-adjoint* or *baseline operator* if

- A_0 is an sc-operator, i.e. $A_0(W_{k+1}) \subset W_k$ and $A_0 \in \mathcal{L}(W_{k+1}, W_k)$
- A_0 is regularizing, i.e. $A_0^{-1}(W_k) = W_{k+1}$
- $A_0 : W_1 \longrightarrow H$ is self-adjoint as an unbounded operator on H

Note that the datum of a self-adjoint sc-operator $A_0 : W_1 \longrightarrow H$ comes with a unique equivalence class of possible norms $\|\cdot\|_{W_1}$:

Lemma 2.16 (Characterising the graph norm of a self-adjoint operator)
Let $A_0 : W_1 \longrightarrow H$ be self-adjoint as an unbounded operator on H and assume that $\|\cdot\|_{W_1}$ is chosen in such a way that both A_0 and the inclusion $\iota : W_1 \hookrightarrow H$ belong to $\mathcal{L}(W_1, H)$.
Then $\|\cdot\|_{W_1}$ is equivalent to the graph norm of A_0

Proof. Since A_0 is a symmetric operator, we observe that for all $\lambda \in \mathbb{C}, w \in W_1$ one has

$$\|(A_0 + \lambda)w\|_H^2 \;=\; \|A_0 w\|_H^2 + |\lambda|^2 \, \|w\|_H^2 + 2\,\mathrm{Re}\lambda \, \langle w, A_0 w\rangle_H \tag{2.2}$$

By inserting $\lambda = i$ we see that the expression

$$\|(A_0 + i)w\|_H^2 \;=\; \|A_0 w\|_H^2 + \|w\|_H^2 \tag{2.3}$$

defines a norm on W_1. This norm is equivalent to the graph norm of A_0, denoted by $\|\cdot\|_{W_1}^{A_0}$. Since A_0 and the inclusion ι belong to $\mathcal{L}(W_1, H)$ we immediately have

$$\|w\|_{W_1}^{A_0} \underset{\text{Def}}{=} \|A_0 w\|_H + \|w\|_H \le \Big[\, \|A_0\|_{\mathcal{L}(W_1,H)} + \|\iota\|_{\mathcal{L}(W_1,H)} \Big] \cdot \|w\|_{W_1}$$

Self-adjointness of A_0 ensures that $A_0 + i : W_1 \longrightarrow H$ is invertible and the Inverse Mapping Theorem guarantees $(A_0 + i)^{-1} \in \mathcal{L}(H, W_1)$. Thus, equation (2.3) provides the missing inequality

$$\|w\|_{W_1} \;=\; \big\|(A_0+i)^{-1}(A_0+i)w\big\|_{W_1} \;\le\; \big\|(A_0+i)^{-1}\big\|_{\mathcal{L}(H,W_1)} \, \|(A_0+i)w\|_H \;\le\; \big\|(A_0+i)^{-1}\big\|_{\mathcal{L}(H,W_1)} \, \|w\|_{W_1}^{A_0}$$

□

On the other hand, the combination of A_0 being a regularizing sc-operator leads to a decreasing filtration of spectra

$$\sigma(A_0 : W_1 \longrightarrow H) \supset \sigma(A_0 : W_2 \longrightarrow W_1) \supset \dots$$

We will use the dual version of this statement:

Lemma 2.17 (Increasing filtration of resolvent sets)
Let $A_0 : (W_{k+1})_{k\in\mathbb{N}} \longrightarrow (W_k)_{k\in\mathbb{N}}$ be a regularizing sc-operator.
Then $\rho\big(A_0 : W_1 \longrightarrow W_0\big) \subset \rho\big(A_0 : W_{k+1} \longrightarrow W_k\big)$ so the resolvent set can only grow as we restrict A_0 to higher levels $1 + k$

Proof. Choosing $\lambda \in \rho\big(A_0 : W_1 \longrightarrow W_0\big)$ means that $A_0 - \lambda : W_1 \longrightarrow W_0$ is invertible, so the restriction $A_0 - \lambda : W_{k+1} \longrightarrow W_k$ is at least injective. In fact it is also surjective because Auxiliary Lemma 2.13 guarantees $(A_0 - \lambda)^{-1}(W_k) \subset W_{k+1}$. Since $A_0 - \lambda$ belongs to $\mathcal{L}(W_{k+1}, W_k)$, the Inverse Mapping Theorem ensures that its inverse $(A_0 - \lambda)^{-1} \in \mathcal{L}(W_k, W_{k+1})$ is a bounded linear operator as well. □

The combination of Lemma 2.16 and Lemma 2.17 can be bootstrapped to establish the conditions for [RS] Thm. 3.10 at every level $W_{k+1} \hookrightarrow W_k$ instead of just $W_1 \hookrightarrow H$:

Proposition 2.18 (Self-adjointness at every level)
Let $H = W_0 \supset W_1 \supset \ldots \supset W_k \supset \ldots$ be an (almost) sc-Hilbert space. Given a baseline operator $A_0 : W_1 \longrightarrow H$ equip W_{k+1} with the inner product $\langle v, w\rangle_{W_{k+1}} := \langle v, w\rangle_{W_k} + \langle A_0 v, A_0 w\rangle_{W_k}$ and write $\|\cdot\|_{W_k, A_0}$ for the norm induced by $\langle \cdot, \cdot\rangle_{W_k}$
Then the following statements hold true for all $k \geq 0$:

1) $\|\cdot\|_{W_k, A_0}$ is equivalent to the norm $\|\cdot\|_{W_k}$ already there

2) $A_0 : W_{k+1} \longrightarrow W_k$ is self-adjoint as an unbounded operator on $\big(W_k, \langle \cdot, \cdot\rangle_{W_k}\big)$

Proof of Proposition 2.18.
For $k = 0$ note that $\|\cdot\|_H$ arises from an inner product $\langle \cdot, \cdot\rangle_H$ and $A_0 : W_1 \longrightarrow H$ is self-adjoint as an unbounded operator on $H = W_0$.

Now assume by induction that conditions 1) and 2) are fulfilled at a given $k \geq 0$.
Condition 1) implies that A_0 and the inclusion $\iota : W_{k+1} \hookrightarrow W_k$ are bounded linear operators from $\big(W_{k+1}, \|\cdot\|_{W_{k+1}}\big)$ to $\big(W_k, \langle \cdot, \cdot\rangle_{W_k}\big)$ whereas condition 2) says that $A_0 : W_{k+1} \longrightarrow W_k$ is self-adjoint as an unbounded operator on $\big(W_k, \langle \cdot, \cdot\rangle_{W_k}\big)$. Since $\|\cdot\|_{W_{k+1}, A_0}$ represents the graph norm of $A_0 : W_{k+1} \longrightarrow \big(W_k, \langle \cdot, \cdot\rangle_{W_k}\big)$, we can apply Lemma 2.16 in the setting "H" $= W_k$ and "W_1" $= W_{k+1}$ to conclude that $\|\cdot\|_{W_{k+1}, A_0}$ is equivalent to the norm $\|\cdot\|_{W_{k+1}}$ already there.
It remains to show that $A_0 : W_{k+2} \longrightarrow W_{k+1}$ is self-adjoint as an unbounded operator on $\big(W_{k+1}, \langle \cdot, \cdot\rangle_{W_{k+1}}\big)$. Since $A_0 : W_{k+1} \longrightarrow W_k$ is symmetric, symmetry of $A_0 :\ W_{k+2} \longrightarrow W_{k+1}$ follows from the calculation

$$\begin{aligned} \langle \underbrace{w}_{W_{k+2}}, \underbrace{A_0 w}_{W_{k+1}} \rangle_{W_{k+1}} &= \langle \underbrace{w}_{W_{k+1}}, A_0 \underbrace{w}_{W_{k+1}} \rangle_{W_k} + \langle \underbrace{A_0 w}_{W_{k+1}}, A_0 \underbrace{A_0 w}_{W_{k+1}} \rangle_{W_k} \\ &= \big\langle A_0\, w, w\big\rangle_{W_k} + \big\langle A_0\, A_0 w, A_0 w\big\rangle_{W_k} = \big\langle A_0 w, w\big\rangle_{W_{k+1}} \end{aligned}$$

To verify self-adjointness it remains to check $\pm i \in \rho\big(A_0 : W_{k+2} \longrightarrow W_{k+1}\big)$
which immediately follows from Lemma 2.17 because the self-adjointness of $A_0 : W_1 \longrightarrow H$ ensures $\pm i \in \rho\big(A_0 : W_1 \longrightarrow H\big)$. □

Remark. Proposition 2.18 shows that every baseline operator $A_0 : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ can be truncated to a baseline operator $A_0 : (W_{k+1})_{k\geq 1} \longrightarrow (W_k)_{k\geq 1}$.

Before invoking [RS], let us mention an additional property that is in a way built into the definition of a baseline operator:

Lemma 2.19 (Horizontal Regularization)
Let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an almost sc-Hillbert space $H \supset W_1 \supset \ldots \supset W_k \supset \ldots$
Then the map $A_0 : L^2(\mathbb{R}, W_1) \longrightarrow L^2(\mathbb{R}, H)$ satisfies $A_0^{-1}\Big(L^2(\mathbb{R}, W_k)\Big) = L^2(\mathbb{R}, W_{k+1})$

Proof. By Auxiliary Lemma 2.12 it suffices to verify that A_0 is an escalator for $\big[L^2(\mathbb{R}, W_k)\big]_{k\in\mathbb{N}}$.
Given $w \in L^2(\mathbb{R}, W_k)$ such that $A_0 w$ belongs to $L^2(\mathbb{R}, W_k)$ as well, $A_0^{-1}(W_k) = W_{k+1}$ shows that we have $w(t) \in W_{k+1}$ almost everywhere. In fact, by setting $w(t) = 0$ on the null set where it would be violated, we can arrange for the statement that $w(t) \in W_{k+1}$ for all $t \in \mathbb{R}$.
Using the modified norms $\|\cdot\|_{W_k, A_0}$ from Proposition 2.18 we observe

$$\|w(t)\|^2_{W_{k+1}} \leq \text{const.} \times \|w(t)\|^2_{W_{k+1}, A_0} = \text{const.} \times \Big[\, \|w(t)\|^2_{W_k, A_0} + \|A_0 w(t)\|^2_{W_k, A_0} \Big] \leq \text{const.} \times \Big[\, \|w(t)\|^2_{W_k} + \|A_0 w(t)\|^2_{W_k} \Big]$$

Integration over $t \in \mathbb{R}$ yields

$$\int \|w(t)\|^2_{W_{k+1}} \leq \text{const.} \times \Big[\int \|w(t)\|^2_{W_k} + \int \|A_0 w(t)\|^2_{W_k} \Big] < \infty$$

and therefore $w \in L^2(\mathbb{R}, W_{k+1})$. □

On the other hand, as a reward for Proposition 2.18, we obtain the following property:

Lemma 2.20 (Vertical Regularization)

Let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an almost sc-Hillbert space $H \supset W_1 \supset \ldots \supset W_k \supset \ldots$ Then for every pair $r, k \geq 0$ the operator

$$D_{A_0} = \frac{d}{dt} - A_0 : \; L^2(\mathbb{R}, W_{k+1}) \cap W^{1,2}(\mathbb{R}, W_k) \longrightarrow L^2(\mathbb{R}, W_k)$$

satisfies $D_{A_0}^{-1}\Big(W^{r,2}(\mathbb{R}, W_k)\Big) \subset W^{r,2}(\mathbb{R}, W_{k+1}) \cap W^{r+1,2}(\mathbb{R}, W_k)$

Proof. Proposition 2.18 shows that A_0 can be regarded as a self-adjoint operator on the Hilbert space $\big(W_k, \langle \cdot, \cdot \rangle_{W_k}\big)$, whose graph norm is equivalent to the norm $\|\cdot\|_{W_{k+1}}$ already there.

Note that [RS] Thm 3.10 (Elliptic regularity) relies on the operator "$A(t)$" being self-adjoint at every $t \in \mathbb{R}$ but does not require existence of invertible endpoints $A_\pm = \lim_{t\to\pm\infty} A(t)$. [RS] Thm 3.13 is a direct consequence of Thm 3.10. Applied in the setting "H" $= W_k$, "W" $= W_{k+1}$ with "$A(t)$" $= A_0$ it shows that $D_{A_0}^{-1}\Big(W^{r,2}(\mathbb{R}, W_k)\Big) \subset W^{r,2}(\mathbb{R}, W_{k+1}) \cap W^{r+1,2}(\mathbb{R}, W_k)$ as claimed. □

Our proof of the key Proposition 2.35 will be a combination of Lemma 2.20 (Vertical Regularization) and Lemma 2.19 (Horizontal Regularization). Note, however, that we need to work with constant A_0 instead of a time-dependent operator family $A(t)$ because the operators $A(t)$ and $A(t_0)$ at different times $t \neq t_0$ will in general no longer commute, making it impossible for $A(t)$ to be symmetric w.r.t. $\langle v, w \rangle_{W_{k+1}} = \langle v, w \rangle_{W_k} + \langle A(t_0)v, A(t_0)w \rangle_{W_k}$.

2.3 Admissible perturbations of a baseline operator

As explained in the previous section, our proof of regularization requires a fixed baseline operator A_0 instead of a time-dependent family $A(t)$. Nonetheless, we can introduce perturbations $B(t) \in \mathcal{L}(H)$ on top of A_0 that are a posteriori compatible with the process of regularization.

Definition 2.21 (Perturbations of a baseline operator)

Let $H = W_0 \supset W_1 \supset \ldots \supset W_k \supset \ldots$ be an almost sc-Hilbert space.
As perturbations to the baseline operator A_0 from Section 2.2
we will consider operator families $B \in C^1_{\text{bounded}}\big(\mathbb{R}, \mathcal{L}(H)\big)$.
Such a perturbation will be called...

- ***symmetric*** if at every $t \in \mathbb{R}$ the operator $B(t) \in \mathcal{L}(H)$ is symmetric
- ***endpoint-regular*** if there exist invertible operators $A_\pm \in \mathcal{L}(W_1, H)$ such that $A_0 + B(t) \longrightarrow A_\pm$ in $\mathcal{L}(W_1, H)$ as $t \to \pm\infty$
- ***localized*** if $B \in C^2\big(\mathbb{R}, \mathcal{L}(H)\big)$ and $\|B'(t)\|_{\mathcal{L}(H)}, \|B''(t)\|_{\mathcal{L}(H)} \longrightarrow 0$ as $t \to \pm\infty$
- ***moderate*** if B is a family of sc-operators $B(t) : (W_k)_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ such that at every $k \geq 0$ the map $\mathbb{R} \longrightarrow \mathcal{L}(W_k)$ is *smooth with bounded derivatives*, i.e. all derivatives $B^{(m)}(t) \in \mathcal{L}(W_k)$, $m \geq 0$ exist and satisfy $\sup_{t\in\mathbb{R}} \|B^{(m)}(t)\|_{\mathcal{L}(W_k)} < \infty$

We will say that $B(t) \in \mathcal{L}(H)$ is a...

- *Robbin-Salamon perturbation* if it is symmetric and endpoint-regular
- *good perturbation* if it is moderate, symmetric and endpoint-regular
- *very good perturbation* if it is moderate, symmetric, endpoint-regular and localized

As a rule of thumb, we will rely on moderate perturbations to ensure that D_A is regularizing, whereas Robbin-Salamon perturbations are used to make D_A a Fredholm operator in the classical sense. Very good perturbations will only be required for the proof of Theorem 4.15.

At the beginning of [RS] section 3, Robbin and Salamon give a list of three in part rather specific properties that are assumed in order to make their proofs work. For the record, let us observe that these properties are automatically covered by our notion of a 'Robbin-Salamon perturbation':

Lemma 2.22 (Applicability of the results from [RS])
Let $A_0 \in \mathcal{L}(W, H)$ be self-adjoint as an unbounded operator on H and assume that $B \in C^1_{bounded}\big(\mathbb{R}, \mathcal{L}(H)\big)$ is a symmetric and endpoint-regular perturbation of A_0. Then $A(t) := A_0 + B(t) \in \mathcal{L}(W, H)$ satisfies the conditions $(A1), (A2), (A3)$ formulated at the beginning of [RS] Section 3.

Proof.

- A slightly stronger and more concise version of $(A1)$ consists in the requirement
$$A \in C^1_{\text{bounded}}\big(\mathbb{R}, \mathcal{L}(W, H)\big)$$
as realized by demanding $B \in C^1_{\text{bounded}}\big(\mathbb{R}, \mathcal{L}(H)\big)$.
- Since $B(t) \in \mathcal{L}(H)$ is symmetric at every $t \in \mathbb{R}$, the Kato-Rellich Theorem ensures that $A(t) : W \longrightarrow H$ is self-adjoint as an unbounded operator on H, which verifies the first part of condition $(A2)$. The second part is covered by the statement that $\|\cdot\|_W$ is equivalent to the graph norm of $A(t)$, with the equivalence being realized by constants independent of t. In our case, we know from the proof of Lemma 2.16 that
$$\|w\|_W \;\leq\; \underbrace{\big\|(A_0 + i)^{-1}\big\|_{\mathcal{L}(H,W)}}_{=:\, c} \big(\|A_0 w\|_H + \|w\|_H \big)$$
so using $\kappa := \sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(H)} < \infty$ we obtain
$$\|w\|_W \;\leq\; c\,\big(\|A(t)w\|_H + (1+\kappa)\,\|w\|_H \big) \;\leq\; c(1+\kappa)\big(\|A(t)w\|_H + \|w\|_H \big)$$
whereas the opposite direction is already settled by $A \in C^0_{\text{bounded}}\big(\mathbb{R}, \mathcal{L}(W, H)\big)$.
- Finally, condition $(A3)$ coincides with the statement that $A(t) = A_0 + B(t)$ is endpoint-regular. $\square$

To increase flexibility and as a preparation for the next section, we briefly generalize the notion of a moderate perturbation from Definition 2.21. In the following let U,V and W be almost sc-Banach spaces.

Definition 2.23 (Moderate families of sc-operators)
We say that a family of sc-operators $A(t) :\; (U_k)_{k\geq 0} \;\longrightarrow\; (V_k)_{k\geq 0}$ is *moderate* if at every level $k \geq 0$ the map $A : \mathbb{R} \longrightarrow \mathcal{L}(U_k, V_k)$ is smooth with bounded derivatives, i.e. all derivatives $A^{(m)}(t) \in \mathcal{L}(U_k, V_k)$, $m \geq 0$ exist and satisfy $\sup_{t\in\mathbb{R}} \big\|A^{(m)}(t)\big\|_{\mathcal{L}(U_k,V_k)} < \infty$.
The set of moderate families $A(t) : U \longrightarrow V$ will be denoted by $\mathcal{M}(U, V)$.

Remark 2.24 (Moderate families are the morphisms of a $\mathbb{C}$-linear category)

- The sum of moderate families $A, B \in \mathcal{M}(U, V)$ is again moderate
- Let $U \xrightarrow{B(t)} V \xrightarrow{A(t)} W$ be moderate families. Then the product rule together with
$$\Big\|A^{(m)}(t)B^{(n)}(t)\Big\|_{\mathcal{L}(U_k,W_k)} \leq \Big\|A^{(m)}(t)\Big\|_{\mathcal{L}(V_k,W_k)} \Big\|B^{(n)}(t)\Big\|_{\mathcal{L}(U_k,V_k)}$$
shows that the composition $A(t) \circ B(t)$ is moderate, too.

Example. Let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an almost sc-Hilbert space $H \supset W_1 \supset \dots$ and denote by $\iota : W_1 \longrightarrow H$ the canonical inclusion.
Then $A_0 : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ and $\iota : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ are constant families of sc-operators. Given a moderate perturbation $B(t) \in \mathcal{L}(H)$ the perturbed

$$A(t) = A_0 + B(t) \circ \iota : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$$

is a moderate family of sc-operators in the sense of Definition 2.23.

2.4 D_A as a sc-operator

2.4.1 Banach-space-valued Sobolev spaces

Before turning to D_A, let us identify the condition under which a general t-family of operators $A(t) : E \longrightarrow F$ induces a map $A : W^{r,2}(\mathbb{R}, E) \longrightarrow W^{r,2}(\mathbb{R}, F)$ between Banach-space-valued Sobolev spaces.

We adopt a slightly unconventional perspective on Sobolev spaces, which as we shall see in Part II Remark 6.1 opens a door to generalization:

Remark 2.25 (Alternative construction of Sobolev spaces)
Let E be a Banach space. Given any test function $\phi \in C_0^\infty(\mathbb{R})$, Young's inequality shows that the expression

$$\delta_\phi(u_0, u_1) := \int u_0\, \partial\phi + \int u_1 \phi$$

defines a bounded linear map $\delta_\phi : L^2(\mathbb{R}, E)^{\oplus 2} \longrightarrow E$. Hence, for every $r \geq 1$

$$\widehat{W}^{r,2}(\mathbb{R}, E) := \left\{ (u_0, u_1, ..., u_r) \in L^2(\mathbb{R}, E)^{\oplus r+1} \,\middle|\, (u_k, u_{k+1}) \in \bigcap_{\phi \in C_0^\infty(\mathbb{R})} \ker \delta_\phi \;\; \forall\, k = 0, ..., r-1 \right\}$$

is a closed subspace of $L^2(\mathbb{R}, E)^{\oplus r+1}$ and thus a Banach space itself.

The standard Sobolev spaces can be recovered as follows:

Lemma 2.26 (Identifying $\widehat{W}^{r,2}$ with $W^{r,2}$)
The projection map $p_0 : \widehat{W}^{r,2}(\mathbb{R}, E) \longrightarrow L^2(\mathbb{R}, E)$, $(u_k)_{k=0,\dots,r} \longmapsto u_0$ is injective.
In particular, we can identify $\widehat{W}^{r,2}(\mathbb{R}, E)$ with its image under p_0, denoted by $W^{r,2}(\mathbb{R}, E) \subset L^2(\mathbb{R}, E)$.

Proof. Consider $\big(u_0 = 0, u_1, \dots, u_r\big) \in \widehat{W}^{r,2}(\mathbb{R}, E)$ and assume by contradiction that there exists a minimal $k \leq r-1$ such that $u_{k+1} \neq 0$. For the convolution with any $\varphi \in C_0^\infty(\mathbb{R})$ we obtain

$$\varphi * u_{k+1} \;=\; \dot{\varphi} * u_k \;=\; 0$$

Now let $\varphi_\epsilon \in C_0^\infty(\mathbb{R})$ be a standard Dirac sequence[1]. Then one has

$$\lim_{\epsilon \to 0} \left\| \varphi_\epsilon * u_{k+1} - u_{k+1} \right\|_{L^2(\mathbb{R}, E)} = 0$$

so having $\varphi_\epsilon * u_{k+1} = 0$ for all $\epsilon > 0$ implies $u_{k+1} = 0$ in contradiction to our assumption. □

Now, as a first step, let us describe transitions between Banach-space-valued Sobolev spaces $W^{1,2}(\mathbb{R}, E)$ and $W^{1,2}(\mathbb{R}, F)$:

Lemma 2.27 (Product rule)
Let E, F be Banach spaces. Assume that we are given $(u, u') \in \widehat{W}^{1,2}(\mathbb{R}, E)$ and $A \in C^1(\mathbb{R}, \mathcal{L}(E, F))$ with $\sup_{t\in\mathbb{R}} \left\| A^{(m)}(t) \right\|_{\mathcal{L}(E,F)} < \infty$ for $m = 0, 1$.

Then one has $Au \in W^{1,2}(\mathbb{R}, F)$ with weak derivative $\frac{d}{dt} Au \;=\; A'u + Au'$

[1] i.e. $\varphi_\epsilon = \frac{1}{\epsilon}\varphi\left(\frac{\cdot}{\epsilon}\right)$ where $\varphi \geq 0$ is a smooth function supported in $[-1, 1]$ such that $\int \varphi = 1$

Proof. Given a standard Dirac sequence $\varphi_\epsilon \in C_0^\infty(\mathbb{R})$ we have $u_\epsilon := \varphi_\epsilon * u \in C^\infty \cap W^{1,2}(\mathbb{R}, E)$ and

$$\lim_{\epsilon\to 0} \big\| u_\epsilon - u \big\|_{W^{1,2}(\mathbb{R},\,E)} = 0$$

By combining Young's inequality with our assumption $\sup_{t\in\mathbb{R}} \big\| A^{(m)}(t) \big\|_{\mathcal{L}(E,\,F)} < \infty$ one verifies that for any test function $\phi \in C_0^\infty(\mathbb{R})$

$$\begin{aligned} \delta_\phi\big(Au, A'u + Au'\big) &= \int Au\,\partial\phi + \int [A'u + Au']\,\phi \\ &= \lim_{\epsilon\to 0} \Big[\int Au_\epsilon\,\partial\phi + [A'u_\epsilon + Au_\epsilon']\,\phi \Big] = \lim_{\epsilon\to 0} \Big[\int \partial\big(Au_\epsilon\phi\big) \Big] = 0 \end{aligned}$$

and therefore $\big(Au, A'u + Au'\big) \in \widehat{W}^{1,2}(\mathbb{R}, F)$. $\square$

Having understood the case $r = 1$, let us turn to higher Sobolev spaces:

Lemma 2.28 ($\widehat{W}^{r,2}$ as a functor)
Let E and F be Banach spaces. Then for every C^r-map $A : \mathbb{R} \longrightarrow \mathcal{L}(E, F)$ whose derivatives are bounded in the sense that $\sup_{t\in\mathbb{R}} \big\| A^{(m)}(t) \big\|_{\mathcal{L}(E,F)} < \infty$ for all $m = 0, ..., r$ there exists a unique bounded linear map $\widehat{W}^{r,2}(A) : \widehat{W}^{r,2}(\mathbb{R}, E) \longrightarrow \widehat{W}^{r,2}(\mathbb{R}, F)$ such that the diagram

$$\begin{array}{ccc} \widehat{W}^{r,2}(\mathbb{R},E) & \xrightarrow{\widehat{W}^{r,2}(A)} & \widehat{W}^{r,2}(\mathbb{R},F) \\ \big\downarrow p_0 & & \big\downarrow p_0 \\ L^2(\mathbb{R},E) & \xrightarrow{A} & L^2(\mathbb{R},F) \end{array}$$

commutes.

Proof. To verify that the bounded linear map

$$\begin{array}{ccc} L^2(\mathbb{R},E)^{\oplus r+1} & \xrightarrow{\widehat{W}^{r,2}(A)} & L^2(\mathbb{R},F)^{\oplus r+1} \\ [u_k]_{k=0,\ldots,r} & \longmapsto & \Big[[Au]_k = \sum_{l=0}^{k} \binom{k}{l} A^{(k-l)}\,u_l \Big]_{k=0,\ldots,r} \end{array}$$

maps $\widehat{W}^{r,2}(\mathbb{R}, E)$ to $\widehat{W}^{r,2}(\mathbb{R}, F)$, let us pick any $\big(u_0, u_1, ..., u_r\big) \in \widehat{W}^{r,2}(\mathbb{R}, E)$.
For $0 \le l, k - l \le k \le r - 1$ we have $A^{(k-l)} \in C^1\big(\mathbb{R}, \mathcal{L}(E, F)\big)$ and $\big(u_l, u_{l+1}\big) \in \widehat{W}^{1,2}(\mathbb{R}, E)$, so Lemma 2.27 shows that $[Au]_k \in W^{1,2}(\mathbb{R}, F)$ with weak derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}[Au]_k = \sum_{l=0}^{k} \binom{k}{l} \big[A^{(k+1-l)}u_l + A^{(k-l)}u_{l+1}\big] = \sum_{l=0}^{k+1} \underbrace{\Big[\binom{k}{l} + \binom{k}{l-1}\Big]}_{\binom{k+1}{l}} A^{(k+1-l)}u_l = [Au]_{k+1}$$

The injectivity of $p_0 : \widehat{W}^{r,2}(\mathbb{R}, F) \longrightarrow L^2(\mathbb{R}, F)$ ensures that $\widehat{W}^{r,2}(A)$ is the unique map lifting $A : L^2(\mathbb{R}, E) \longrightarrow L^2(\mathbb{R}, F)$. $\square$

Observe that according to our proof of Lemma 2.28, the operator norm of $A : W^{r,2}(\mathbb{R}, E) \longrightarrow W^{r,2}(\mathbb{R}, F)$ is bounded by

$$2^{r+1} \cdot \sum_{m=0}^{r} \sup_{t\in\mathbb{R}} \big\| A^{(m)}(t) \big\|_{\mathcal{L}(E,\,F)}$$

Typically, we will work with smooth families $A(t)$ to induce a bounded linear map at every regularity level $r \geq 0$:

Corollary 2.29
Assume that $A : \mathbb{R} \longrightarrow \mathcal{L}(E,F)$ is smooth with bounded derivatives, i.e. all derivatives $A^{(m)}(t) \in \mathcal{L}(E,F)$, $m \geq 0$ exist and satisfy $\sup_{t\in\mathbb{R}} \left\|A^{(m)}(t)\right\|_{\mathcal{L}(E,F)} < \infty$
Then A defines an sc-operator

$$A : \left(W^{r,2}(\mathbb{R},E)\right)_{r\geq 0} \longrightarrow \left(W^{r,2}(\mathbb{R},F)\right)_{r\geq 0}$$

Proof. Given $u \in W^{r,2}(\mathbb{R},E)$ we can write $u = p_0(\hat{u})$ with a unique $\hat{u} \in \widehat{W}^{r,2}(\mathbb{R},E)$ and the norm satisfies $\left\|u\right\|_{W^{r,2}(\mathbb{R},E)} = \left\|\hat{u}\right\|_{\widehat{W}^{r,2}(\mathbb{R},E)}$.
The commutative diagram from Lemma 2.28 shows that $Au \in W^{r,2}(\mathbb{R},F)$ with

$$\left\|Au\right\|_{W^{r,2}(\mathbb{R},F)} = \left\|A\, p_0\, \hat{u}\right\|_{W^{r,2}(\mathbb{R},F)} = \left\|p_0\, \widehat{W}(A)\, \hat{u}\right\|_{W^{r,2}(\mathbb{R},F)} = \left\|\widehat{W}(A)\, \hat{u}\right\|_{\widehat{W}^{r,2}(\mathbb{R},F)} \leq \left\|\widehat{W}(A)\right\| \cdot \left\|u\right\|_{W^{r,2}(\mathbb{R},E)}$$

so we conclude that $A : L^2(\mathbb{R},E) \longrightarrow L^2(\mathbb{R},F)$ restricts to a bounded linear map from $W^{r,2}(\mathbb{R},E)$ to $W^{r,2}(\mathbb{R},F)$. □

2.4.2 Construction of the bifiltration W_k^r and nested Sobolev spaces $\mathcal{W}_n$

Having worked with a fixed pair of Banach spaces E and F, we now turn to the "bifiltration" $W_k^r = W^{r,2}(\mathbb{R}, W_k)$ of Sobolev spaces associated to an almost sc-Banach space $W_0 \supset W_1 \supset \dots$ Given an arbitrary subset $\mathcal{S} \subset \mathbb{N}^2$, we describe a general recipe to regard the intersection

$$\mathcal{W}_{\mathcal{S}} := \bigcap_{(k,r)\in\mathcal{S}} W_k^r$$

as a Banach space in its own right. Our recipe consists in taking successive pullbacks in the category $\mathcal{B} = \big[$Banach spaces, bounded linear maps$\big]$. In fact, the following special case will be sufficient for our purposes:

Auxiliary Lemma 2.30 (Intersection of Banach spaces)
Given bounded linear inclusions of Banach spaces $X \hookrightarrow \mathcal{H} \hookleftarrow Y$ equip $X \cap Y$ with the norm $\|w\|_{X\cap Y} := \|w\|_X + \|w\|_Y$. Then $(X\cap Y, \|\cdot\|_{X\cap Y})$ is a Banach space with bounded linear inclusions $X \hookleftarrow X \cap Y \hookrightarrow Y$

Proof. As the inclusions $X \cap Y \hookrightarrow X$ and $X \cap Y \hookrightarrow Y$ are bounded linear, every Cauchy sequence $w_n \in X \cap Y$ gets mapped to Cauchy sequences $w_n \in X$ and $w_n \in Y$, which by completeness of X and Y converge to limits $x \in X$ and $y \in Y$, respectively. Since the inclusions $X \hookrightarrow \mathcal{H}$ and $Y \hookrightarrow \mathcal{H}$ are bounded linear, we observe that $x = \lim_{n\to\infty} w_n = y$ so the limits agree and belong to $X \cap Y$. Going back to the norm $\|\cdot\|_{X\cap Y}$ we have

$$\|w_n - x\|_{X\cap Y} = \|w_n - x\|_X + \|w_n - y\|_Y \longrightarrow 0$$

□

Remark 2.31 (Pullbacks in $\mathcal{B}$)
By adapting the proof of Auxiliary Lemma 2.30 one can show that the Cartesian pullback of any two bounded linear maps $X \xrightarrow{\alpha} Z \xleftarrow{\beta} Y$ is represented by the closed subspace

$$\ker\left(\alpha \circ \mathrm{pr}_0 - \beta \circ \mathrm{pr}_1\right) \subset X \oplus Y$$

This explains our choice of the norm $\|\cdot\|_X + \|\cdot\|_Y$

Now we are ready to summarize our basic setup:

Construction (Bifiltration of Banach-space-valued Sobolev spaces)

- Given an almost sc-Banach space $H \supset W_1 \supset \ldots \supset W_k \supset \ldots$
  the subspaces $W^{r,2}(\mathbb{R}, W_k) \subset L^2(\mathbb{R}, W_k) \subset L^2(\mathbb{R}, H)$ organise into a bifiltration
  $$W_k^r := W^{r,2}(\mathbb{R}, W_k)$$
  with $W_{k'}^{r'} \subset W_k^r$ whenever $r' \geq r$ and $k' \geq k$.
- Given a finite set $\mathcal{S} \subset \mathbb{N}^2$ Auxiliary Lemma 2.30 shows that
  $$\mathcal{W}_{\mathcal{S}} := \bigcap_{(k,r)\in\mathcal{S}} W^{r,2}(\mathbb{R}, W_k)$$
  is a Banach space with norm $\|w\|_{\mathcal{S}} := \sum_{(k,r)\in\mathcal{S}} \|w\|_{W^{r,2}(\mathbb{R},W_k)}$
- Of particular interest are the diagonals
  $$\mathcal{W}_n := \bigcap_{\substack{(k,r)\in\mathbb{N}^2 \\ k+r=n}} W^{r,2}(\mathbb{R}, W_k) \;=\; L^2(\mathbb{R}, W_n) \cap W^{1,2}(\mathbb{R}, W_{n-1}) \cap \ldots \cap W^{n,2}(\mathbb{R}, H)$$
- The lowest levels read
  $$\mathcal{H} := L^2(\mathbb{R}, H) \qquad \mathcal{W}_1 = L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H) \qquad \mathcal{W}_2 = L^2(\mathbb{R}, W_2) \cap W^{1,2}(\mathbb{R}, W_1) \cap W^{2,2}(\mathbb{R}, H)$$

It is important to know how the maps $\frac{\mathrm{d}}{\mathrm{d}t}$ and $A = A_0 + B$ operate on the bifiltration W_k^r. As it turns out, this is question is easily settled by using our work from section 2.4.1:

Lemma 2.32 (Maps operating on the bifiltration)

1) *The map* $\frac{d}{dt} : W^{1,2}(\mathbb{R}, H) \longrightarrow L^2(\mathbb{R}, H)$ *satisfies* $\frac{d}{dt}\Big(W_k^{r+1}\Big) \subset W_k^r$
2) *Given a moderate family of sc-operators* $A(t)$: $(W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ *the map* $A : L^2(\mathbb{R}, W_1) \longrightarrow L^2(\mathbb{R}, H)$ *obeys* $A\Big(W_{k+1}^r\Big) \subset W_k^r$
3) *Given a moderate family of sc-operators* $B(t)$: $(W_k)_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ *the map* $B : L^2(H) \longrightarrow L^2(H)$ *preserves the bifiltration, i.e.* $B\Big(W_k^r\Big) \subset W_k^r$

Proof. Fix any $k \geq 0$. Statement (1) is a result of the commutative diagram

$$\begin{tikzcd} W^{r+1,2}(\mathbb{R}, W_k) \arrow[r, phantom, "\subset"] \arrow[d, "\frac{\mathrm{d}}{\mathrm{d}t}"] & W^{1,2}(\mathbb{R}, W_k) \arrow[r, phantom, "\subset"] \arrow[d, "\frac{\mathrm{d}}{\mathrm{d}t}"] & W^{1,2}(\mathbb{R}, H) \arrow[d, "\frac{\mathrm{d}}{\mathrm{d}t}"] \\ W^{r,2}(\mathbb{R}, W_k) \arrow[r, phantom, "\subset"] & L^2(\mathbb{R}, W_k) \arrow[r, phantom, "\subset"] & L^2(\mathbb{R}, H) \end{tikzcd}$$

Moreover, the maps $A : \mathbb{R} \longrightarrow \mathcal{L}(W_{k+1}, W_k)$ and $B : \mathbb{R} \longrightarrow \mathcal{L}(W_k)$ being smooth with bounded derivatives, statements (2) and (3) follow directly from Corollary 2.29. $\square$

Now that we know how its components operate, let us describe suitable domains for the operator D_A.

Notation. By $h, v : \mathbb{N}^2 \longrightarrow \mathbb{N}^2$ we mean the maps
$h : (k,r) \longmapsto (k+1, r)$ and $v : (k,r) \longmapsto (k, r+1)$

Lemma 2.33 (Flexible target)
Let $A(t)\colon \big(W_{k+1}\big)_{k\geq 0} \longrightarrow \big(W_k\big)_{k\geq 0}$ *be a moderate family of sc-operators.*
Then for every finite subset $\mathcal{S} \subset \mathbb{N}^2$ *the operator*

$$D_A = \frac{d}{dt} - A : \; L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H) \longrightarrow L^2(\mathbb{R}, H)$$

restricts to a bounded linear map $\;D_A : \mathcal{W}_{h(\mathcal{S}) \cup v(\mathcal{S})} \longrightarrow \mathcal{W}_{\mathcal{S}}$

Proof. Combining $\frac{\mathrm{d}}{\mathrm{d}t}\Big(W_k^{r+1}\Big) \subset W_k^r$ and $A\Big(W_{k+1}^r\Big) \subset W_k^r$ we have

$$D_A\Big(W_{k+1}^r \cap W_k^{r+1}\Big) \subset W_k^r$$

Regarding the norms we observe that $\left\|\frac{\mathrm{d}}{\mathrm{d}t} w\right\|_{W_k^r} \leq \|w\|_{W_k^{r+1}}$ and $\|Aw\|_{W_k^r} \leq \mathcal{C}_k^r \, \|w\|_{W_{k+1}^r}$
where the constants $\mathcal{C}_k^r > 0$ arise from Lemma 2.28
Thus, by writing $\mathcal{C}_{\mathcal{S}} := \max\limits_{(k,r)\in\mathcal{S}} \mathcal{C}_k^r$ we find

$$\|D_A w\|_{\mathcal{S}} = \sum_{(k,r)\in\mathcal{S}} \|D_A w\|_{W_k^r} \leq \sum_{(k,r)\in\mathcal{S}} \|w\|_{W_k^{r+1}} + \sum_{(k,r)\in\mathcal{S}} \mathcal{C}_k^r \, \|w\|_{W_{k+1}^r} \leq \|w\|_{v(\mathcal{S})} + \mathcal{C}_{\mathcal{S}} \, \|w\|_{h(\mathcal{S})}$$

□

Iterating the prescription from Lemma 2.33 with $\mathcal{S}_0 = L^2(\mathbb{R}, H)$ and $\mathcal{S}_{n+1} = h(\mathcal{S}_n) \cup v(\mathcal{S}_n)$, we arrive at the following conclusion:

Corollary 2.34 (D_A as a sc-operator)
The map $D_A = \dfrac{d}{dt} - A : \; L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H) \longrightarrow L^2(\mathbb{R}, H)$ *defines a sc-operator*

$$D_A : \big(\mathcal{W}_{n+1}\big)_{n\geq 0} \longrightarrow \big(\mathcal{W}_n\big)_{n\geq 0}$$

Proof. Let $\mathcal{D}_n := \big\{(k,r) \in \mathbb{N}^2 \,\big|\, k + r = n\big\}$ denote the n-th diagonal.
Then $h(\mathcal{D}_n) \cup v(\mathcal{D}_n) = \mathcal{D}_{n+1}$ so Lemma 2.33 shows that $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ restricts to bounded linear maps $D_A \in \mathcal{L}(\mathcal{W}_{n+1}, \mathcal{W}_n)$. □

2.5 Further properties of D_A

2.5.1 ... in the case of a moderate perturbation

Having identified $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ as a sc-operator, one may ask whether it is also regularizing. In the case where our family $A(t)$ is replaced by a constant baseline operator A_0, this question can be answered by combining Lemmas 2.19 and 2.20:

Proposition 2.35 ($D_{A_0} : \mathcal{W}^1 \longrightarrow \mathcal{W}$ is regularizing)
Let A_0 *be a baseline operator on an almost sc-Hilbert space* $H \supset W_1 \supset \ldots \supset W_k \supset \ldots$
Then the operator $D_{A_0} = \frac{d}{dt} - A_0 : \mathcal{W}_1 \longrightarrow \mathcal{H}$ *satisfies* $D_{A_0}^{-1}\big(\mathcal{W}_n\big) = \mathcal{W}_{n+1}$

Proof of Proposition 2.35.
By Auxiliary Lemma 2.12 it suffices to verify that D_{A_0} is an escalator for $(\mathcal{W}_n)_{n\geq 0}$.
Thus, given $u \in \mathcal{W}_n$ with $D_{A_0}u \in \mathcal{W}_n$ we have to show that $u \in \mathcal{W}_{n+1}$
Step 1 (Vertical Shift). At every $k = 0, ..., n-1$ we argue as follows:
The property $u \in \mathcal{W}_n \subset L^2(\mathbb{R}, W_n) \cap W^{1,2}(\mathbb{R}, W_{n-1})$ can be weakened to saying that

$$u \in L^2(\mathbb{R}, W_{k+1}) \cap W^{1,2}(\mathbb{R}, W_k)$$

Meanwhile, we have $D_{A_0}u \in \mathcal{W}_n \subset W^{n-k,2}(\mathbb{R}, W_k)$ so Vertical Regularization (Lemma 2.20) implies that $u \in W^{n-k,2}(\mathbb{R}, W_{k+1}) \cap W^{n-k+1,2}(\mathbb{R}, W_k)$.
By repeating this argument for all $k = 0, .., n-1$ we arrive at

$$u \in \bigcap_{k=0}^{n} W^{n+1-k,2}(\mathbb{R}, W_k)$$

but it still remains to show $u \in L^2(\mathbb{R}, W_{n+1})$
Step 2 (Horizontal Extension).
By Step 1 we have $u \in W^{1,2}(\mathbb{R}, W_n)$ so the weak derivative $\dot{u}$ belongs to $L^2(\mathbb{R}, W_n)$. Combined with our assumption that $D_{A_0}u \in \mathcal{W}_n \subset L^2(\mathbb{R}, W_n)$, this implies

$$A_0 u = \dot{u} - D_{A_0}u \in L^2(\mathbb{R}, W_n)$$

so Horizontal Regularization (Lemma 2.19) yields $u \in L^2(\mathbb{R}, W_{n+1})$ and we are done. □

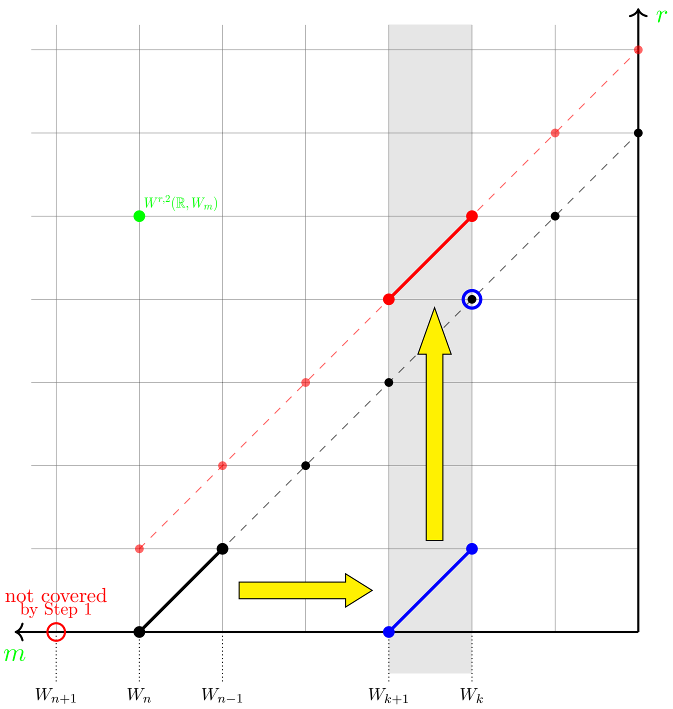


Figure 2.1: Illustration of Proposition 2.35. To any collection of points $\mathcal{S} \subset \mathbb{N}^2$ we associate an intersection of subsets
$$\bigcap_{(m,r)\in\mathcal{S}} W^{r,2}(\mathbb{R}, W_m) \subset L^2(\mathbb{R}, H).$$

Our claim $\mathcal{W}_n \cap D_{A_0}^{-1}(\mathcal{W}_n) \subset \mathcal{W}_{n+1}$ is proven by a combination of 'Vertical Regularization'
$$D_{A_0}^{-1}(W_k^r)|_{W_{k+1}^0 \cap W_k^1} \subset W_{k+1}^r \cap W_k^{r+1} \quad \text{for } k = 0, ..., n-1$$

and 'Horizontal Regularization' $A_0^{-1}(W_n^0) \subset W_{n+1}^0$.

The result of Proposition 2.35 carries over to time-dependent operator families $A(t)=A_0+B(t)$, provided that $B(t)\in\mathcal{L}(H)$ is a moderate perturbation in the sense of Definition 2.21:

Proposition 2.36 ($D_A:\mathcal{W}^1\longrightarrow\mathcal{W}$ is regularizing)
Assume that $A(t)\colon W_1\longrightarrow H$ arises from a moderate perturbation of our baseline operator A_0. Then the modified operator $D_A=\frac{d}{dt}-A:\mathcal{W}_1\longrightarrow\mathcal{H}$ still satisfies $D_A^{-1}\big(\mathcal{W}_n\big)=\mathcal{W}_{n+1}$

Proof. The perturbation $A(t)=A_0+B(t)$ leads us to decompose $D_A=D_{A_0}-B$. In Proposition 2.35 we have shown that D_{A_0} is regularizing, i.e. $D_{A_0}^{-1}\big(\mathcal{W}_n\big)=\mathcal{W}_{n+1}$. On the other hand, Lemma 2.32(3) guarantees that $B\in\mathcal{L}(H)$ preserves the filtration in the sense that $B\big(\mathcal{W}_n\big)\subset\mathcal{W}_n$. Thus, by Auxiliary Lemma 2.13 it cannot alter our conclusion that D_A is regularizing. □

2.5.2 ... in the case of a symmetric perturbation

Let A_0 be a baseline operator on an almost sc-Hilbert space $H\supset W{=}W_1\supset\dots$ and consider $A(t)=A_0+B(t)\in\mathcal{L}(W,H)$ in the case where $B(t)\in\mathcal{L}(H)$ is a *symmetric* perturbation. Then we have the following result which is merely a reformulation of [RS] Thm. 3.10 (Elliptic Regularity):

Lemma 2.37 ($D^*_{\text{-}A}=-D_A$)
The operators $D_{\text{-}A}:\mathcal{W}_1\longrightarrow\mathcal{H}$ and $-D_A:\mathcal{W}_1\longrightarrow\mathcal{H}$ are mutually adjoint.

Proof. Recall that the adjoint of an unbounded operator $D_{\text{-}A}$ is defined on the domain

$$\mathcal{D}(D^*_{\text{-}A}):=\big\{\xi\in\mathcal{H}\,\big|\,\langle\xi,D_{\text{-}A}\cdot\rangle_{\mathcal{H}}\text{ is a bounded functional on }(\mathcal{W}_1,\|\cdot\|_{\mathcal{H}})\big\}$$

We claim that

$$\mathcal{D}(D^*_{\text{-}A})\underset{(1)}{\subset}\big\{\xi\in\mathcal{H}\,\big|\,\exists\eta\in\mathcal{H}:\langle\xi,D_{\text{-}A}\phi\rangle_{\mathcal{H}}+\langle\eta,\phi\rangle_{\mathcal{H}}=0\ \forall\phi\in C_0^\infty(\mathbb{R},W)\big\}\underset{(2)}{\subset}\mathcal{W}_1\underset{(3)}{\subset}\mathcal{D}(D^*_{\text{-}A})$$

To verify inclusion (1) pick any $\xi\in\mathcal{D}(D^*_{\text{-}A})$. Since $\langle\xi,D_{\text{-}A}\cdot\rangle_{\mathcal{H}}$ is a bounded functional on $(\mathcal{W}_1,\|\cdot\|_{\mathcal{H}})$, the Hahn-Banach theorem shows that there exists an extension $\lambda\in\mathcal{H}^*$ with $\lambda|_{\mathcal{W}_1}=\langle\xi,D_{\text{-}A}\cdot\rangle_{\mathcal{H}}$. By the Riesz represention theorem we can find $\eta\in\mathcal{H}$ such that $\lambda=-\langle\eta,\cdot\rangle_{\mathcal{H}}$. Thus, for any $\phi\in C_0^\infty(\mathbb{R},W)\subset\mathcal{W}_1$ we have $\langle\xi,D_{\text{-}A}\phi\rangle_{\mathcal{H}}+\langle\eta,\phi\rangle_{\mathcal{H}}=0$.

Inclusion (2) is the statement of [RS] Thm. 3.10 (Elliptic regularity).

Regarding inclusion (3) let us first establish a partial integration formula for weak derivatives. Given $\xi,\rho\in\mathcal{W}_1$ we can find approximating sequences $\phi_n,\psi_n\in C_0^\infty(\mathbb{R},H)$ such that $\phi_n\to\xi$ and $\psi_n\to\rho$ in $W^{1,2}(\mathbb{R},H)$. At fixed $n\in\mathbb{N}$ the functions ϕ_n and ψ_n are smooth and compactly supported, so the Fundamental Theorem of Calculus gives

$$0=\int\frac{\mathrm{d}}{\mathrm{d}t}\langle\phi_n,\psi_n\rangle_H=\int\langle\phi_n',\psi_n\rangle_H+\int\langle\phi_n,\psi_n'\rangle_H$$

On the other hand, using Young's inequality we can identify the limit as

$$\langle\xi',\rho\rangle_{\mathcal{H}}+\langle\xi,\rho'\rangle_{\mathcal{H}}=\lim_{n\to\infty}\int\langle\phi_n',\psi_n\rangle_H+\int\langle\phi_n,\psi_n'\rangle_H=0$$

With this done, recall that $A(t):W\longrightarrow H$ is a symmetric operator at every $t\in\mathbb{R}$, so given $\xi(t),\ \rho(t)\in W$ we obtain $\big\langle\xi(t),A(t)\rho(t)\big\rangle_H=\big\langle A(t)\xi(t),\rho(t)\big\rangle_H$.
Taking both ingredients together we conclude that for $\xi,\rho\in\mathcal{W}_1$ one has

$$\langle\,D_A\xi,\rho\,\rangle_{\mathcal{H}}+\langle\,\xi,D_{\text{-}A}\rho\,\rangle_{\mathcal{H}}\;=\;0\tag{2.4}$$

In particular, $\langle\xi,D_{\text{-}A}\cdot\rangle_{\mathcal{H}}=-\langle D_A\xi,\cdot\rangle_{\mathcal{H}}$ is a bounded functional on $(\mathcal{W}_1,\|\cdot\|_{\mathcal{H}})$ so $\xi\in\mathcal{D}(D^*_{\text{-}A})$.

Now that we have shown $\mathcal{D}(D^*_{\text{-}A})=\mathcal{W}_1$, note that according to formula (2.4) the adjoint of $D_{\text{-}A}$ is given by $D^*_{\text{-}A}=-D_A$. This choice is unique because $\mathcal{W}_1$ is dense in $\mathcal{H}$. □

2.6 The sc-Fredholm property of $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$

Building on our observations in sections 2.5.1 and 2.5.2, the sc-Fredholm property of D_A arises from a nice general pattern:

Theorem 2.38 (Criterion for being sc-Fredholm)
Let $\mathcal{H} \supset \mathcal{W}_1 \supset \ldots \supset \mathcal{W}_n \supset \ldots$ be an almost sc-Hilbert space.
Assume we are given regularizing sc-operators $\varphi_\pm : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ such that at the lowest level $n = 0$ the operators $\varphi_\pm : \mathcal{W}_1 \longrightarrow \mathcal{H}$ are Fredholm and mutually adjoint.
Then the operators $\varphi_\pm$ are sc-Fredholm.

Proof of Theorem 2.38.
Let us begin by the following observation, which works for φ_+ and φ_- individually:

- Since $\varphi_\pm$ is regularizing, we have $\ker \varphi_\pm = \varphi_\pm^{-1}(0) \subset \bigcap_{n\geq 0} \mathcal{W}_n$
- Moreover, $\varphi_\pm : \mathcal{W}_1 \longrightarrow \mathcal{H}$ being Fredholm ensures that $\ker \varphi_\pm$ is finite-dimensional and $\varphi_\pm(\mathcal{W}_1) \subset \mathcal{H}$ is a closed subspace.

Next let us study the interplay of φ_+ and φ_-:
As the operators $\varphi_\pm : \mathcal{W}_1 \longrightarrow \mathcal{H}$ are mutually adjoint, we have $\varphi_\pm(\mathcal{W}_1)^\perp = \ker \varphi_\mp$ so with $\varphi_\pm(\mathcal{W}_1) \subset \mathcal{H}$ being closed subspaces there are orthogonal decompositions

$$\mathcal{H} = \varphi_\pm(\mathcal{W}_1) \oplus \ker \varphi_\mp$$

As mentioned above, $\ker \varphi_\pm$ is contained in every $\mathcal{W}_n$ and therefore taking the intersection with $\mathcal{W}_n$ yields decompositions

$$\mathcal{W}_n = \big[\mathcal{W}_n \cap \varphi_\pm(\mathcal{W}_1)\big] \oplus \ker \varphi_\mp = \varphi_\pm(\mathcal{W}_{n+1}) \oplus \ker \varphi_\mp \tag{2.5}$$

where for the second equality we have used that $\varphi_\pm^{-1}(\mathcal{W}_n) = \mathcal{W}_{n+1}$.
Corollary 2.6 shows that $X_n := \varphi_-(\mathcal{W}_{n+1})$ and $Y_n := \varphi_+(\mathcal{W}_{n+1})$ are honest sc-subspaces of $\mathcal{W} = (\mathcal{W}_n)_{n\geq 0}$. Setting $K := \ker \varphi_+$ and $C := \ker \varphi_-$ we have ticked all boxes to ensure that $\varphi_+ : \mathcal{W}^1 \longrightarrow \mathcal{W}$ is sc-Fredholm in the sense of Definition 2.7. By symmetry of our construction $\varphi_- : \mathcal{W}^1 \longrightarrow \mathcal{W}$ is sc-Fredholm as well. $\square$

Remark 2.39 (Double Helix I)
As an important takeaway from the proof of Theorem 2.38, note that, according to the decompositions (2.5), our maps $\varphi_\pm$ give rise to a "double helix" of isomorphisms

$$\begin{tikzcd}
\ker\varphi_+ \oplus \varphi_-(\mathcal{W}_1) \arrow[d, dotted, no head] & = & \mathcal{H} & = & \varphi_+(\mathcal{W}_1) \oplus \ker\varphi_- \arrow[d, dotted, no head] \\
\ker\varphi_+ \oplus \varphi_-(\mathcal{W}_2) \arrow[urrrr, "\varphi_+"] \arrow[d, dotted, no head] & = & \mathcal{W}_1 & = & \varphi_+(\mathcal{W}_2) \oplus \ker\varphi_- \arrow[ullll, red, "\varphi_-"'] \arrow[d, dotted, no head] \\
\ker\varphi_+ \oplus \varphi_-(\mathcal{W}_3) \arrow[urrrr, "\varphi_+"] \arrow[d, dotted, no head] & = & \mathcal{W}_2 & = & \varphi_+(\mathcal{W}_3) \oplus \ker\varphi_- \arrow[ullll, red, "\varphi_-"'] \arrow[d, dotted, no head] \\
\ker\varphi_+ \oplus \varphi_-(\mathcal{W}_4) \arrow[urrrr, "\varphi_+"] \arrow[d, dotted, no head] & = & \mathcal{W}_3 \arrow[d, dotted, no head] & = & \varphi_+(\mathcal{W}_4) \oplus \ker\varphi_- \arrow[ullll, red, "\varphi_-"'] \arrow[d, dotted, no head] \\
{} & & {} & & {}
\end{tikzcd}$$

This picture will be revisited in Remark 3.16 and provides guidance for Theorem 4.15.

Let us put all pieces together and use Theorem 2.38 to exhibit $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ as a sc-Fredholm operator between almost sc-Banach spaces:

Theorem 2.40 (D_A as an sc-Fredholm operator in the unweighted case)
Let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an honest sc-Hilbert space $H \supset W_1 \supset \ldots \supset W_k \supset \ldots$ Assume that $B(t) \in \mathcal{L}(\mathcal{H})$ is a good perturbation (moderate, symmetric, endpoint-regular) and write $A(t) = A_0 + B(t)$.

Then $D_{\pm A} : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ are sc-Fredholm operators between almost sc-Banach spaces.

Proof. All we have to do is verify that the pair "$\varphi_\pm$" $= \mp D_{\pm A}$ satisfies the conditions of Theorem 2.38: With $A(t) : (W_{k+1})_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ being a moderate family of sc-operators, Corollary 2.34 ensures that also $D_{\pm A} : (\mathcal{W}_{n+1})_{\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ are sc-operators. Since more specifically $A(t) = A_0 + B(t)$ is a moderate perturbation of a baseline operator, we have seen in Proposition 2.36 that $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ and similarly $D_{\text{-}A}$ is regularizing. Moreover, we have observed in Lemma 2.37 that for a symmetric perturbation $B(t)$ the operators $D_{\text{-}A}$ and $-D_A$ are mutually adjoint.

Hence, it remains to explain why $D_{\pm A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$ are Fredholm operators.
This, however, is the main statement of the paper [RS] (see [RS] Thm. A or [RS] Thm. 3.12) and requires $A(t) = A_0 + B(t) \in \mathcal{L}(W_1, H)$ to be a symmetric, endpoint-regular perturbation of A_0 as well as our assumption that $W_1 \hookrightarrow H$ is a compact inclusion.[2] □

Corollary 2.41 (Classical Fredholm property at every level)
The assumptions of Theorem 2.40 ensure that
$D_{\pm A} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ is a Fredholm operator at every level $n \geq 0$.

Proof. Combine Theorem 2.40 with Remark 2.8. □

[2]Compactness of the inclusion $W_1 \hookrightarrow H$ enters [RS] Lem. 3.8 which by the inequality

$$\|\xi\|_{\mathcal{W}_1} \leq \text{const.} \times \left[\, \|\xi\|_{\mathcal{H}(T)} + \|D_A\xi\|_{\mathcal{H}} \,\right] \tag{2.6}$$

from [RS] Lem. 3.9 makes it possible to apply the 'Abstract Closed Range Lemma' [RS] Lem. 3.7
Note that the inequality (2.6) is the only step in the proof of [RS] Thm. 3.12 that requires invertible endpoints

$$A_\pm \in \mathcal{L}(W, H)$$

2.7 $(\mathcal{W}_n)_{n\geq 0}$ as an almost sc-Banach space

Let $H \supset W_1 \supset \ldots$ be an almost sc-Banach space. Then the following result confirms that the "suspension"
$$\mathcal{W}_n = \bigcap_{r+k=n} W^{r,2}(\mathbb{R}, W_k)$$
constructed in section 2.4.2 is indeed an almost sc-Banach space (resp. almost sc-Hilbert space if H was a Hilbert space). What we will prove is actually a bit stronger and will therefore become important in section 4.1:

Proposition 2.42 (Density axiom for $(\mathcal{W}_n)_{n\geq 0}$)
The space $C_0^\infty(\mathbb{R})W_\infty \subset \mathcal{W}_\infty$ consisting of finite sums $\sum_i' w_i f_i(\cdot)$ with $f_i \in C_0^\infty(\mathbb{R})$ and $w_i \in W_\infty$ is dense in every $\mathcal{W}_n$

Proof. As a warm-up let us prove that $\mathcal{S} := C_0^\infty(\mathbb{R})W_\infty$ is dense in every $W_k^r = W^{r,2}(\mathbb{R}, W_k)$. To approximate a given $u \in W_k^r$ pick a Dirac sequence $\varphi_\delta \in C_0^\infty(\mathbb{R})$. Given a prescribed accuracy $\epsilon > 0$, we fix δ such that
$$\left\|\varphi_\delta * u - u\right\|_{W^{r,2}(\mathbb{R},W_k)} < \epsilon/3$$
Now that δ has been chosen, the expression $\kappa := \sum_{l=0}^{r} \|\varphi_\delta^{(l)}\|_{L^1(\mathbb{R})}$ can be treated as a constant. Auxiliary Lemma 2.43 shows that for any $w \in L^2(\mathbb{R}, W_k)$ we have $\varphi_\delta * w \in W^{r,2}(\mathbb{R}, W_k)$ with
$$\left\|\varphi_\delta * w\right\|_{W^{r,2}(\mathbb{R},W_k)} = \sum_{l=0}^{r} \left\|\varphi_\delta^{(l)} * w\right\|_{L^2(\mathbb{R},W_k)} \leq \kappa \left\|w\right\|_{L^2(\mathbb{R},W_k)}$$
In our case, u can be approximated by a step function $v = \sum_{i=1}^{m} v_i \chi_{I_i}$ with $v_i \in W_k$ and $\overline{I_i}$ compact such that $\left\|v - u\right\|_{L^2(\mathbb{R},W_k)} < \epsilon/3\kappa$. Moreover, since W_∞ is dense in W_k, we can consider a modified step function $\tilde{v} = \sum_{i=1}^{m} \tilde{v}_i \chi_{I_i}$ with $\tilde{v}_i \in W_\infty$ such that $\left\|\tilde{v}_i - v_i\right\|_{W_k}^2 < \frac{(\epsilon/3\kappa)^2}{m^2 \cdot |I_i|}$ and therefore $\|\tilde{v} - v\|_{L^2(\mathbb{R},W_k)} < \epsilon/3\kappa$.
Note that $\varphi_\delta * \tilde{v} \in C_0^\infty(\mathbb{R})W_\infty$ and
$$\begin{aligned}\left\|\varphi_\delta * \tilde{v} - u\right\|_{W^{r,2}(\mathbb{R},W_k)} &\leq \left\|\varphi_\delta * [\tilde{v} - v]\right\|_{W^{r,2}(\mathbb{R},W_k)} + \left\|\varphi_\delta * [v - u]\right\|_{W^{r,2}(\mathbb{R},W_k)} + \left\|\varphi_\delta * u - u\right\|_{W^{r,2}(\mathbb{R},W_k)} \\ &\leq \kappa \left\|\tilde{v} - v\right\|_{L^2(\mathbb{R},W_k)} + \kappa \left\|v - u\right\|_{L^2(\mathbb{R},W_k)} + \left\|\varphi_\delta * u - u\right\|_{W^{r,2}(\mathbb{R},W_k)} < \epsilon\end{aligned}$$
To prove that $\mathcal{S} = C_0^\infty(\mathbb{R})W_\infty$ is dense in every $\mathcal{W}_n$ let us repeat the above proof for $u \in L^2(\mathbb{R}, W_n) \cap W^{1,2}(\mathbb{R}, W_{n-1}) \cap \ldots \cap W^{n,2}(\mathbb{R}, H)$. This time we fix δ such that
$$\left\|\varphi_\delta * u - u\right\|_{W^{r,2}(\mathbb{R},W_{n-r})} < \epsilon/3 \qquad \text{for all } r = 0, ..., n$$
and define $\kappa := \sum_{l=0}^{n} \|\varphi_\delta^{(l)}\|_{L^1(\mathbb{R})}$. Then as before we choose a step function $v = \sum_i' v_i \chi_{I_i}$ with $v_i \in W_n$ such that $\left\|v - u\right\|_{L^2(\mathbb{R},W_n)} < \epsilon/3\kappa$ and therefore automatically $\left\|v - u\right\|_{L^2(\mathbb{R},W_{n-r})} < \epsilon/3\kappa$ for all $r = 0, ..., n$. Since W_∞ is dense in W_n, we can find a modified step function $\tilde{v} = \sum_i' \tilde{v}_i \chi_{I_i}$ with $\tilde{v}_i \in W_\infty$ such that $\left\|\tilde{v} - v\right\|_{L^2(\mathbb{R},W_n)} < \epsilon/3\kappa$ and hence $\left\|\tilde{v} - v\right\|_{L^2(\mathbb{R},W_{n-r})} < \epsilon/3\kappa$ for all $r = 0, ..., n$.
With these adaptions we see that
$$\left\|\varphi_\delta * \tilde{v} - u\right\|_{W^{r,2}(\mathbb{R},W_{n-r})} \leq \kappa\left\|\tilde{v} - v\right\|_{L^2(\mathbb{R},W_{n-r})} + \kappa\left\|v - u\right\|_{L^2(\mathbb{R},W_{n-r})} + \left\|\varphi_\delta * u - u\right\|_{W^{r,2}(\mathbb{R},W_{n-r})} < \epsilon$$
holds simultaneously for all $r = 0, ..., n$. Thus, we have found $\varphi_\delta * \tilde{v} \in C_0^\infty(\mathbb{R})W_\infty$ with
$$\left\|\varphi_\delta * \tilde{v} - u\right\|_{\mathcal{W}_n} < (n+1)\,\epsilon$$
□

Auxiliary Lemma 2.43 (Convolution inequality)
Let E be a Banach space. Then the convolution of $\varphi \in C_0^\infty(\mathbb{R})$ and $u \in L^2(\mathbb{R}, E)$ satisfies

$$\left\|\varphi * u\right\|_{L^2(\mathbb{R},E)} \leq \left\|\varphi\right\|_{L^1(\mathbb{R})} \left\|u\right\|_{L^2(\mathbb{R},E)}$$

Proof. The pointwise estimate

$$\left\|\varphi * u(x)\right\|_E \leq \int \mathrm{d}y \, |\varphi(x-y)|^{1/2} \, |\varphi(x-y)|^{1/2} \left\|u(y)\right\|_E \leq \left[\int \mathrm{d}y \, |\varphi(x-y)|\right]^{1/2} \left[\int \mathrm{d}y \, |\varphi(x-y)| \cdot \left\|u(y)\right\|_E^2\right]^{1/2}$$

shows that

$$\left\|\varphi * u(x)\right\|_E^2 \leq \left\|\varphi\right\|_{L^1(\mathbb{R})} \int \mathrm{d}y \; |\varphi(x-y)| \cdot \left\|u(y)\right\|_E^2$$

so by Tonelli's theorem we obtain

$$\int \mathrm{d}x \left\|\varphi * u(x)\right\|_E^2 \leq \left\|\varphi\right\|_{L^1(\mathbb{R})}^2 \left\|u\right\|_{L^2(\mathbb{R},E)}^2$$

and therefore $\left\|\varphi * u\right\|_{L^2(\mathbb{R},E)} \leq \left\|\varphi\right\|_{L^1(\mathbb{R})} \left\|u\right\|_{L^2(\mathbb{R},E)}$. □

Let us conclude this chapter by a simple argument showing that unboundedness of the domain $I = \mathbb{R}$ poses an obstruction to $\mathcal{W}_1 \hookrightarrow \mathcal{H}$ being compact. Thus, $(\mathcal{W}_n)_{n\geq 0}$ may be an almost sc-Banach space, but it fails to be an honest sc-Banach space, by lack of compact inclusions:

Lemma 2.44 (Escape argument)
The inclusion operator $L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H) \hookrightarrow L^2(\mathbb{R}, H)$ is non-compact.

Proof. Consider a sequence of bump functions $\phi_n = \phi(\cdot - n) \in C_0^\infty(\mathbb{R}, W_1)$ escaping to infinity. The shift map being an isometry, this sequence remains bounded in $L^2(\mathbb{R}, W_1) \cap W^{1,2}(\mathbb{R}, H)$. However, since for N large enough ϕ_n and ϕ_{n+N} have disjoint support, we cannot find a Cauchy subsequence even in $L^2(\mathbb{R}, H)$. □

Before resolving this issue in Section 4.1, let us focus on the spectral-theoretic consequences of our findings so far. This will provide useful tools for our main effort in Chapter 4.

Chapter 3

Spectral Techniques

In this interlude chapter, we explore consequences of Corollary 2.41. This will give us the necessary tools to prove Theorem 4.17 as an analogon to Theorem 2.40 in the 'weighted' case.

From now on let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an honest sc-Hilbert space $H \supset W_1 \supset \dots$ Further assume that $B(t) \in \mathcal{L}(\mathcal{H})$ is a good perturbation (moderate, symmetric, endpoint-regular) and write $A(t) = A_0 + B(t)$. These assumptions ensure that all results from Chapter 2 apply at once.

3.1 The self-adjoint Fredholm operator $D_{-A}D_A : \mathcal{W}_2 \longrightarrow \mathcal{H}$

Before applying our own results from Chapter 2, let us explain how observations from [RS] lead to $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ being a closed operator. Note that closed operators are characterized by completeness of their graph norm:

Auxiliary Lemma 3.1 (Completeness of the graph norm)
Let $T : \mathcal{D}(T) \longrightarrow \mathcal{H}$ be an unbounded operator on a Hilbert space $\mathcal{H}$.
Then T is a closed operator if and only if its graph norm makes $\mathcal{D}(T)$ a Banach space.

Proof. Using the injective map $\mathcal{D}(T) \overset{\mathrm{id}_{\mathcal{H}}\oplus T}{\longrightarrow} \mathcal{H} \oplus \mathcal{H}$ to identify $\mathcal{D}(T)$ with a subspace $\mathrm{graph}(T) \subset \mathcal{H} \oplus \mathcal{H}$, the graph norm of T can be understood as pullback of $\|\cdot\|_{\mathcal{H}\oplus\mathcal{H}}$. Now T being a closed operator is synonymous to $\mathrm{graph}(T) \subset \mathcal{H}\oplus\mathcal{H}$ being a closed subspace, which again is equivalent to $\big(\mathrm{graph}(T), \|\cdot\|_{\mathcal{H}\oplus\mathcal{H}}\big)$ being complete. $\square$

Lemma 3.2 *$D_{\pm A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is a closed operator.*

Proof. All assumptions are invariant under "$A \to$ -A", so it suffices to consider D_{+A}.
By [RS] Lem 3.9 we can find a constant $c_0 > 0$ such that

$$c_0 \cdot \|\xi\|_{\mathcal{W}_1} \leq \|\xi\|_{\mathcal{H}} + \|D_A\xi\|_{\mathcal{H}} \qquad \forall\, \xi \in \mathcal{W}_1$$

Conversely, both D_A and the inclusion $\iota : \mathcal{W}_1 \hookrightarrow \mathcal{H}$ are bounded linear operators from $\mathcal{W}_1$ to $\mathcal{H}$, so with another constant $c_1 > 0$ we have

$$\|\xi\|_{\mathcal{H}} + \|D_A\xi\|_{\mathcal{H}} \leq c_1 \cdot \|\xi\|_{\mathcal{W}_1} \qquad \forall\, \xi \in \mathcal{W}_1$$

The combination of these inequalities means that the graph norm of D_A is equivalent to the norm $\|\cdot\|_{\mathcal{W}_1}$ already there, which by the constructions from Section 2.4.2 is known to be complete. Using Auxiliary Lemma 3.1 completeness of the graph norm can be rephrased as saying that "T" $= D_A$ is closed. $\square$

The following rather well-known result provides a recipe to build a self-adjoint operator from any closed operator:

Lemma 3.3 (von Neumann's theorem)
Let $T : \mathcal{D}(T) \longrightarrow \mathcal{H}$ be a densely defined, closed operator on a Hilbert space $\mathcal{H}$ and write

$$\mathcal{D}(T^*T) := T^{-1}(\mathcal{D}(T^*)) \subset \mathcal{D}(T)$$

Then $S = T^*T : \mathcal{D}(T^*T) \longrightarrow \mathcal{H}$ is self-adjoint and non-negative. Its kernel equals $\ker T$

Proof. For $x \in \mathcal{D}(S) = T^{-1}(\mathcal{D}(T^*)) \subset \mathcal{D}(T)$ one has

$$\langle x, Sx \rangle = \langle \underbrace{x}_{\mathcal{D}(T)}, T^* \underbrace{Tx}_{\mathcal{D}(T^*)} \rangle = \|Tx\|^2 \geq 0 \tag{3.1}$$

so S is non-negative. Note that every non-negative operator is symmetric because to prove that S is a symmetric operator it suffices to verify $\langle x, Sx \rangle \in \mathbb{R}$ for all $x \in \mathcal{D}(S)$.

Equation (3.1) shows that $Sx = 0$ implies $Tx = 0$, so we have $\ker(S) \subset \ker(T)$.
With the reverse inclusion trivially satisfied, we obtain $\ker(S) = \ker(T)$.

The statement about self-adjointness is known as "von Neumann's Theorem" and can be found in [Te] Problem 2.12 (p.73). As no proof is given there let us provide one here.

By [Te] Lemma 2.3 (p.63) it suffices to verify $\operatorname{ran}(S + 1) = \mathcal{H}$.
Since $T : \mathcal{D}(T) \to \mathcal{H}$ is a closed operator, the inner product

$$(\,\cdot, \cdot\,)_T := \langle T\cdot, T\,\cdot\, \rangle_{\mathcal{H}} + \langle \cdot, \cdot \rangle_{\mathcal{H}}$$

induces a complete norm on $\mathcal{D}(T)$ and gives $\mathcal{D}(T)$ itself the structure of a Hilbert space. Given any $z \in \mathcal{H}$ we observe that $\langle z, \cdot \rangle|_{\mathcal{D}(T)}$ is a bounded linear functional on $\big(\mathcal{D}(T), \|\cdot\|_T\big)$ so by the Riesz Representation Theorem there exists $\tilde{z} \in \mathcal{D}(T)$ with

$$\langle z, \cdot \rangle\big|_{\mathcal{D}(T)} = (\tilde{z}, \cdot)_T = \langle T\tilde{z}, T\,\cdot\, \rangle_{\mathcal{H}} + \langle \tilde{z}, \cdot \rangle_{\mathcal{H}} \tag{3.2}$$

This shows that $\langle T\tilde{z}, T\cdot \rangle_{\mathcal{H}} = \langle z - \tilde{z}, \cdot \rangle_{\mathcal{H}}$ is a bounded linear functional on $\big(\mathcal{D}(T), \|\cdot\|_{\mathcal{H}}\big)$ so we get $T\tilde{z} \in \mathcal{D}(T^*)$. Thus, we have $\tilde{z} \in \mathcal{D}(T^*T)$ and (3.2) can be rewritten as

$$\langle z, \cdot \rangle|_{\mathcal{D}(T)} = \langle (T^*T + 1)\tilde{z}, \cdot \rangle$$

As $\mathcal{D}(T)$ is dense in $\mathcal{H}$, this implies $z = (T^*T + 1)\tilde{z} \in \operatorname{ran}(S + 1)$ and we are done. $\square$

Remark. Our proof of Lemma 3.3 was inspired by the Friedrichs extension theorem as in [Te] Section 2.3 (p.67) and in a sense bypasses the construction of a Friedrichs extension to T^*T.

The above observations suggest that the operator $S = D_{-A}D_A$, while sharing features of D_A and D_{-A}, has better properties than the original D_A:

Theorem 3.4 (Characterisation of $S = D_{-A}D_A$)
The operator $S = D_{-A}D_A : \mathcal{W}_2 \longrightarrow \mathcal{H}$ is non-positive, self-adjoint and Fredholm.
It has the same kernel as D_A and the same range as D_{-A} .

Proof. Corollary 2.41 tells us that not only $D_{-A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$ but also $D_A : \mathcal{W}_2 \longrightarrow \mathcal{W}_1$ is Fredholm, so as the composition of Fredholm operators $D_{-A}D_A : \mathcal{W}_2 \longrightarrow \mathcal{H}$ is again Fredholm.

For the self-adjointness part let us apply Lemma 3.3 in the case "T" $= D_A$:
From Proposition 2.42 we know that $\mathcal{W}_1 \subset \mathcal{H}$ is a dense subspace and Lemma 3.2 verifies that $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is a closed operator. In Lemma 2.37 we have seen that the adjoint operator D_A^* is simply $-D_{-A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$, which exhibits $S = D_{-A}D_A : D_A^{-1}(\mathcal{W}_1) \longrightarrow \mathcal{H}$ as

minus the operator T^*T from Lemma 3.3 . Using Proposition 2.36 to identify the domain as

$$\mathcal{D}(T^*T) = D_A^{-1}\big(\mathcal{W}_1\big) = \mathcal{W}_2$$

we conclude that $D_{\text{-}A}D_A : \mathcal{W}_2 \longrightarrow \mathcal{H}$ is non-positive self-adjoint with kernel $\ker D_A$.

Last but not least, recall that our operators $\varphi_\pm = \pm D_{\pm A}$ satisfy the conditions of Theorem 2.38 so by the proof of that theorem $\mathcal{W}_1$ admits a decomposition

$$\mathcal{W}_1 = \big[\mathcal{W}_1 \cap D_A(\mathcal{W}_1)\big] \oplus \ker D_{\text{-}A} = D_A(\mathcal{W}_2) \oplus \ker D_{\text{-}A}$$

As a direct result, we obtain $D_{\text{-}A}(\mathcal{W}_1) = D_{\text{-}A}D_A(\mathcal{W}_2)$. □

3.2 The spectrum of self-adjoint Fredholm operators

In this section, we describe the conditions that the properties from Theorem 3.4 impose on the spectrum of $S = D_{\text{-}A}D_A$. The consequences of being non-positive and self-adjoint are well-known:

Lemma 3.5 *The spectrum of a self-adjoint, non-positive operator* $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ *satisfies*

$$\sigma(S) \subset (\text{-}\infty, 0]$$

Proof. See [Te] Theorem 2.19 (p.77) □

Exploiting the combination of 'Fredholm' and 'self-adjoint' requires a bit more work: The following statement is equivalent to [Wa] Lemma 2.2.5 (p.27). However, we give different, possibly more intuitive proof.

Lemma 3.6 (Isolated Origin)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ *be Fredholm and self-adjoint.*
Then we can find $\epsilon > 0$ *such that* $B_\epsilon(0) \cap \sigma(S) = \{0\} \cap \sigma_p(S)$.

Proof. Self-adjointness implies that S is closed. S being closed has the following advantage: When equipped with the graph norm of S the domain $\mathcal{D}(S)$ is a Banach space in its own right and the inclusion ι is a bounded map from $\mathcal{D}(S)$ to $\mathcal{H}$.

In particular, if for some $\lambda \in \mathbb{C}$ it turns out that $S - \lambda \in \mathcal{L}\left(\mathcal{D}(S), \mathcal{H}\right)$ is invertible, the Inverse Mapping Theorem guarantees that $(S-\lambda)^{-1} \in \mathcal{L}\left(\mathcal{H}, \mathcal{D}(S)\right) \subset \mathcal{L}(\mathcal{H})$ is bounded as well. This shows that the resolvent set can be described as

$$\sigma(S)^c = \{\lambda \in \mathbb{C} |\ S - \lambda : \mathcal{D}(S) \longrightarrow \mathcal{H} \text{ is invertible } \}$$

Taking into account that self-adjoint operators satisfy $\sigma(S) \subset \mathbb{R}$, the spectrum of our operator S becomes

$$\sigma(S) = \{\lambda \in \mathbb{R} |\ S - \lambda : \mathcal{D}(S) \longrightarrow \mathcal{H} \text{ is not invertible } \} \tag{3.3}$$

Next recall that the set of Fredholm operators from $\mathcal{D}(S)$ to $\mathcal{H}$ is open in $\mathcal{L}\left(\mathcal{D}(S), \mathcal{H}\right)$, so with $\epsilon > 0$ small enough $S - \lambda : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is Fredholm for all $\lambda \in B_\epsilon(0)$. We claim that $B_\epsilon(0) \cap \sigma(S) \subset \sigma_p(S)$. Indeed, the Fredholm property guarantees that $\operatorname{ran}(S-\lambda) \subset \mathcal{H}$ is a closed subspace and therefore $\mathcal{H} = \operatorname{ran}(S-\lambda) \oplus \operatorname{ran}(S-\lambda)^\perp$. Since S is self-adjoint, we have $\operatorname{ran}(S-\lambda)^\perp = \ker(S^* - \bar{\lambda}) = \ker(S - \bar{\lambda})$. Specifying to $\lambda \in \mathbb{R} \cap B_\epsilon(0)$ one gets

$$S - \lambda \text{ invertible } \iff \ker(S-\lambda) = 0$$

so going back to (3.3) we find

$$\sigma(S) \cap B_\epsilon(0) \subset \{\lambda \in \mathbb{R} |\ \ker(S-\lambda) \neq 0\,\} = \sigma_p(S) \tag{3.4}$$

We claim that it is possible to further shrink $\epsilon > 0$ so that also $\sigma(S) \cap B_\epsilon(0) \subset \{0\}$.
Assume by contradiction that there exists a sequence $\lambda_n \in \sigma(S)$ with $\lambda_n \neq 0$ and $\lambda_n \longrightarrow 0$. Inclusion (3.4) ensures that for $n \in \mathbb{N}$ large enough the λ_n are contained in the point spectrum $\sigma_p(S)$, so we can find "eigenvectors" $x_n \in \mathcal{D}(S)$ satisfying $\|x_n\|_{\mathcal{D}(S)} = 1$ and $(S - \lambda_n)x_n = 0$. In particular, $\|Sx_n\|_{\mathcal{H}} = |\lambda_n| \cdot \|x_n\|_{\mathcal{H}} \leq |\lambda_n|$ shows that $Sx_n \longrightarrow 0$ in $\mathcal{H}$.
By Atkinson's Theorem our operator S comes with a parametrix $\tilde{S} \in \mathcal{L}(\mathcal{H}, \mathcal{D}(S))$ such that $\tilde{S}S = \mathrm{Id}_{\mathcal{D}(S)} - K$ with $K : \mathcal{D}(S) \to \mathcal{D}(S)$ compact. Passing to a subsequence we can assume that $Kx_n \longrightarrow x$ converges in $\mathcal{D}(S)$, so

$$x_n = \underbrace{\tilde{S}}_{\mathcal{L}(\mathcal{H},\mathcal{D}(S))}(\overbrace{Sx_n}^{\to 0}) + Kx_n \longrightarrow x$$

converges as well. Since S is a bounded operator from $\left(\mathcal{D}(S), \|\cdot\|_{\mathcal{D}(S)}\right)$ to $\mathcal{H}$, we have $Sx = \lim_{n\to\infty} Sx_n = 0$ and therefore $x \in \ker(S)$. As for a self-adjoint operator eigenspaces with different eigenvalues are always orthogonal, we obtain $\langle x, x\rangle_{\mathcal{H}} = \lim_{n\longrightarrow\infty} \langle x_n, x\rangle_{\mathcal{H}} = 0$ and thus $x = 0$. This is in contradiction to $\|x\|_{\mathcal{D}(S)} = \lim_{n\to\infty} \|x_n\|_{\mathcal{D}(S)} = 1$. □

Taking all three properties together, we arrive at the following picture about the spectrum of S:

Corollary 3.7 (Characterising the spectrum of $S = D_{\text{-}A}D_A$)
Let $S : \mathcal{D}(S) \longrightarrow S$ be non-positive, self-adjoint and Fredholm.
Then there exists a constant $\epsilon > 0$ such that the spectrum of S satisfies $\sigma(S) \setminus \{0\} \subset (-\infty, -\epsilon]$.
The maximum possible such ϵ will be called the spectral gap of S.

Proof. Combine Lemmas 3.5 and 3.6 □

In particular, $0 \in \mathbb{C}$ will be isolated from the rest of the spectrum.

3.3 The operator norm of the resolvent

In section 3.4 Lemma 3.14(ii) we will show that around isolated points of the spectrum the resolvent map $R_\bullet(S)$ can only have simple poles. The key ingredient will be Proposition 3.11 for which we will need Lemmas 3.8 and 3.10 as preparations. Proposition 3.12 is an immediate consequence of Proposition 3.11, but will not be needed until chapter 4 Theorem 4.15.

We begin by the following result which is a simplification of [Ka] Theorem III. 6.15 and Problem III.6.16 (p.177):

Lemma 3.8 (Spectral mapping for $\boldsymbol{z \mapsto z^{-1}}$)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be a closed (unbounded) operator on $\mathcal{H}$.

i) If S is invertible, we can treat its inverse S^{-1} as a bounded operator from $\mathcal{H}$ to itself. The spectra $\sigma\big(S : \mathcal{D}(S) \to \mathcal{H}\big) \setminus \{0\}$ and $\sigma\big(S^{-1} \in \mathcal{L}(\mathcal{H})\big) \setminus \{0\}$ are related by $z \mapsto z^{-1}$

ii) Given $\lambda \in \rho(S) = \sigma(S)^c$ we have $dist(\lambda, \sigma(S)) = \inf |\sigma(S) - \lambda| > 0$ and the resolvent $R_\lambda(S) := (S - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ has spectral radius $spr(R_\lambda(S)) = \frac{1}{dist(\lambda, \sigma(S))}$

Proof. Part (i). Recall from our proof of Lemma 3.6 that the resolvent set of a closed operator S is simply $\rho(S) = \{z \in \mathbb{C} |\ S - z : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is bijective $\}$. Moreover, for S invertible the inverse $S^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{D}(S))$ is a bounded operator. Since the inclusion $\iota : \mathcal{D}(S) \hookrightarrow \mathcal{H}$ is bounded as well, we can consider "S^{-1}" $= \iota \circ S^{-1} \in \mathcal{L}(\mathcal{H})$ as a bounded operator from $\mathcal{H}$ to itself. So as far as S^{-1} is concerned we will use the resolvent formalism of bounded operators. Now pick any $z \in \mathbb{C} \setminus \{0\}$. The calculation

$$\iota \circ S^{-1} - z^{-1}\,\mathrm{id}_{\mathcal{H}} = (z\,\iota - S) \circ (zS)^{-1}$$

translates into a commutative diagram

$$\begin{tikzcd} \mathcal{H} & & \mathcal{H} \arrow[ll, "S^{-1} - z^{-1}"'] \arrow[dl, "(zS)^{-1}"] \\ & \mathcal{D}(S) \arrow[ul, "z - S"] & \end{tikzcd}$$

Since $(zS)^{-1} : \mathcal{H} \longrightarrow \mathcal{D}(S)$ is invertible, we find that

$$S^{-1} - z^{-1} : \mathcal{H} \to \mathcal{H} \text{ bijective} \qquad \Longleftrightarrow \qquad z - S : \mathcal{D}(S) \to \mathcal{H} \text{ bijective}$$

so $\sigma\big(S \in \mathcal{L}(\mathcal{D}(S), \mathcal{H})\big) \setminus \{0\}$ and $\sigma\big(S^{-1} \in \mathcal{L}(\mathcal{H})\big) \setminus \{0\}$ are related by $z \mapsto z^{-1}$.

Part (ii). Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be closed but not necessarily invertible and consider $\lambda \in \rho(S)$. As the resolvent set $\rho(S) \subset \mathbb{C}$ is open, there exists $\epsilon > 0$ such that

$$\Omega := \sigma(S - \lambda) = \sigma(S) - \lambda \ \subset \ \mathbb{C} \setminus B_\epsilon(0)$$

In particular, $\operatorname{dist}(\lambda, \sigma(S)) = \inf |\Omega| \geq \epsilon > 0$. Since the property of being closed is stable under perturbation by bounded operators, we observe that $S - \lambda : \mathcal{D}(S) \to \mathcal{H}$ is a closed invertible operator. By part (i) we find that $\frac{1}{\Omega} = \sigma((S - \lambda)^{-1}) \setminus \{0\}$ and therefore $\sup \left|\frac{1}{\Omega}\right| = \sup \left|\sigma\big((S - \lambda)^{-1}\big)\right|$. For general $\Omega \subset \mathbb{C} \setminus \{0\}$ one has $\left|\frac{1}{\Omega}\right| = \frac{1}{|\Omega|} \subset (0, \infty)$ and $\sup \frac{1}{|\Omega|} = \frac{1}{\inf |\Omega|} \in [0, \infty]$, so in our case

$$\operatorname{spr}\big(R_\lambda(S)\big) \underset{\text{def.}}{=} \sup \left|\sigma\big((S - \lambda)^{-1}\big)\right| = \frac{1}{\operatorname{dist}(\lambda, \sigma(S))}$$

$\square$

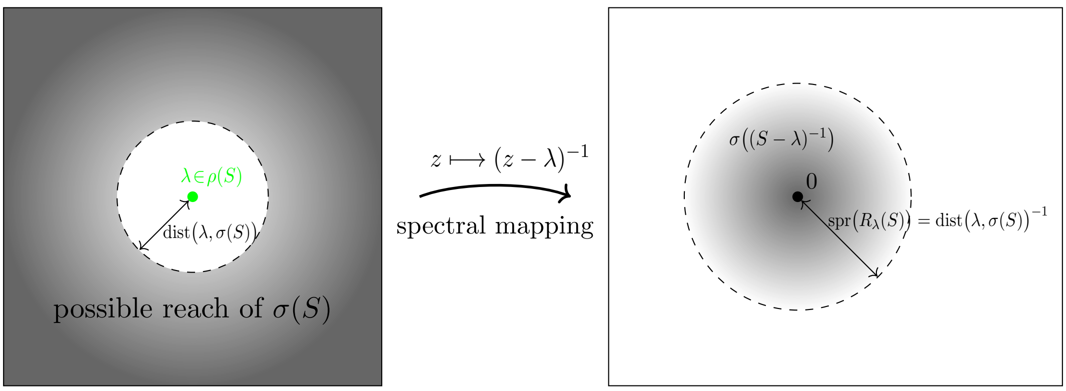


Figure 3.1: Proof idea of Lemma 3.8(ii) for a closed operator $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$.

In the remainder of this section we investigate whether not only $\operatorname{spr}\big(R_\lambda(S)\big)$ but also $\big\|R_\lambda(S)\big\|_{\mathcal{L}(\mathcal{H})}$ can be identified with $\frac{1}{\operatorname{dist}(\lambda,\sigma(S))}$. Unfortunately, if S is a general closed operator, we only get a lower bound:

Corollary 3.9 (Resolvent norm of a closed operator)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be a closed operator. Given $\lambda \in \rho(S)$ the resolvent satisfies

$$\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} \geq \frac{1}{dist(\lambda, \sigma(S))}$$

Proof. The spectral radius of a bounded operator $R \in \mathcal{L}(\mathcal{H})$ is constrained by

$$\|R\|_{\mathcal{L}(\mathcal{H})} \geq \operatorname{spr}(R)$$

so our claim follows from Lemma 3.8(ii) □

Part (ii) of our next result shows that [ReSi] Thm. VI.7 (p.192) extends to closed unbounded operators. Moreover, as stated in part (iii), self-adjointness of S can be used to guarantee that the resolvent is a normal operator. This will be key to improving on Corollary 3.9.

Lemma 3.10 (Adjoint of the resolvent)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be a closed, densely defined operator.

i) If S is invertible, then so is $S^* : \mathcal{D}(S^*) \longrightarrow \mathcal{H}$. Its inverse is given by $(S^*)^{-1} = (S^{-1})^*$ where in taking the adjoint we consider $S^{-1} \in \mathcal{L}(\mathcal{H})$ as a bounded operator.

ii) If $\lambda \in \mathbb{C}$ is in the resolvent set of S, then $\bar{\lambda}$ is in the resolvent set of S^* and we have

$$R_{\bar{\lambda}}(S^*) = \big(R_\lambda(S)\big)^*$$

where in taking the adjoint we consider $R_\lambda(S) \in \mathcal{L}(\mathcal{H})$ as a bounded operator.

iii) If $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is self-adjoint and $\lambda \in \rho(S)$ is in the resolvent set, then $R_\lambda(S) \in \mathcal{L}(\mathcal{H})$ is a bounded normal operator.

Proof. Part (i). Assume that $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is invertible and consider $x \in \mathcal{D}(S^*)$.
Then for all $y \in \mathcal{H}$ we have

$$\big\langle x, y\big\rangle = \Big\langle \underbrace{x}_{\mathcal{D}(S^*)} ,\, S \underbrace{S^{-1}y}_{\mathcal{D}(S)} \Big\rangle = \Big\langle S^*x \,,\, \underbrace{S^{-1}}_{\mathcal{L}(\mathcal{H})}y \Big\rangle = \Big\langle (S^{-1})^* S^* x,\, y \Big\rangle$$

which implies $(S^{-1})^*S^* = \mathrm{id}_{\mathcal{D}(S^*)}$. Conversely, consider $y \in \mathcal{H}$.
Then for all $w \in \mathcal{D}(S)$ we calculate

$$\big\langle y, w\big\rangle = \big\langle y, \underbrace{S^{-1}}_{\mathcal{L}(\mathcal{H})} Sw \big\rangle = \big\langle (S^{-1})^* y, Sw \big\rangle$$

so $\big\langle (S^{-1})^* y, S\,\cdot\, \big\rangle = \big\langle y, \cdot\big\rangle$ is a bounded linear functional on $\big(\mathcal{D}(S), \|\cdot\|_{\mathcal{H}}\big)$. This shows that given $y \in \mathcal{H}$ we have $(S^{-1})^*y \in \mathcal{D}(S^*)$. Since $\mathcal{D}(S) \subset \mathcal{H}$ is dense, our observation that $\langle S^*(S^{-1})^*y, w\rangle = \langle y, w\rangle$ holds for all $y \in \mathcal{H}$ and $w \in \mathcal{D}(S)$ implies $S^*(S^{-1})^* = \mathrm{id}_{\mathcal{H}}$.

Part (ii). Assume $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is closed and densely defined but not necessarily invertible. Consider $\lambda \in \rho(S)$ from the resolvent set. Then $S - \lambda : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is closed and invertible. Its adjoint is $(S - \lambda)^* = S^* - \bar{\lambda} : \mathcal{D}(S^*) \longrightarrow \mathcal{H}$ and with part (i) we obtain

$$(S^* - \bar{\lambda})^{-1} = \big((S-\lambda)^*\big)^{-1} = \big((S-\lambda)^{-1}\big)^* \in \mathcal{L}(\mathcal{H})$$

Part (iii) The resolvent formula

$$(\mu - \lambda)\, R_\lambda(S)\, R_\mu(S) = R_\lambda(S)\ [(S-\lambda) - (S-\mu)]\ R_\mu(S) = R_\mu(S) - R_\lambda(S) \tag{3.5}$$

shows that for any pair $\mu, \lambda \in \rho(S)$ the operators $R_\mu(S), R_\lambda(S) \in \mathcal{L}(\mathcal{H})$ commute.
For $S = S^*$ self-adjoint and $\lambda \in \rho(S)$ part (ii) yields $R_\lambda(S)^* = R_{\bar{\lambda}}(S) \in \mathcal{L}(\mathcal{H})$
so $R_\lambda(S)$ commutes with its adjoint. □

After these preparations, we are ready to prove that, for a self-adjoint operator, the norm $\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})}$ is completely determined in terms of the distance to the spectrum $\sigma(S)$. This generalizes the formulae obtained in [Te] Thm. 2.18 (p.77).

Proposition 3.11 (Norm of the resolvent I)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be self-adjoint. Given $\lambda \in \rho(S)$ the resolvent satisfies

$$\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{dist(\lambda, \sigma(S))} \tag{3.6}$$

Proof. Lemma 3.10(iii) shows that $R_\lambda(S) \in \mathcal{L}(\mathcal{H})$ is a bounded normal operator. In view of Lemma 3.8(ii) it remains to prove that the spectral radius of any bounded normal operator $R \in \mathcal{L}(H)$ is given by

$$\operatorname{spr}(R) = \|R\|_{\mathcal{L}(\mathcal{H})}$$

To do so, we combine ideas from the discussion at "https://math.stackexchange.com/q/1052614" (version: 2017-11-23). By [ReSi] Thm IV.3(f) (p.186) every bounded operator $R \in \mathcal{L}(\mathcal{H})$ satisfies $\|R^*R\| = \|R\|^2$. If R is normal, this generalizes to $\|(R^*R)^n\| = \|(R^n)^*R^n\| = \|R^n\|^2$. Since $R^*R \in \mathcal{L}(\mathcal{H})$ is self-adjoint, [ReSi] Thm. VI.6 (p.192) tells us that

$$\|R\|^2 = \|R^*R\| = \lim_{n\to\infty} \|(R^*R)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|R^n\|^{\frac{2}{n}} = \Big(\lim_{n\to\infty} \|R^n\|^{\frac{1}{n}}\Big)^2 = \operatorname{spr}(R)^2$$

so $\operatorname{spr}(R) = \|R\|$. □

Formula (3.6) can be used to constrain the spectrum of perturbed operators $S-K\colon \mathcal{D}(S) \longrightarrow \mathcal{H}$ with $K \in \mathcal{L}(\mathcal{H})$: As illustrated by Figure 4.2, the perturbed spectrum $\sigma(S-K)$ will be contained in a $\|K\|_{\mathcal{L}(\mathcal{H})}$-thickening of $\sigma(S)$. In the proof of Theorem 4.15, however, we will be dealing with unbounded perturbations $K \in \mathcal{L}\big(\mathcal{D}(S), \mathcal{H}\big)$ which means that spectral perturbation theory relies on an upper bound for $\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H}, \mathcal{D}(S))}$ instead of just $\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})}$:

Proposition 3.12 (Norm of the resolvent II)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be self-adjoint and assume that $\mathcal{W} := \mathcal{D}(S)$ is equipped with a complete norm such that S and the inclusion $\iota : \mathcal{W} \longrightarrow \mathcal{H}$ belong to $\mathcal{L}(\mathcal{W}, \mathcal{H})$. Then for any $\lambda \in \rho(S)$ the resolvent satisfies

$$\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W})} \le \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W})} \left(1 + \frac{1+|\lambda|}{dist\,\big(\lambda, \sigma(S)\big)}\right) \tag{3.7}$$

Proof. Self-adjointness of S ensures that $(S-i) \in \mathcal{L}(\mathcal{W}, \mathcal{H})$ is invertible. Since $\|\cdot\|_{\mathcal{W}}$ is complete, the Inverse Mapping Theorem shows that $(S-i)^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{W})$ is bounded as well. Thus, for $\lambda \in \rho(S)$ and $u \in \mathcal{H}$ we obtain

$$\begin{aligned}
\big\|(S-\lambda)^{-1}u\big\|_{\mathcal{W}} &\le \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W})} \big\|\overbrace{(S-i)}^{(S-\lambda)+(\lambda-i)}(S-\lambda)^{-1}u\big\|_{\mathcal{H}} \\
&\le \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W})} \Big(1 + |\lambda - i| \cdot \|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})}\Big) \|u\|_{\mathcal{H}}
\end{aligned}$$

Using Proposition 3.11 to identify $\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{\operatorname{dist}\big(\lambda,\sigma(S)\big)}$ we arrive at

$$\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W})} \le \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W})} \left(1 + \frac{|\lambda - i|}{\operatorname{dist}\,\big(\lambda, \sigma(S)\big)}\right) \tag{3.8}$$

so the claimed formula follows with $|\lambda - i| \le 1 + |\lambda|$. □

Note that the r.h.s. of (3.7) is a continuous function of $\lambda \in \rho(S)$ and will be uniformly bounded on compact subsets of $\rho(S)$. We will come back to this point in Theorem 4.15. For the next section, however, formula (3.6) will be enough.

3.4 An operator-valued Laurent expansion and its consequences

Even if $\lambda_0 = 0$ belongs to the spectrum of $S = D_{\text{-}A}D_A$, it will be an isolated point, with the rest of the spectrum satisfying $\sigma(S) \setminus \{0\} \subset (\text{-}\infty, \text{-}\epsilon]$. In this section, we prove that the resolvent of a general closed operator S can be Laurent-expanded around isolated points of the spectrum, with the stronger requirement of S being a self-adjoint operator leading to a truncation of higher poles in the Laurent expansion. This implies a set of algebraic conditions on the coefficients, allowing us to dramatically simplify the expansion and geometrically interpret the two 'fundamental coefficients' P and Q.[1]

In proving Lemma 3.14, we will repeatedly apply the following calculation rule:

Auxiliary Lemma 3.13 (Permutation of limit and Bochner-integral)
Let $\mathbb{B}$ be a Banach space and consider a sequence of continuous functions $f_n : S^1 \longrightarrow \mathbb{B}$, uniformly convergent to $f : S^1 \longrightarrow \mathbb{B}$. Then f and all the f_n are Bochner-integrable and we have

$$\lim_{n\to\infty} \int_{S^1} f_n = \int_{S^1} f \in \mathbb{B}$$

Proof. As the uniform limit of a sequence of continuous functions, $f : S^1 \longrightarrow \mathbb{B}$ is continuous itself. Since S^1 is compact, all continuous functions $S^1 \to \mathbb{B}$ are Bochner-integrable. Moreover, one has

$$\Big\| \int_{S^1} f_n - \int_{S^1} f \Big\|_{\mathbb{B}} \le \int_{S^1} \|f_n - f\|_{\mathbb{B}} \le \|f_n - f\|_\infty \cdot \mu(S^1) \longrightarrow 0 \qquad \text{as } n \to \infty$$

□

As we will see next, analyticity of the resolvent map allows us to deform integration contours, leading to the following fundamental but still to be refined result:

Lemma 3.14 (Laurent expansion I)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be a closed, densely defined operator.

i) The resolvent map $R_\bullet(S) : \rho(S) \longrightarrow \mathcal{L}(\mathcal{H}, \mathcal{D}(S))$ is analytic. In the vicinity of an isolated point of the spectrum $\lambda_0 \in \sigma(S)$ it admits the Laurent expansion

$$R_{\lambda_0+\mu}(S) = \sum_{n=1}^{\infty} \frac{1}{\mu^n} Q_{-n} + \sum_{n=0}^{\infty} \mu^n Q_n$$

where the coefficients $Q_n \in \mathcal{L}(\mathcal{H}, \mathcal{D}(S))$ are uniquely given by Bochner contour integrals

$$Q_n = \frac{1}{2\pi i} \int_{\circlearrowleft} \frac{d\mu}{\mu} \frac{1}{\mu^n} R_{\lambda_0+\mu}(S) \qquad \textit{around the origin } \mu = 0.$$

*ii) If S is self-adjoint, all Laurent coefficients with $n \le -2$ vanish.
This means isolated points of the spectrum $\lambda_0 \in \sigma(S)$ correspond to simple poles of the resolvent and the Laurent expansion reads*

$$R_{\lambda_0+\mu}(S) = -\frac{1}{\mu} P + \sum_{n=0}^{\infty} \mu^n Q_n \qquad \textit{with } P = -Q_{\text{-}1}$$

The coefficients $P, Q_n \in \mathcal{L}(\mathcal{H}, \mathcal{D}(S))$ obey

$$\begin{aligned} (S - \lambda_0)\, P &= 0 & P\,(S - \lambda_0) &= 0 \\ (S - \lambda_0)\, Q_0 &= id_{\mathcal{H}} - P & Q_0\,(S - \lambda_0) &= [id - P]_{\mathcal{D}(S)} \\ (S - \lambda_0)\, Q_{n+1} &= Q_n & Q_{n+1}\,(S - \lambda_0) &= Q_n|_{\mathcal{D}(S)} \end{aligned}$$

where the bottom row holds for $n \ge 0$.

When considered as bounded operators from $\mathcal{H}$ to itself, all coefficients "Q_n" $= \iota \circ Q_n \in \mathcal{L}(\mathcal{H})$ are self-adjoint, i.e. for all $n \in \mathbb{Z}$ we have $Q_n^* = Q_n$

[1] Disclaimer: This has nothing to do with position and momentum.

Proof. Part (i). First, let us study how $R_\bullet(S)$ behaves in the vicinity of a point $\mu \in \rho(S)$. By Corollary 3.9 we already know that

$$r_\mu := \frac{1}{\|R_\mu(S)\|_{\mathcal{L}(\mathcal{H})}} \leq \operatorname{dist}\big(\mu, \sigma(S)\big)$$

and therefore $\mu + B_{r_\mu}(0) \subset \rho(S)$. Now fix a constant $0 < q < 1$. Choosing $\lambda \in B_{q \cdot r_\mu}(0)$ guarantees $|\lambda| \, \|R_\mu(S)\|_{\mathcal{L}(\mathcal{H})} < q$, so we observe that the series expansion

$$R_{\mu+\lambda}(S) = [S - \mu - \lambda]^{-1} = \underbrace{R_\mu(S)}_{\mathcal{L}(\mathcal{H},\mathcal{D}(S))} \big[\mathrm{id}_{\mathcal{H}} - \lambda \underbrace{R_\mu(S)}_{\mathcal{L}(\mathcal{H})}\big]^{-1} = R_\mu(S) \sum_{n=0}^{\infty} \big[\lambda \, R_\mu(S)\big]^n$$

converges uniformly on $B_{qr_\mu}(0)$. This shows that $R_\bullet(S) : \rho(S) \longrightarrow \mathcal{L}(\mathcal{H}, \mathcal{D}(S))$ is analytic.

Note that analytic functions $f : U \longrightarrow \mathbb{B}$ (where $U \subset \mathbb{C}$ is an open subset and $\mathbb{B}$ a Banach space) automatically satisfy Cauchy's integral formula. Indeed, when $\Gamma : S^1 \longrightarrow U \setminus \{\mu\}$ is a contour of winding number $w(\Gamma, \mu) = 1$ that stays within the radius of convergence of $f(\mu + \lambda) = \sum_{n=0}^{\infty} f_n \, \lambda^n$, we simply calculate

$$\frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\lambda}{\lambda} f(\mu + \lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\lambda}{\lambda} \sum_{n=0}^{\infty} f_n \lambda^n = \sum_{n=0}^{\infty} f_n \, \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\lambda}{\lambda} \lambda^n = f_0 = f(\mu)$$

where by Auxiliary Lemma 3.13 we were allowed to commute limit and integral.

Similarly, $\frac{1}{2\pi i} \int_\Gamma \mathrm{d}\lambda \, f(\mu + \lambda) = 0$ can be used to prove that contour integrals of analytic functions are homotopy-invariant.

To obtain the Laurent expansion around an isolated point of the spectrum $\lambda_0 \in \sigma(S)$ pick $\epsilon > 0$ such that $B_\epsilon(\lambda_0) \cap \sigma(S) = \{\lambda_0\}$ and consider the contour shown in Figure 3.2. Using Cauchy's integral formula and homotopy-invariance of the contour integral we get

$$R_{\lambda_0+\mu}(S) = \frac{1}{2\pi i} \int_{\Gamma_\mu} \mathrm{d}\lambda \, \frac{R_{\lambda_0+\lambda}(S)}{\lambda - \mu} = \frac{1}{2\pi i} \left(\int_{-\Gamma_-} + \int_{\Gamma_+} \right) \mathrm{d}\lambda \, \frac{R_{\lambda_0+\lambda}(S)}{\lambda - \mu}$$

Along Γ_+ we have $|\lambda| > |\mu|$, so a geometric series expansion yields

$$\frac{1}{2\pi i} \int_{\Gamma_+} \mathrm{d}\lambda \, \underbrace{\frac{1}{\lambda - \mu}}_{\frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{\mu}{\lambda}\right)^n} R_{\lambda_0+\lambda}(S) = \sum_{n=0}^{\infty} \mu^n \underbrace{\frac{1}{2\pi i} \int_{\Gamma_+} \frac{\mathrm{d}\lambda}{\lambda} \, \frac{R_{\lambda_0+\lambda}(S)}{\lambda^n}}_{\text{def. } Q_n}$$

where we have used Auxiliary Lemma 3.13 to commute limit and integral.
Along Γ_- one has $|\lambda| < |\mu|$, so we obtain

$$\frac{1}{2\pi i} \int_{-\Gamma_-} \mathrm{d}\lambda \, \frac{1}{\lambda - \mu} R_{\lambda_0+\lambda}(S) = \frac{1}{2\pi i} \int_{\Gamma_-} \mathrm{d}\lambda \, \underbrace{\frac{1}{\mu - \lambda}}_{\frac{1}{\mu} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} R_{\lambda_0+\lambda}(S) = \sum_{n=0}^{\infty} \frac{1}{\mu^{n+1}} \underbrace{\frac{1}{2\pi i} \int_{\Gamma_-} \frac{\mathrm{d}\lambda}{\lambda} \lambda^{n+1} R_{\lambda_0+\lambda}(S)}_{Q_{-(n+1)}}$$

In summary we have found that at each $\mu \in B_\epsilon(0) \setminus \{0\}$ the resolvent can be written as

$$R_{\lambda_0+\mu}(S) = \sum_{n=1}^{\infty} \frac{Q_{-n}}{\mu^n} + \sum_{n=0}^{\infty} \mu^n Q_n \tag{3.9}$$

Our derivation shows that convergence of (3.9) is uniform on sets $B_{q\epsilon}(0) \setminus B_{q'\epsilon}(0)$ with $0 < q' < q < 1$. Thus, using Auxiliary Lemma 3.13, the coefficients can be uniquely extracted by contour integrals

$$Q_n = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\mu}{\mu} \frac{R_{\lambda_0+\mu}(S)}{\mu^n} \in \mathcal{L}\big(\mathcal{H}, \mathcal{D}(S)\big)$$

where Γ is a sufficiently small circle around the origin.

Part (ii). Now assume in addition that $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ is self-adjoint. If $\epsilon > 0$ is such that $B_\epsilon(\lambda_0) \cap \sigma(S) = \{\lambda_0\}$, then for every $\mu \in B_{\epsilon/2}(0)$ we have $\mathrm{dist}(\lambda_0 + \mu, \sigma(S)) = |\mu|$ so using Proposition 3.11 we obtain $\|R_{\lambda_0+\mu}(S)\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{|\mu|}$. Choosing Γ to be the contour $\mu(\theta) = re^{2\pi i\theta}$ with $\theta \in [0, 1]$ and constant $0 < r < \epsilon/2$, we observe that

$$\|Q_n\|_{\mathcal{L}(\mathcal{H})} = \left\| \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}\mu}{\mu} \frac{R_{\lambda_0+\mu}}{\mu^n} \right\| = \left\| \int_0^1 \mathrm{d}\theta \frac{R_{\lambda_0+\mu}}{\mu^n} \right\| \leq \int_0^1 \mathrm{d}\theta \frac{\|R_{\lambda_0+\mu}\|_{\mathcal{L}(\mathcal{H})}}{|\mu|^n} = \frac{1}{r^{n+1}}$$

For $n \leq -2$ we have $\|Q_n\|_{\mathcal{L}(\mathcal{H})} \leq r^{-n-1} \longrightarrow 0$ as $r \to 0$, so in this case $\|Q_n\|_{\mathcal{L}(\mathcal{H})} = 0$ and therefore $Q_n = 0$ even in $\mathcal{L}(\mathcal{H}, \mathcal{D}(S))$.
Thus, for a self-adjoint operator the Laurent expansion reads

$$R_{\lambda_0+\mu}(S) = \frac{Q_{-1}}{\mu} + \sum_{n=0}^{\infty} \mu^n Q_n$$

Multiplication by $S - \lambda_0 = \text{const.} \in \mathcal{L}(\mathcal{D}(S), \mathcal{H})$ produces competing Laurent expansions

$$(S - \lambda_0)\, R_{\lambda_0+\mu}(S) = \frac{1}{\mu}(S - \lambda_0)\, Q_{-1} + (S - \lambda_0)\, Q_0 + \sum_{n=0}^{\infty} \mu^{n+1}(S - \lambda_0)\, Q_{n+1}$$

$$(S - \lambda_0)\, R_{\lambda_0+\mu}(S) = \mathrm{id}_{\mathcal{H}} + \mu\, R_{\lambda_0+\mu}(S) = 0 + \left[\mathrm{id}_{\mathcal{H}} + Q_{-1}\right] + \sum_{n=0}^{\infty} \mu^{n+1} Q_n$$

As the coefficients are unique, we obtain the relations

$$(S - \lambda_0)\, Q_{-1} = 0\,, \qquad (S - \lambda_0)\, Q_0 = \mathrm{id}_{\mathcal{H}} + Q_{-1} \quad \text{and} \quad (S - \lambda_0)\, Q_{n+1} = Q_n \text{ for } n \geq 0$$

Similarly, comparison of the Laurent expansions

$$R_{\lambda_0+\mu}(S)\,(S - \lambda_0) = \frac{1}{\mu} Q_{-1}\,(S - \lambda_0) + Q_0\,(S - \lambda_0) + \sum_{n=0}^{\infty} \mu^{n+1} Q_{n+1}\,(S - \lambda_0)$$

$$R_{\lambda_0+\mu}(S)\,(S - \lambda_0) = \mathrm{id}_{\mathcal{D}(S)} + \mu\, R_{\lambda_0+\mu}(S)|_{\mathcal{D}(S)} = 0 + \left[\mathrm{id}_{\mathcal{D}(S)} + Q_{-1}|_{\mathcal{D}(S)}\right] + \sum_{n=0}^{\infty} \mu^{n+1} Q_n|_{\mathcal{D}(S)}$$

produces relations

$$Q_{-1}\,(S - \lambda_0) = 0\,, \quad Q_0\,(S - \lambda_0) = \mathrm{id}_{\mathcal{D}(S)} + Q_{-1}|_{\mathcal{D}(S)} \quad \text{and} \quad Q_{n+1}\,(S - \lambda_0) = Q_n|_{\mathcal{D}(S)} \text{ for } n \geq 0$$

Recall that the spectrum of a self-adjoint operator satisfies $\sigma(S) \subset \mathbb{R}$, so we necessarily are in the situation $\lambda_0 \in \mathbb{R}$. Lemma 3.10(ii) shows that if we consider $R_{\lambda_0+\mu}(S) \in \mathcal{L}(\mathcal{H}, \mathcal{D}(S)) \subset \mathcal{L}(\mathcal{H})$ as a bounded operator from $\mathcal{H}$ to itself, the adjoint is simply $R_{\lambda_0+\mu}(S)^* = R_{\lambda_0+\bar{\mu}}(S)$.
A straightforward calculation involving the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ shows that the Bochner integral of an $\mathcal{L}(\mathcal{H})$-valued function commutes with the operation of taking adjoints.
So in our case we find

$$Q_n^* = \left(\frac{1}{2\pi i} \int_{\circlearrowleft} \frac{\mathrm{d}\mu}{\mu} \frac{R_{\lambda_0+\mu}(S)}{\mu^n} \right)^* = -\frac{1}{2\pi i} \int_{\circlearrowleft} \frac{\mathrm{d}\bar{\mu}}{\bar{\mu}} \frac{R_{\lambda_0+\bar{\mu}}(S)}{\bar{\mu}^n}$$

$$= -\frac{1}{2\pi i} \int_{\circlearrowright} \frac{\mathrm{d}\mu}{\mu} \frac{R_{\lambda_0+\mu}(S)}{\mu^n} = +\frac{1}{2\pi i} \int_{\circlearrowleft} \frac{\mathrm{d}\mu}{\mu} \frac{R_{\lambda_0+\mu}(S)}{\mu^n} = Q_n$$

which verifies our claim that the coefficients $Q_n \in \mathcal{L}(\mathcal{H})$ are bounded self-adjoint operators.

□

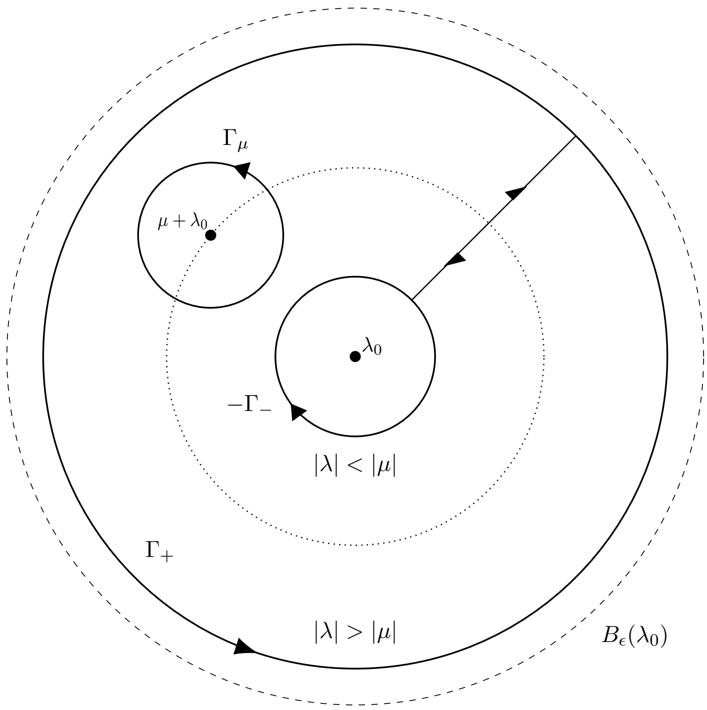


Figure 3.2: Integration contour used to derive the Laurent expansion of Lemma 3.14(i)

The algebraic relations from Lemma 3.14(ii) contain enough information to obtain the following refinement, together with a useful interpretation of the Laurent coefficients:

Proposition 3.15 (Laurent expansion II)
Let $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ be a densely defined, self-adjoint operator and assume that $\lambda_0 \in \mathbb{C}$ is an isolated point of the spectrum, i.e. $B_\epsilon(\lambda_0) \cap \sigma(S) = \{\lambda_0\}$ for some $\epsilon > 0$.
Then the Laurent expansion from Lemma 3.14(ii) leads to the following consequences:

i) λ_0 belongs to the point spectrum, $ran(S - \lambda_0) \subset \mathcal{H}$ is a closed subspace and we have an orthogonal decomposition $\mathcal{H} = ran(S - \lambda_0) \oplus \ker(S - \lambda_0)$.
$P = -Q_{-1}$ and $(S - \lambda_0)\, Q_0$ are the orthogonal projections onto $\ker(S - \lambda_0)$ and $ran(S - \lambda_0)$, respectively.

ii) For all $n \geq 0$ the operators $P, Q_n \in \mathcal{L}(\mathcal{H})$ satisfy

$$PQ_n = Q_n P = 0$$
$$Q_n = Q_0^{n+1}$$

In particular, the Laurent expansion can be rewritten as

$$R_{\lambda_0+\mu}(S) = -\frac{P}{\mu} + Q_0 \sum_{n=0}^{\infty} (\mu Q_0)^n = -\frac{P}{\mu} + Q_0 \cdot (id_{\mathcal{H}} - \mu Q_0)^{-1}$$

Proof. The spectrum of a self-adjoint operator satisfies $\sigma(S) \subset \mathbb{R}$ so we necessarily have $\lambda_0 \in \mathbb{R}$. The shifted operator $S - \lambda_0$ is self-adjoint with spectrum $\sigma(S - \lambda_0) = \sigma(S) - \lambda_0$ so there is no loss of generality in assuming $\lambda_0 = 0$. All our claims will be derived from the algebraic relations of Lemma 3.14(ii) which upon setting $\lambda_0 = 0$ take a more appealing form.

Part (i). The relation "$Q_0 S = \mathrm{id}_{\mathcal{D}(S)} - P|_{\mathcal{D}(S)}$" shows that $P|_{\ker(S)} = \mathrm{id}$. On the other hand, "$PS = 0$" means that $P|_{\mathrm{ran}(S)} = 0$. Since $P \in \mathcal{L}(\mathcal{H})$ is a bounded operator, this implies $P|_{\overline{\mathrm{ran}(S)}} = 0$. Recall that every self-adjoint operator comes with an orthogonal decomposition $\mathcal{H} = \overline{\mathrm{ran}(S)} \oplus \ker(S)$ so by the above remarks $P = -Q_{-1}$ is the orthogonal projection operator onto $\ker(S)$. As a result, the relation "$SQ_0 = \mathrm{id}_{\mathcal{H}} - P$" shows that SQ_0 is the orthogonal projector onto $\overline{\mathrm{ran}(S)} = \ker(S)^{\perp}$. In particular, we have $SQ_0|_{\overline{\mathrm{ran}(S)}} = \mathrm{id}$, which implies $\overline{\mathrm{ran}(S)} \subset \mathrm{ran}(S)$ and therefore proves that $\mathrm{ran}(S) \subset \mathcal{H}$ is a closed subspace. The decomposition $\mathcal{H} = \mathrm{ran}(S) \oplus \ker(S)$ shows that in our case $\ker(S) \neq 0$ because otherwise S would be bijective and $\lambda_0 = 0$ would belong to the resolvent set.

Part (ii). For $n \geq 0$ we can argue as follows: "$Q_{n+1} S = Q_n|_{\mathcal{D}(S)}$" implies $Q_n|_{\ker(S)} = 0$ and "$SQ_{n+1} = Q_n$" shows $Q_n(\mathcal{H}) \subset \mathrm{ran}(S)$. Since by Part (i) P satisfies $P(\mathcal{H}) \subset \ker(S)$ and $P|_{ran(S)} = 0$, we obtain $Q_n P = PQ_n = 0$.
To prove the second part of our claim we iteratively define

$$\mathcal{D}_0(S) = \mathcal{H} \qquad \mathcal{D}_1(S) = \mathcal{D}(S) \qquad \mathcal{D}_{n+1}(S) = S^{-1}(\mathcal{D}_n(S)) \subset \mathcal{D}_n(S)$$

and consider the filtration

$$..... \subset \mathrm{ran}(S) \cap \mathcal{D}_n(S) \subset ... \subset \mathrm{ran}(S) \cap \mathcal{D}_2(S) \subset \mathrm{ran}(S) \cap \mathcal{D}(S) \subset \mathrm{ran}(S)$$

At each $n \geq 0$ the relation "$SQ_0 = \mathrm{id}_{\mathcal{H}} - P$" restricts to $SQ_0|_{ran(S) \cap \mathcal{D}_n(S)} = \mathrm{id}_{ran(S) \cap \mathcal{D}_n(S)}$ and similarly the relation "$Q_0 S = \mathrm{id}_{\mathcal{D}(S)} - P|_{\mathcal{D}(S)}$" yields $Q_0 S|_{ran(S) \cap \mathcal{D}_{n+1}(S)} = \mathrm{id}_{ran(S) \cap \mathcal{D}_{n+1}(S)}$.
Thus, S and Q_0 provide a ladder of mutually inverse isomorphisms

$$..... \underset{Q_0}{\overset{S}{\rightleftarrows}} \mathrm{ran}(S) \cap \mathcal{D}_2(S) \underset{Q_0}{\overset{S}{\rightleftarrows}} \mathrm{ran}(S) \cap \mathcal{D}(S) \underset{Q_0}{\overset{S}{\rightleftarrows}} \mathrm{ran}(S)$$

In particular, we get $Q_0^{n+1}S^{n+1}|_{\mathrm{ran}(S)\cap\mathcal{D}_{n+1}(S)} = \mathrm{id}_{\mathrm{ran}(S)\cap\mathcal{D}_{n+1}(S)}$ which means that Q_0^{n+1} is a left-inverse to S^{n+1}. On the other hand, iterative application of "$SQ_{n+1} = Q_n$" shows that $Q_n(\mathcal{H}) \subset S^{-1}(\mathcal{D}_n(S)) = \mathcal{D}_{n+1}(S)$ and

$$\begin{aligned} Q_0 &= SQ_1 = S^2Q_2 = \ldots = S^nQ_n = \ldots. \\ \mathrm{id}_{\mathcal{H}} - P = SQ_0 &= S^{n+1}Q_n \end{aligned}$$

The bottom line restricts to $S^{n+1}Q_n|_{\mathrm{ran}(S)} = \mathrm{id}_{\mathrm{ran}(S)}$ so $Q_n : \mathrm{ran}(S) \to \mathrm{ran}(S) \cap \mathcal{D}_{n+1}(S)$ is a right-inverse to S^{n+1}. The idea behind our proof is that we can identify right-inverse and left-inverse. Indeed,

$$Q_n \;=\; \underbrace{Q_0^{n+1}\mathrlap{\overbrace{\phantom{S^{n+1}Q_n}}^{\mathrm{id}_{\mathcal{H}}-P}}S^{n+1}}_{\mathrm{id}_{\mathrm{ran}(S)\cap\mathcal{D}_{n+1}(S)}}Q_n \;=\; Q_0^{n+1} - \underbrace{Q_0^{n+1}P}_{=0} \;=\; Q_0^{n+1}$$

□

Remark 3.16 (Double Helix II)
Note that the condition of $A(t) = A_0 + B(t) : W_1 \longrightarrow H$ being a good perturbation is invariant under $A \to -A$. Hence Theorem 3.4 shows that the operators $S_+ = D_{-A}D_A$ and $S_- = D_AD_{-A}$ are non-positive, self-adjoint and Fredholm. We have

$$\begin{aligned} \ker S_\pm &= \ker D_{\pm A} \\ \mathrm{ran}\; S_\pm &= D_{\mp A}(\mathcal{W}_1) \end{aligned}$$

and for $S = S_\pm$ the spaces $\mathcal{D}_{n+1}(S) = S^{-1}\big(\mathcal{D}_n(S)\big)$ and $\mathrm{ran}(S) \cap \mathcal{D}_n(S)$ are given as

$$\begin{aligned} \mathcal{D}_n(S) &= \mathcal{W}_{2n} \\ \mathrm{ran}\,(S) \cap D_n(S) &= D_{\mp A}(\mathcal{W}_{2n+1}) \end{aligned}$$

Writing $Q_\pm := Q_0(S_\pm)$ we can interpret the proof of Proposition 3.15(ii) as providing a tower of mutually inverse maps

$$\begin{tikzcd} \cdots\cdots \arrow[r, shift left, "S_\pm"] \arrow[r, phantom, "\cong"] & D_{\mp A}(\mathcal{W}_5) \arrow[l, shift left, "Q_\pm"] \arrow[r, shift left, "S_\pm"] \arrow[r, phantom, "\cong"] & D_{\mp A}(\mathcal{W}_3) \arrow[l, shift left, "Q_\pm"] \arrow[r, shift left, "S_\pm"] \arrow[r, phantom, "\cong"] & D_{\mp A}(\mathcal{W}_1) \arrow[l, shift left, "Q_\pm"] \end{tikzcd}$$

Recall from Remark 3.16 that the maps $\varphi_\pm = \pm D_{\pm A}$ can be organised into a double helix of isomorphisms

$$\begin{tikzcd} \ker D_A \oplus D_{-A}(\mathcal{W}_1) \arrow[d, dotted, no head] & = & \mathcal{H} & = & D_A(\mathcal{W}_1) \oplus \ker D_{-A} \arrow[d, dotted, no head] \\ \ker D_A \oplus D_{-A}(\mathcal{W}_2) \arrow[urrrr, "D_A"'] \arrow[d, dotted, no head] & = & \mathcal{W}_1 & = & D_A(\mathcal{W}_2) \oplus \ker D_{-A} \arrow[ullll, red, "D_{-A}"] \arrow[d, dotted, no head] \\ \ker D_A \oplus D_{-A}(\mathcal{W}_3) \arrow[urrrr, "D_A"'] \arrow[d, dotted, no head] & = & \mathcal{W}_2 & = & D_A(\mathcal{W}_3) \oplus \ker D_{-A} \arrow[ullll, red, "D_{-A}"] \arrow[d, dotted, no head] \\ \ker D_A \oplus D_{-A}(\mathcal{W}_4) \arrow[urrrr, "D_A"'] & = & \mathcal{W}_3 & = & D_A(\mathcal{W}_4) \oplus \ker D_{-A} \arrow[ullll, red, "D_{-A}"] \\ \vdots & & \vdots & & \vdots \end{tikzcd}$$

from which we now extract a triangle of bijective maps

$$\begin{tikzcd}
 & D_A(\mathcal{W}_1) \arrow[dd, "\cong"', "Q_-"] \\
D_{\text{-}A}(\mathcal{W}_2) \arrow[ur, "D_A", "\cong"'] & \\
 & D_A(\mathcal{W}_3) \arrow[ul, "\cong"', "D_{\text{-}A}"]
\end{tikzcd}$$

The new insight from Lemma 3.14 and Proposition 3.15 is that $Q = S|_{\mathrm{ran}(S)}^{-1}$ can be expressed as a Bochner integral

$$Q_- = \frac{1}{2\pi i} \int_{S^1} \frac{\mathrm{d}\mu}{\mu} \, [D_A D_{-A} - \mu]^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{W}_2)$$

where $S^1 \subset \rho(S_-)$ is a small circle around the origin. This observation will be key to our proof of Theorem 4.15 Step 1.

To conlude this section, let us remark that the coefficient Q_- can be used to obtain an integral representation for a quasi-inverse to D_A, in a way similar to a Green's function:

Corollary 3.17 (Parametrix)
The operator $D_{\text{-}A}Q_- : \mathcal{H} \longrightarrow \mathcal{W}_1$ is a parametrix to $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$.
In particular, we have $D_{\text{-}A}Q_{\text{-}}\, D_A\big|_{D_{-A}(\mathcal{W}_2)} = id$

Proof. In Remark 3.16 we have seen that

$$S_- Q_- \,=\, D_A D_{\text{-}A} Q_{\text{-}} \,=\, \mathrm{id}_{D_A(\mathcal{W}_1)}$$

fits into a triangle of bijective maps. Thus, with Auxiliary Lemma 3.18 (applied in the category **Sets**) we obtain

$$D_{\text{-}A} Q_{\text{-}}\, D_A \,=\, \mathrm{id}_{D_{-A}(\mathcal{W}_2)} \tag{3.10}$$

Recall from Proposition 3.15 that w.r.t. the decomposition $\mathcal{W}_1 = \ker D_A \oplus D_{\text{-}A}(\mathcal{W}_2)$
the Laurent coefficient $P_- := P(S_-)$ serves as the projector onto $\ker D_A$.
Thus, Eq. (3.10) can be augmented to

$$D_{\text{-}A} Q_{\text{-}}\, D_A \,=\, \mathrm{id}_{\mathcal{W}_1} - P_-$$

Note that P_- is finite rank and therefore compact. □

Auxiliary Lemma 3.18 (Cyclic reshuffling)
Assume that in any category we are given a triangle of isomorphisms

$$\begin{tikzcd}
 & A \arrow[dd, "\cong"', "\gamma"] \\
B \arrow[ur, "\alpha", "\cong"'] & \\
 & C \arrow[ul, "\cong"', "\beta"]
\end{tikzcd}$$

Then $\alpha\beta\gamma = id_A$ implies $\beta\gamma\alpha = id_B$.

Proof. Since α is invertible, our claim follows from the simple calculation

$$\mathrm{id}_B \,=\, \alpha^{-1} \circ \mathrm{id}_A \circ \alpha \,=\, \alpha^{-1} \circ \alpha\beta\gamma \circ \alpha \,=\, \beta\gamma\alpha$$

□

Chapter 4

APS operators on weighted Floer path spaces

4.1 An abstract twisting procedure turning almost into honest sc-Banach spaces

As mentioned at the end of section 2.7, our filtration $(\mathcal{W}_n)_{n\geq 0}$ fails to be an honest sc-Banach space, by lack of compact inclusions. In this section, however, we describe a systematic procedure by which any almost sc-Banach space $(\mathcal{W}_n)_{n\geq 0}$ gives rise to a k-family of honest sc-Banach spaces $(\mathcal{W}^{\delta_n}_{n+k})_{n\geq 0}$. This involves adopting an inverse perspective on the standard technique of 'weight factors' that is used for example in [FW].

Given $\delta > 0$ and $\eta \in C^\infty(\mathbb{R})$ such that $\eta(t) = |t|$ for $|t| \geq 1$, we consider the inverse weight function

$$\gamma_{\text{-}\delta}(t) = e^{-\delta\eta(t)} \leq 1$$

Working at fixed $t \in \mathbb{R}$, we observe that $\lambda = \gamma_{\text{-}\delta}(t) \in \mathbb{R}$ defines a sc-operator $\lambda\colon (W_k)_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ with $\|\lambda\|_{\mathcal{L}(W_k)} = |\lambda|$. Moreover, $t \longmapsto \gamma_{\text{-}\delta}(t) \in \mathbb{R}$ being smooth with bounded derivatives ensures that $\gamma_{\text{-}\delta}(t) : (W_k)_{k\geq 0} \longrightarrow (W_k)_{k\geq 0}$ is a *moderate family of sc-operators* in the sense of Definition 2.23. As a result, the map $\gamma_{\text{-}\delta} : \mathcal{H} \longrightarrow \mathcal{H}$ preserves the bifiltration $W^r_k = W^{r,2}(\mathbb{R}, W_k) \subset L^2(\mathbb{R}, H)$ in the sense that

$$\gamma_{\text{-}\delta}\big(W^r_k\big) \subset W^r_k \quad \text{and} \quad \gamma_{\text{-}\delta} \in \mathcal{L}(W^r_k)$$

where the operator norms $\big\|\gamma_{\text{-}\delta}\big\|_{\mathcal{L}(W^r_k)}$ can be constructed with Lemma 2.28.

By a suitable restriction, $\gamma_{\text{-}\delta}$ can be regarded as a compact operator:

Lemma 4.1 ($\gamma_{\text{-}\delta}$ as a compact operator between tiles of the bifiltration)
At every $r, k \geq 0$ the map $\gamma_{\text{-}\delta} : \mathcal{H} \longrightarrow \mathcal{H}$ restricts to a compact operator $\gamma_{\text{-}\delta} : W^{r+1}_k \longrightarrow W^r_k$

Proof. The statement that for a fixed Banach space $\mathbb{B} = W_k$ the map

$$\gamma_{\text{-}\delta} : W^{r+1,2}(\mathbb{R}, W_k) \longrightarrow W^{r,2}(\mathbb{R}, W_k)$$

is compact, can be seen as a reinterpretation of [FW] Lem. 8.4 and [FW] Lem. 8.5 .
For instance, to account for the case $r = 0$, the proof of [FW] Lem. 8.4 can be rephrased as follows: At finite $T \geq 1$ the "truncation map"

$$c_T : W^{1,2}(\mathbb{R}, \mathbb{B}) \longrightarrow W^{1,2}\big((-T, T), \mathbb{B}\big) \longrightarrow L^2\big((-T, T), \mathbb{B}\big) \longrightarrow L^2(\mathbb{R}, \mathbb{B})$$

is compact and the calculation

$$\Big\|\gamma_{\text{-}\delta} \cdot \big(v - v|_{(-T,T)}\big)\Big\|_{L^2(\mathbb{R},\mathbb{B})} \leq e^{-\delta T} \|v\|_{L^2(\mathbb{R},\mathbb{B})} \leq e^{-\delta T} \|v\|_{W^{1,2}(\mathbb{R},\mathbb{B})}$$

shows that $\gamma_{\text{-}\delta} \circ c_T \longrightarrow \gamma_{\text{-}\delta}$ converges in the operator norm. Hence, $\gamma_{\text{-}\delta}\colon W^{1,2}(\mathbb{R}, \mathbb{B}) \longrightarrow L^2(\mathbb{R}, \mathbb{B})$ is compact itself.

The case $r \geq 1$, as covered by [FW] Lem. 8.5 , can be seen by a subsequence argument similar to the one encountered in Auxiliary Lemma 4.2(b) and relies on the fact that all derivatives $\gamma^{(m)}_{\text{-}\delta}(t)$ are again multiples of the exponentially decaying $\gamma_{\text{-}\delta}(t)$. $\square$

As a follow-up, Lemma 4.3 will show that not only $\gamma_{\text{-}\delta} : W_k^{r+1} \longrightarrow W_k^r$
but also $\gamma_{\text{-}\delta} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ is a compact operator. This requires a two-step process:

Auxiliary Lemma 4.2 (Restriction of the domain and restriction of the target)
Let $\mathcal{H}, X, Y, Z$ and W be Banach spaces.
Assume we are given bounded linear inclusions $X \hookrightarrow \mathcal{H} \hookleftarrow Y$
and consider the canonical inclusions $X \overset{\iota_X}{\hookleftarrow} X \cap Y \overset{\iota_Y}{\hookrightarrow} Y$ from Auxiliary Lemma 2.30.

a) If $f : X \longrightarrow Z$ is compact, then so is the restricted map $X \cap Y \hookrightarrow X \xrightarrow{f} Z$.

b) If $g : W \longrightarrow X \cap Y$ is a bounded linear map such that $\iota_X \circ g$ and $\iota_Y \circ g$ are compact, then g is compact itself.

Proof. a) Since f is compact and ι_X bounded linear, the composition $f \circ \iota_X$ is again compact.
b) Consider a bounded sequence $w_n \in W$. As $\iota_X \circ g$ is compact, we can find a subsequence w_{n_k} such that $g(w_{n_k})$ converges in X. Since $\iota_Y \circ g$ is compact, we can pass to a subsequence of w_{n_k} whose image converges in Y. The resulting sequence converges w.r.t. $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$ □

Lemma 4.3 ($\gamma_{\text{-}\delta}$ as a compact operator between nested Sobolev spaces)
Fix an arbitrary $\delta > 0$. Then $\gamma_{\text{-}\delta} : \mathcal{H} \longrightarrow \mathcal{H}$ restricts to compact operators $\gamma_{\text{-}\delta} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$

Proof. Consider a fixed W_k^r with $k + r = n$. Lemma 4.1 shows that $\gamma_{\text{-}\delta} \in \mathcal{L}(\mathcal{H})$ restricts to a compact operator $\gamma_{\text{-}\delta} : W_k^{r+1} \longrightarrow W_k^r$, so by Auxiliary Lemma 4.2(a) the restriction

$$\gamma_{\text{-}\delta} : \mathcal{W}_{n+1} \subset W_k^{r+1} \longrightarrow W_k^r$$

is compact as well. Now that $\gamma_{\text{-}\delta} : \mathcal{W}_{n+1} \longrightarrow W_k^r$ is compact for all $r + k = n$, we can use Auxiliary Lemma 4.2(b) to conclude that also

$$\gamma_{\text{-}\delta} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n = \bigcap_{k+r=n} W_k^r \qquad \text{is compact.}$$

□

The operator $\gamma_{\text{-}\delta} : (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ serves as an inspiration for the following abstract, but very useful definition:

Definition 4.4 (Twisting sequence)
Let $\mathcal{W} = (\mathcal{W}_n)_{n\geq 0}$ be an almost sc-Banach space. Given a sequence of injective sc-operators

$$\alpha_i : (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}\,, \qquad i \in \mathbb{N}$$

we say that $(\alpha_i)_{i\in\mathbb{N}}$ is a *twisting sequence* on $\mathcal{W}$ if

1. For all $i, n \geq 0$ the restricted operator $\alpha_i : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ is compact.
2. There exists a subset $\mathcal{S} \subset \mathcal{W}_\infty$ such that $\mathcal{S}$ is dense in every $\mathcal{W}_n$
   and all α_i satisfy $\alpha_i(\mathcal{S}) = \mathcal{S}$

Let us elaborate on the implications of condition (2):

Remark 4.5 (Cumulated twisting sequence)
Along with a twisting sequence $(\alpha_i)_{i\in\mathbb{N}}$ we consider the injective sc-operators
$\beta_i : (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ defined by $\beta_0 = \mathrm{id}_{\mathcal{H}}\,,\ \beta_{i+1} = \beta_i \circ \alpha_i$

The collection $(\beta_i)_{i\in\mathbb{N}}$ will be called the *cumulated twisting sequence* associated to $(\alpha_i)_{i\in\mathbb{N}}$

Observe that Definition 4.4 immediately implies $\beta_i(\mathcal{S}) = \mathcal{S}$ for all $i \geq 0$.
In fact, by bootstrapping condition (2) we get $\mathcal{S} \subset \alpha_0(\mathcal{S}) \subset \alpha_0(\alpha_1(\mathcal{S})) \subset \dots$
as well as $\mathcal{S} \supset \alpha_0(\mathcal{S}) \supset \alpha_0(\alpha_1(\mathcal{S})) \supset \dots$

Moreover, we claim that, when $\beta_i(\mathcal{W}_n)$ is equipped with the norm coming from $\beta_i : \mathcal{W}_n \xrightarrow{\sim} \beta_i(\mathcal{W}_n)$, $\mathcal{S} \subset \beta_i(\mathcal{W}_n)$ is a dense subset for all $i, n \geq 0$. Indeed, since $\mathcal{S} \subset \mathcal{W}_n$ is dense subset, every $x \in \mathcal{W}_n$ can be approximated by a sequence $x_m \in \mathcal{S}$. Using the injective map $\beta_i : \mathcal{W}_n \longrightarrow \mathcal{W}_n$ to identify $\mathcal{W}_n$ with $\beta_i(\mathcal{W}_n)$, this can be rephrased as having $\beta_i(x_m) \longrightarrow \beta_i(x)$ converge w.r.t the norm $\left\|\beta_i^{-1}(\cdot)\right\|_{\mathcal{W}_n}$ coming from above.
Note that by $\beta_i(\mathcal{S}) \subset \mathcal{S}$ not only the x_m but also the $\beta_i(x_m)$ belong to $\mathcal{S}$.

Let us check that the conditions from Definition 4.4 are fulfilled in our case of interest:

Lemma 4.6 (Verifying the conditions of Definition 4.4)
Consider a "weight sequence" $0 = \delta_0 < \delta_1 < \ldots\ldots$ *and write* $\Delta\delta_i := \delta_{i+1} - \delta_i > 0$
Then $\alpha_i := \gamma_{\text{-}\Delta\delta_i} : (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ *is a twisting sequence in the sense of Definition 4.4 and we have* $\beta_i = \gamma_{\text{-}\delta_i}$

Proof. Having $\gamma_{\text{-}\delta}(t) = e^{-\delta\eta(t)} > 0$ at every $t \in \mathbb{R}$ ensures that $\gamma_{\text{-}\delta} : \mathcal{H} \longrightarrow \mathcal{H}$ is injective. Moreover, Lemma 4.3 shows that $\gamma_{\text{-}\Delta\delta_i} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ is a compact operator for all $i, n \geq 0$, which accounts for the first condition of Definition 4.4.

So it remains to verify the second condition: In Proposition 2.42 we have seen that the set

$$\mathcal{S} := C_0^\infty(\mathbb{R})W_\infty$$

is dense in every $\mathcal{W}_n$. Since multiplication by $\gamma_{\pm\delta} \in C^\infty(\mathbb{R})$ preserves $C_0^\infty(\mathbb{R})$, we immediately find $\gamma_{\text{-}\delta}(\mathcal{S}) \subset \mathcal{S}$ and $\mathcal{S} = \gamma_{\text{-}\delta}\big(\gamma_{+\delta}(\mathcal{S})\big) \subset \gamma_{\text{-}\delta}(\mathcal{S})$. $\square$

Next we have to explain how the compact operators $\gamma_{\text{-}\Delta\delta} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ translate into compact inclusions $\mathcal{W}_{n+1}^{\delta+\Delta\delta} \hookrightarrow \mathcal{W}_n^{\delta}$:

Remark 4.7 (Keeping track of compact inclusions)
Let us consider the following category, denoted by $(\mathcal{B}/\mathcal{H})_{inj}$:

- The objects are pairs (X, ρ) where X is a Banach space and $\rho : X \hookrightarrow \mathcal{H}$ a bounded linear injective map to our favorite ambient Banach space $\mathcal{H}$
- A morphism $(X, \rho) \xrightarrow{f} (Y, \kappa)$ is bounded linear map $f : X \longrightarrow Y$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\rho"'] & & Y \arrow[dl, "\kappa"] \\ & \mathcal{H} & \end{tikzcd}$$

commutes.

The reason for introducing $(\mathcal{B}/\mathcal{H})_{inj}$ is the following functor $(\mathcal{B}/\mathcal{H})_{inj} \longrightarrow \mathcal{B}$ to the category of Banach spaces and bounded linear maps:

- To a bounded linear injective map $\rho : X \hookrightarrow \mathcal{H}$ we associate the subspace $\rho(X) \subset \mathcal{H}$. The linear isomorphism $\rho : X \xrightarrow{\sim} \rho(X)$ makes $\rho(X)$ a Banach space such that the inclusion $\rho(X) \subset \mathcal{H}$ is a bounded linear map.
- Crucially, every commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\rho"'] & & Y \arrow[dl, "\kappa"] \\ & \mathcal{H} & \end{tikzcd}$$

  induces a bounded linear inclusion $\rho(X) \subset \kappa(Y)$ whose operator norm equals $\|f\|_{\mathcal{L}(X,Y)}$.
  If the operator $f : X \longrightarrow Y$ is compact, then so is the inclusion $\rho(X) \subset \kappa(Y)$.

Notation. Given a bounded linear injective map $\rho : X \hookrightarrow \mathcal{H}$ we abbreviate $X^\rho := \rho(X)$

Example. Going back to our operator $\gamma_{\text{-}\delta} \in \mathcal{L}(\mathcal{H})$, the maps $\gamma_{\text{-}\delta} : \mathcal{W}_k \longrightarrow \mathcal{H}$ induce linear isomorphisms $\mathcal{W}_k \xrightarrow{\sim} \gamma_{\text{-}\delta}(\mathcal{W}_k) \subset \mathcal{H}$ by which we can regard each

$$\mathcal{W}_k^\delta := \gamma_{\text{-}\delta}(\mathcal{W}_k)$$

as a Banach space with norm $\|\cdot\|_{\mathcal{W}_k^\delta} = \|\gamma_{+\delta}(\cdot)\|_{\mathcal{W}_k}$.

Given $\Delta\delta := \delta - \delta' > 0$ Lemma 4.3 shows that $\gamma_{\text{-}\Delta\delta} : \mathcal{W}_{n+1} \longrightarrow \mathcal{W}_n$ is compact, so the commutative triangle

$$\begin{tikzcd} \mathcal{W}_{n+1} \arrow[rr, "\gamma_{\text{-}\Delta\delta}"] \arrow[dr, "\gamma_{\text{-}\delta}"'] & & \mathcal{W}_n \arrow[dl, "\gamma_{\text{-}\delta'}"] \\ & \mathcal{H} & \end{tikzcd}$$

induces a compact inclusion $\mathcal{W}_{n+1}^\delta \subset \mathcal{W}_n^{\delta'}$.

Returning to the abstract picture of Definition 4.4 and Remark 4.5, we observe that every twisting sequence $\alpha_i : (\mathcal{W}_n)_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ comes with a commutative diagram

$$
\begin{tikzcd}
\cdots & (\mathcal{W}_3, \mathrm{id}_{\mathcal{H}}) \arrow[r, hook, "\iota"] & (\mathcal{W}_2, \mathrm{id}_{\mathcal{H}}) \arrow[r, hook, "\iota"] & (\mathcal{W}_1, \mathrm{id}_{\mathcal{H}}) \arrow[r, hook, "\iota"] & (\mathcal{H}, \mathrm{id}_{\mathcal{H}}) \\
\cdots & (\mathcal{W}_3, \beta_1) \arrow[r, hook, "\iota"] \arrow[u, "\alpha_0"] \arrow[ur, red] & (\mathcal{W}_2, \beta_1) \arrow[r, hook, "\iota"] \arrow[u, "\alpha_0"] \arrow[ur, red] & (\mathcal{W}_1, \beta_1) \arrow[r, hook, "\iota"] \arrow[u, "\alpha_0"] \arrow[ur, red] & (\mathcal{H}, \beta_1) \arrow[u, "\alpha_0"] \\
\cdots & (\mathcal{W}_3, \beta_2) \arrow[r, hook, "\iota"] \arrow[u, "\alpha_1"] \arrow[ur, red] & (\mathcal{W}_2, \beta_2) \arrow[r, hook, "\iota"] \arrow[u, "\alpha_1"] \arrow[ur, red] & (\mathcal{W}_1, \beta_2) \arrow[r, hook, "\iota"] \arrow[u, "\alpha_1"] \arrow[ur, red] & (\mathcal{H}, \beta_2) \arrow[u, "\alpha_1"] \\
 & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

that under the functor $(\mathcal{B}/\mathcal{H})_{inj} \longrightarrow \mathcal{B}$ translates to a bifiltration

$$
\begin{tikzcd}
\cdots & \mathcal{W}_3 \arrow[r, phantom, "\subset"] & \mathcal{W}_2 \arrow[r, phantom, "\subset"] & \mathcal{W}_1 \arrow[r, phantom, "\subset"] & \mathcal{H} \\
\cdots & \beta_1(\mathcal{W}_3) \arrow[r, phantom, "\subset"] \arrow[u, phantom, "\cup"] \arrow[ur, red] & \beta_1(\mathcal{W}_2) \arrow[r, phantom, "\subset"] \arrow[u, phantom, "\cup"] \arrow[ur, red] & \beta_1(\mathcal{W}_1) \arrow[r, phantom, "\subset"] \arrow[u, phantom, "\cup"] \arrow[ur, red] & \beta_1(\mathcal{H}) \arrow[u, phantom, "\cup"] \\
\cdots & \beta_2(\mathcal{W}_3) \arrow[r, phantom, "\subset"] \arrow[u, phantom, "\cup"] \arrow[ur, red] & \beta_2(\mathcal{W}_2) \arrow[r, phantom, "\subset"] \arrow[u, phantom, "\cup"] \arrow[ur, red] & \beta_2(\mathcal{W}_1) \arrow[r, phantom, "\subset"] \arrow[u, phantom, "\cup"] \arrow[ur, red] & \beta_2(\mathcal{H}) \arrow[u, phantom, "\cup"] \\
 & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}
\tag{4.1}
$$

It turns out that the diagonals of this bifiltration are the desired 'honest sc-Banach spaces':

Lemma 4.8 (Honest sc-Banach spaces from a twisting sequence)
Let $(\alpha_i)_{i\in\mathbb{N}}$ be a twisting sequence on an almost sc-Banach space $\mathcal{W} = (\mathcal{W}_n)_{n\geq 0}$ and denote by $(\beta_i)_{i\in\mathbb{N}}$ the cumulated twisting sequence from Remark 4.5
Then at every $k \geq 0$ we have an honest sc-Banach space $\big(\beta_n(\mathcal{W}_{n+k})\big)_{n\in\mathbb{N}}$

Proof. From the definition of a twisting sequence we get a string of compact operators

$$\mathcal{W}_k \xleftarrow{\alpha_0} \mathcal{W}_{k+1} \xleftarrow{\alpha_1} \mathcal{W}_{k+2} \longleftarrow \ldots$$

Writing $\beta_0 = \mathrm{id}_{\mathcal{H}}$, $\beta_{i+1} = \beta_i \circ \alpha_i$ this can be augmented to a string of morphisms in the category $(\mathcal{B}/\mathcal{H})_{\mathrm{inj}}$:

$$(\mathcal{W}_k, \beta_0) \xleftarrow{\alpha_0} (\mathcal{W}_{k+1}, \beta_1) \xleftarrow{\alpha_1} (\mathcal{W}_{k+2}, \beta_2) \longleftarrow \ldots$$

By applying the functor $(\mathcal{B}/\mathcal{H})_{\mathrm{inj}} \longrightarrow \mathcal{B}$ from Remark 4.7 we conclude that

$$\mathcal{W}_k \supset \beta_1(\mathcal{W}_{k+1}) \supset \beta_2(\mathcal{W}_{k+2}) \supset \ldots$$

is a filtration of Banach spaces with compact inclusions.

Now let $\mathcal{S} \subset \mathcal{W}_\infty$ be the mysterious set from the second condition of Definition 4.4. In Remark 4.5 we have seen that $\mathcal{S}$ is densely contained in $\beta_i(\mathcal{W}_n)$ for all combinations $i, n \geq 0$. Thus, we have found a subset $\mathcal{S} \subset \bigcap_{n\geq 0} \beta_n(\mathcal{W}_{n+k})$ such that $\mathcal{S}$ is dense in every $\beta_n(\mathcal{W}_{n+k})$. $\square$

4.2 Twistable and twist-regularizing operators

We introduce a class of operators $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ that act as sc-operators on the bifiltration (4.1):

Definition 4.9 (Twistable operator)
Let $(\alpha_i)_{i\in\mathbb{N}}$ be a twisting sequence on an almost sc-Banach space $\mathcal{W} = (\mathcal{W}_n)_{n\geq 0}$ and denote by $(\beta_i)_{i\in\mathbb{N}}$ the cumulated twisting sequence from Remark 4.5

An operator $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ will be called *twistable* if for every $i \in \mathbb{N}$ there exists a sc-operator $D_i : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ such that

$$\begin{array}{ccc} \mathcal{W}_1 & \xrightarrow{D_i} & \mathcal{H} \\ \downarrow\beta_i & & \downarrow\beta_i \\ \mathcal{W}_1 & \xrightarrow{D} & \mathcal{H} \end{array} \quad \text{commutes.}$$

Moreover, we will say that $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is *strongly twistable* if the $D_i : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ are regularizing sc-operators.

Remark. $\beta_0 = \mathrm{id}_{\mathcal{H}}$ implies $D = D_0$, so only (regularizing) sc-operators can be (strongly) twistable.

Lemma 4.10 (Families of sc-operators induced by a twistable operator)
Let $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ be a twistable operator in the sense of Definition 4.9
Then for all $i, n \geq 0$ we have $D\big(\beta_i(\mathcal{W}_{n+1})\big) \subset \beta_i(\mathcal{W}_n)$ and $\|D\|_{\mathcal{L}(\mathcal{W}^{\beta_i}_{n+1},\, \mathcal{W}^{\beta_i}_n)} = \|D_i\|_{\mathcal{L}(\mathcal{W}_{n+1},\, \mathcal{W}_n)}$
In particular, we observe that

- *for every fixed $i \geq 0$,*
$$D : \big[\beta_i(\mathcal{W}_{n+1})\big]_{n\geq 0} \longrightarrow \big[\beta_i(\mathcal{W}_n)\big]_{n\geq 0}$$
is a sc-operator between almost sc-Banach spaces
- *for every fixed $k \geq 0$,*
$$D : \big[\beta_n(\mathcal{W}_{n+k+1})\big]_{n\geq 0} \longrightarrow \big[\beta_n(\mathcal{W}_{n+k})\big]_{n\geq 0}$$
is a sc-operator between honest sc-Banach spaces

Proof. Using $D_i(\mathcal{W}_{n+1}) \subset \mathcal{W}_n$ we obtain $D \circ \beta_i(\mathcal{W}_{n+1}) = \beta_i \circ D_i(\mathcal{W}_{n+1}) \subset \beta_i(\mathcal{W}_n)$.
Given $x \in \mathcal{W}_{n+1}$ we combine the calculations
$$\|D\beta_i(x)\|_{\beta_i(\mathcal{W}_n)} = \|\beta_i D_i(x)\|_{\beta_i(\mathcal{W}_n)} = \|D_i(x)\|_{\mathcal{W}_n} \quad \text{and} \quad \|\beta_i(x)\|_{\beta_i(\mathcal{W}_{n+1})} = \|x\|_{\mathcal{W}_{n+1}}$$
to conclude that the operator norms $\|D\| = \|D_i\|$ agree. □

While being 'twistable' is enough to make D a sc-operator, D has to be 'strongly twistable' to become a regularizing sc-operator. This kind of regularization, however, only works at fixed weight level β_i, i.e. for the horizontal filtrations in (4.1):

Lemma 4.11 (Regularization property of 'strongly twistable' operators)
Let $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ be strongly twistable. Then for every fixed $i \geq 0$
$$D : \big[\beta_i(\mathcal{W}_{n+1})\big]_{n\geq 0} \longrightarrow \big[\beta_i(\mathcal{W}_n)\big]_{n\geq 0}$$
is a regularizing sc-operator between almost sc-Banach spaces.

Proof. We work at fixed $i \geq 0$. With Auxiliary Lemma 2.12 in mind let us verify that $D : \beta_i(\mathcal{W}_1) \longrightarrow \beta_i(\mathcal{H})$ is an escalator for $\big[\beta_i(\mathcal{W}_n)\big]_{n\geq 0}$:

Given $n \geq 1$ assume that for $u \in \beta_i(\mathcal{W}_n)$ it so happens that Du belongs to $\beta_i(\mathcal{W}_n)$ instead of just $\beta_i(\mathcal{W}_{n-1})$. This means can find $v, w \in \mathcal{W}_n$ with
$$u = \beta_i v \quad \text{and} \quad Du = \beta_i w$$
By definition of a strongly twistable operator there exists a regularizing sc-operator $D_i : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ such that
$$\begin{array}{ccc} \mathcal{W}_1 & \xrightarrow{D_i} & \mathcal{H} \\ \downarrow\beta_i & & \downarrow\beta_i \\ \mathcal{W}_1 & \xrightarrow{D} & \mathcal{H} \end{array} \quad \text{commutes.}$$
Combining the two identities above we obtain $\beta_i\, w = Du = \beta_i\, D_i v$.
As $\beta_i : \mathcal{H} \longrightarrow \mathcal{H}$ is injective, this implies $D_i v = w \in \mathcal{W}_n$. Now since D_i is regularizing, we conclude that $v \in \mathcal{W}_{n+1}$ and therefore $u = \beta_i v \in \beta_i(\mathcal{W}_{n+1})$. □

In order to increase the weight level, we need an additional, more subtle property:

Definition 4.12 (Twist-regularizing operator)
Let $(\alpha_i)_{i\in\mathbb{N}}$ be a twisting sequence on an almost sc-Banach space $\mathcal{W} = (\mathcal{W}_n)_{n\geq 0}$ and denote by $(\beta_i)_{i\in\mathbb{N}}$ the cumulated twisting sequence from Remark 4.5
An operator $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ will be called *twist-regularizing* if it satisfies

$$D^{-1}\big(\beta_i(\mathcal{H})\big) \subset \beta_i(\mathcal{W}_1) \qquad \text{for all } i \geq 0$$

Finally, we have accumulated enough structure to conclude that operators $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ which are both 'strongly twistable' and 'twist-regularizing' lead to regularizing sc-operators between honest sc-Banach spaces:

Proposition 4.13 (Combination of 'strongly twistable' and 'twist-regularizing')
Let $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ be strongly twistable and twist-regularizing.
Then $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ satisfies $D^{-1}\big(\beta_i(\mathcal{W}_n)\big) = \beta_i(\mathcal{W}_{n+1})$ for all pairs $i, n \geq 0$.
In particular, for every fixed $k \geq 0$

$$D : \big[\beta_n(\mathcal{W}_{n+k+1})\big]_{n\geq 0} \longrightarrow \big[\beta_n(\mathcal{W}_{n+k})\big]_{n\geq 0}$$

is a regularizing sc-operator between honest sc-Banach spaces.

Proof. Fix any $i \geq 0$. By Lemma 4.11 we know that

$$D : \big[\beta_i(\mathcal{W}_{n+1})\big]_{n\geq 0} \longrightarrow \big[\beta_i(\mathcal{W}_n)\big]_{n\geq 0}$$

is a regularizing sc-operator. Hence, the original operator $D : \mathcal{W}_1 \longrightarrow \mathcal{H}$ satisfies

$$\beta_i(\mathcal{W}_1) \cap D^{-1}\big(\beta_i(\mathcal{W}_n)\big) = \beta_i(\mathcal{W}_{n+1})$$

Since D is twist-regularizing, we have $\beta_i(\mathcal{W}_1) = D^{-1}(\beta_i(\mathcal{H}))$ and conclude that

$$\beta_i(\mathcal{W}_{n+1}) = D^{-1}\big(\beta_i(\mathcal{H})\big) \cap D^{-1}\big(\beta_i(\mathcal{W}_n)\big) = D^{-1}\big(\beta_i(\mathcal{W}_n)\big)$$

□

4.3 The sc-Fredholm property of $D_A : (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_{n})_{n\geq 0}$

Inspired by Theorem 2.38, let us investigate whether not only $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ but in fact also $D_A : (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_{n})_{n\geq 0}$ is a regularizing sc-operator.

According to Proposition 4.13 we have to verify that $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is both 'strongly twistable' and 'twist-regularizing'.

First of all, being 'strongly twistable' is fairly straightforward:

Lemma 4.14 (D_A is strongly twistable)
Let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an almost sc-Hilbert space $H \supset W_1 \supset \dots$ Assume that $B(t) \in \mathcal{L}(H)$ is a moderate perturbation and write $A(t) = A_0 + B(t)$.

Then given any weight sequence $0 = \delta_0 < \delta_1 < \dots$ (bounded or not) the operator $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is strongly twistable w.r.t. $\beta_i = \gamma_{\text{-}\delta_i} = e^{-\delta_i \eta}$

Proof. Fix any $\delta > 0$. Given $\xi \in \mathcal{W}_1$ we have $D_A\, \gamma_{\text{-}\delta}\xi = \gamma_{\text{-}\delta}\big[D_A - \delta\eta'\big]\xi$, so the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[d, "\gamma_{\text{-}\delta}"] & \mathcal{W}_1 \arrow[l, "D_A - \delta\eta'"'] \arrow[d, "\gamma_{\text{-}\delta}"] \\ \mathcal{H} & \mathcal{W}_1 \arrow[l, "D_A"] \end{tikzcd}$$

commutes.

It remains to verify that $D_A - \delta\eta'$ is a regularizing sc-operator.
Indeed, for moderate perturbations $B(t) \in \mathcal{L}(\mathcal{H})$ the combination of Corollary 2.34 and Proposition 2.36 shows that $D_A : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ is a regularizing sc-operator.
On the other hand, its derivative $\frac{\mathrm{d}}{\mathrm{d}t}\delta\eta' \in C^\infty_0(\mathbb{R})$ being a bump supported in $[-1, 1]$ ensures that the function $\delta\eta'$ is smooth with bounded derivatives, so multiplication by $\delta\eta'$ preserves the bifiltration $W^r_k = W^{r,2}(\mathbb{R}, W_k)$. In particular, we have $\delta\eta'(\mathcal{W}_n) \subset \mathcal{W}_n$ and $\delta\eta' \in \mathcal{L}(\mathcal{W}_n)$, so with Auxiliary Lemma 2.13 we conclude that the perturbed $D_A - \delta\eta' : (\mathcal{W}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}_n)_{n\geq 0}$ is a regularizing sc-operator as well. □

Being twist-regularizing, on the other hand, poses constraints on the weight sequence

$$0 \;=\; \delta_0 < \delta_1 < \dots$$

in a way that depends on the specific operator family $A(t) : W_1 \longrightarrow H$.
This requires a subtle two-step proof, probably the most interesting of this thesis:

Theorem 4.15 (D_A is twist-regularizing)
Let $A_0 : W_1 \longrightarrow H$ be a baseline operator on an honest sc-Hilbert space $H \supset W_1 \supset \dots$ Assume that $B(t) \in \mathcal{L}(H)$ is a very good perturbation and write $A(t) = A_0 + B(t)$.
Then there exists $\delta_\infty > 0$ such that the operator $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ satisfies

$$D_A^{-1}(\mathcal{H}^\delta) \subset \mathcal{W}^\delta_1 \quad \textit{for all } \delta \in [0, \delta_\infty)$$

In particular, for any weight sequence $0 = \delta_0 < \delta_1 < \dots$ bounded by δ_∞ the operator $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is twist-regularizing w.r.t. $\beta_i = \gamma_{\text{-}\delta_i}$

Proof strategy. The "double helix" of Remark 3.16 contains a triangle

$$\begin{tikzcd}
 & \mathcal{H} = D_A(\mathcal{W}_1) \oplus \ker D_{-A} \arrow[dd, shift left, "Q \in \mathcal{L}(\mathcal{H}, \mathcal{W}_2)"] \\
\ker D_A \oplus D_{\text{-}A}(\mathcal{W}_2) = \mathcal{W}_1 \arrow[ur, "D_A"] & \\
 & \mathcal{W}_2 = D_A(\mathcal{W}_3) \oplus \ker D_{-A} \arrow[ul, "D_{\text{-}A}"] \arrow[uu, shift left, "D_A D_{\text{-}A}"]
\end{tikzcd}$$

where the Laurent coefficient

$$Q = \frac{1}{2\pi i} \int_{S^1} \frac{\mathrm{d}\lambda}{\lambda} \, [D_A D_{\text{-}A} - \lambda]^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{W}_2)$$

serves as a parametrix (quasi-inverse) to $S = D_A D_{\text{-}A}$.

Our proof of Theorem 4.15 consists of two independent steps: Given $u \in \mathcal{W}_1$ such that $D_A u \in \mathcal{H}^\delta$, we decompose $u = v + w$ into $v \in \ker D_A$ and $w \in D_{\text{-}A}(\mathcal{W}_2)$, allowing us to verify $v \in \mathcal{W}_1^\delta$ and $w \in \mathcal{W}_1^\delta$ individually.

- Step 1 (Spectral perturbation theory)
  Corollary 3.17 shows that $w = D_{\text{-}A} Q D_A u$. Since from Lemma 4.14 we know that $D_{\text{-}A}(\mathcal{W}_2^\delta) \subset \mathcal{W}_1^\delta$, it suffices to prove $Q(\mathcal{H}^\delta) \subset \mathcal{W}_2^\delta$. We will find this condition to hold as long as δ stays below a threshold δ_{max} related to the spectral gap of $S = D_A D_{\text{-}A}$.

- Step 2 (Exponential decay of solutions to $D_A v = 0$)
  Note that $D_A v = 0$ is agnostic about our particular choice of $\delta > 0$ in $\mathcal{H}^\delta \subset \mathcal{H}$. Instead, v will belong to $\mathcal{W}_1^\delta$ for an intrinsic reason related to the operator family $A(t) \in \mathcal{L}(W_1, H)$ approaching invertible endpoints $A_\pm$ as $t \longrightarrow \pm\infty$. More precisely, choosing $\bar{\delta} = \min(\delta_\pm)$ where $\delta_\pm = r(A_\pm)$ denotes the invertibility radius at $A_\pm$ we will observe that $\ker D_A \subset \mathcal{W}_1^\delta$ for all $\delta \in [0, \bar{\delta})$.

After steps 1 and 2 have been accomplished, Theorem 4.15 holds with $\delta_\infty := \min(\bar{\delta}, \delta_{max})$. Step 2 will be the only part of our argument that relies on $B(t) \in \mathcal{L}(H)$ being a localized perturbation in the sense of Definition 2.21. $\square$

Let us now carry out the two steps of Theorem 4.15:

Step 1. There exists $\delta_{max} > 0$ such that $Q(\mathcal{H}^\delta) \subset \mathcal{W}_2^\delta$ for all $\delta \in [0, \delta_{max})$.

Proof. Fix any $\delta > 0$. Given $\xi \in \mathcal{W}_1$ we have $D_{\pm A}\,\gamma_{\text{-}\delta}\xi = \gamma_{\text{-}\delta}\big[D_{\pm A} - \delta\eta'\big]\xi$.
Since for $\xi \in \mathcal{W}_2$ the term $\big[D_{\text{-}A} - \delta\eta'\big]\xi$ itself belongs to $\mathcal{W}_1$, iterating this formula yields

$$\begin{aligned} D_A D_{\text{-}A}\,\gamma_{\text{-}\delta}\xi &= \gamma_{\text{-}\delta}\big[D_A - \delta\eta'\big]\big[D_{\text{-}A} - \delta\eta'\big]\xi \\ &= \gamma_{\text{-}\delta}\big[D_A D_{\text{-}A} - \delta\eta'\,D_{\text{-}A} - D_A\,\delta\eta' + (\delta\eta')^2\big]\xi \;=\; \gamma_{\text{-}\delta}\big[D_A D_{\text{-}A} - \delta\cdot K_\delta\big]\xi \end{aligned} \tag{4.2}$$

where the perturbation

$$K_\delta := \eta'\,\underbrace{\big[D_A + D_{\text{-}A}\big]}_{2\frac{\mathrm{d}}{\mathrm{d}t}} + \eta'' - \delta(\eta')^2 \in \mathcal{L}(\mathcal{W}_2, \mathcal{H})$$

is universal in the sense that it no longer depends on $A(t)$ but only on the chosen η and $\delta > 0$. Given any $\lambda \in \mathbb{C}$, Equation (4.2) shows that by restricting the domain of $D_A D_{\text{-}A} - \lambda$ to $\mathcal{W}_2^\delta \subset \mathcal{W}_2$ we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{H} \arrow[d, phantom, sloped, "\supset"] & & \mathcal{W}_2 \arrow[ll, "D_A D_{\text{-}A} - \lambda"'] \arrow[d, phantom, sloped, "\supset"] \\ \mathcal{H}^\delta & & \mathcal{W}_2^\delta \arrow[ll, "D_A D_{\text{-}A} - \lambda"'] \arrow[llu, dotted] \\ \mathcal{H} \arrow[u, "\gamma_{\text{-}\delta}"', "\wr\|"] & & \mathcal{W}_2 \arrow[u, "\gamma_{\text{-}\delta}"', "\wr\|"] \arrow[ll, "D_A D_{\text{-}A} - \delta\cdot K_\delta - \lambda"] \end{tikzcd} \tag{4.3}$$

Thus, treating $D_A D_{\text{-}A}$ on the subspaces $\mathcal{W}_2^\delta$, $\mathcal{H}^\delta$ is equivalent to considering a perturbed operator $D_A D_{\text{-}A} - \delta\cdot K_\delta$ on the original spaces $\mathcal{W}_2$, $\mathcal{H}$. In particular, $D_A D_{\text{-}A} - \lambda : \mathcal{W}_2^\delta \longrightarrow \mathcal{H}^\delta$ being invertible is equivalent to $D_A D_{\text{-}A} - \delta\cdot K_\delta - \lambda : \mathcal{W}_2 \longrightarrow \mathcal{H}$ being invertible, so the two operators share the same resolvent set

$$\rho\big(D_A D_{\text{-}A} : \mathcal{W}_2^\delta \longrightarrow \mathcal{H}^\delta\big) \;=\; \rho\big(D_A D_{\text{-}A} - \delta\cdot K_\delta : \mathcal{W}_2 \longrightarrow \mathcal{H}\big) \subset \mathbb{C}$$

As we will see below, this set inherits the relevant features of $\rho\big(D_A D_{\text{-}A} : \mathcal{W}_2 \longrightarrow \mathcal{H}\big)$ provided that $\delta > 0$ is small enough. Recall from Theorem 3.4 that $S = D_A D_{\text{-}A} : \mathcal{W}_2 \longrightarrow \mathcal{H}$ is self-adjoint and Fredholm, so by Lemma 3.6 its spectrum admits a constant $\epsilon > 0$, called *spectral gap*, such that $\sigma(S) \cap B_\epsilon(0) \subset \{0\}$. For instance, we have

$$S^1_{\epsilon/2} := \big\{\, |\lambda| = \epsilon/2 \big\} \subset \rho(S)$$

Given any $\lambda \in \mathbb{C}$ from the resolvent set of the unperturbed $S = D_A D_{\text{-}A} : \mathcal{W}_2 \longrightarrow \mathcal{H}$, consider the resolvent $R_\lambda(S) = (S - \lambda)^{-1}$ as an operator in $\mathcal{L}(\mathcal{H}, \mathcal{W}_2)$ and observe that

$$D_A D_{\text{-}A} - \lambda - \delta\cdot K_\delta \;=\; \big[\mathrm{id}_{\mathcal{H}} - \delta\cdot \overbrace{K_\delta\; R_\lambda(S)}^{\mathcal{L}(\mathcal{H})}\big](D_A D_{\text{-}A} - \lambda)$$

Now assume that the given combination of $\delta > 0$ and $\lambda \in \rho(S)$ satisfies

$$\delta\cdot \|K_\delta R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} < 1 \tag{4.4}$$

Then $\mathrm{id}_{\mathcal{H}} - \delta \cdot K_\delta R_\lambda(S) \in \mathcal{L}(\mathcal{H})$ is invertible with inverse

$$\big[\mathrm{id}_{\mathcal{H}} - \delta \cdot K_\delta R_\lambda(S)\big]^{-1} = \sum_{n=0}^{\infty} \big[\delta \cdot K_\delta R_\lambda(S)\big]^n \in \mathcal{L}(\mathcal{H})$$

and therefore also $D_A D_{\text{-}A} - \lambda - \delta \cdot K_\delta \in \mathcal{L}(\mathcal{W}_2, \mathcal{H})$ is invertible with inverse

$$\big[D_A D_{\text{-}A} - \lambda - \delta \cdot K_\delta\big]^{-1} \;=\; \underbrace{R_\lambda(S)}_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \sum_{n=0}^{\infty} \big[\delta \cdot K_\delta R_\lambda(S)\big]^n \in \mathcal{L}(\mathcal{H}, \mathcal{W}_2)$$

We claim that with a suitable δ_{max} the constraint (4.4) can be simultaneously satisfied for all $\delta \in (0, \delta_{max})$ and $\lambda \in S^1_{\epsilon/2}$, hence proving that $S^1_{\epsilon/2} \subset \rho\big(D_A D_{-A} - \delta \cdot K_\delta\big)$.
First of all, by fixing an arbitrary $\delta_{cut} > 0$ we guarantee that

$$\|K_\delta\|_{\mathcal{L}(\mathcal{W}_2,\mathcal{H})} \;\leq\; \left\| 2\eta' \frac{\mathrm{d}}{\mathrm{d}t} + \eta'' \right\|_{\mathcal{L}(\mathcal{W}_2,\mathcal{H})} + \delta \cdot \big\|(\eta')^2\big\|_{\mathcal{L}(\mathcal{W}_2,\mathcal{H})} \;\leq\; \kappa = \text{const.}$$

is uniformly bounded for all $\delta \in (0, \delta_{cut})$. Now the bound

$$\|K_\delta R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} \;\leq\; \|K_\delta\|_{\mathcal{L}(\mathcal{W}_2,\mathcal{H})} \|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \;\leq\; \kappa \, \|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)}$$

forces us to study $\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)}$ instead of the better-behaved $\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} = \frac{1}{\mathrm{dist}(\lambda, \sigma(S))}$.
However, in Proposition 3.12 we have found the formula

$$\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \;\leq\; \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \left(1 + \frac{1 + |\lambda|}{\mathrm{dist}\,\big(\lambda, \sigma(S)\big)}\right) \tag{4.5}$$

which for $\lambda \in S^1_{\epsilon/2}$ turns into a uniform bound

$$\|R_\lambda(S)\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \;\leq\; \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \left(2 + \frac{2}{\epsilon}\right)$$

Hence, a suitable $0 < \delta_{max} < \delta_{cut}$ can be obtained by satisfying the requirement

$$\delta_{max} \cdot \|K_\delta R_\lambda(S)\|_{\mathcal{L}(\mathcal{H})} \;\leq\; \delta_{max} \cdot \kappa \, \big\|(S-i)^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{W}_2)} \left(2 + \frac{2}{\epsilon}\right) \;\overset{!}{<}\; 1$$

Now that we have achieved $S^1_{\epsilon/2} \subset \rho\big(D_A D_{\text{-}A} : \mathcal{W}^\delta_2 \longrightarrow \mathcal{H}^\delta\big)$ for $\delta \in (0, \delta_{max})$, recall that, with $D_A D_{\text{-}A} : \mathcal{W}^\delta_2 \longrightarrow \mathcal{H}^\delta$ and the inclusion $\iota : \mathcal{W}^\delta_2 \longrightarrow \mathcal{H}^\delta$ being bounded operators between Banach spaces, we immediately know that the map

$$\lambda \in \rho\big(D_A D_{\text{-}A} : \mathcal{W}^\delta_2 \longrightarrow \mathcal{H}^\delta\big) \longmapsto (D_A D_{\text{-}A} - \lambda)\big|^{-1}_{\mathcal{W}^\delta_2} \in \mathcal{L}(\mathcal{H}^\delta, \mathcal{W}^\delta_2)$$

is analytic and therefore continuous. As a result, the expression

$$\bar{Q} \;=\; \frac{1}{2\pi i} \int_{S^1_{\epsilon/2}} \frac{\mathrm{d}\lambda}{\lambda} \, (D_A D_{\text{-}A} - \lambda)\big|^{-1}_{\mathcal{W}^\delta_2}$$

can be defined as a Bochner integral in $\mathcal{L}(\mathcal{H}^\delta, \mathcal{W}^\delta_2)$. The idea behind our proof is that this $\bar{Q} \in \mathcal{L}(\mathcal{H}^\delta, \mathcal{W}^\delta_2)$ can be compared with the Laurent coefficient

$$Q \;=\; \frac{1}{2\pi i} \int_{S^1_{\epsilon/2}} \frac{\mathrm{d}\lambda}{\lambda} \, (D_A D_{\text{-}A} - \lambda)^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{W}_2)$$

To do so, let us focus on the upper half of the diagram (4.3).

Given $\lambda \in S^1_{\epsilon/2} \subset \rho\big(D_A D_{\text{-}A} : \mathcal{W}_2 \longrightarrow \mathcal{H}\big) \cap \rho\big(D_A D_{\text{-}A} : \mathcal{W}_2^\delta \longrightarrow \mathcal{H}^\delta\big)$ we can invert the horizontal arrows to obtain a commutative square

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "(D_A D_{\text{-}A} - \lambda)^{-1}"] & \mathcal{W}_2 \\
\mathcal{H}^\delta \arrow[u, "\iota_{\mathcal{H}\mathcal{H}^\delta}"] \arrow[r, "(D_A D_{\text{-}A} - \lambda)\big|_{\mathcal{W}_2^\delta}^{-1}"'] \arrow[ur, dashed] & \mathcal{W}_2^\delta \arrow[u, "\iota_{\mathcal{W}_2\mathcal{W}_2^\delta}"']
\end{tikzcd}$$

showing that

$$(D_A D_{\text{-}A} - \lambda)^{-1} \circ \iota_{\mathcal{H}\mathcal{H}^\delta} \;=\; \iota_{\mathcal{W}_2\mathcal{W}_2^\delta} \circ (D_A D_{\text{-}A} - \lambda)\big|_{\mathcal{W}_2^\delta}^{-1} \tag{4.6}$$

Observe that the two sides of Equation (4.6) arise from bounded linear maps

$$\mathcal{L}(\mathcal{H}, \mathcal{W}_2) \xrightarrow{\;\_ \circ\, \iota_{\mathcal{H}\mathcal{H}^\delta}\;} \mathcal{L}(\mathcal{H}^\delta, \mathcal{W}_2) \xleftarrow{\;\iota_{\mathcal{W}_2\mathcal{W}_2^\delta} \circ \,\_\;} \mathcal{L}(\mathcal{H}^\delta, \mathcal{W}_2^\delta)$$

Since Bochner integrals commute with bounded linear maps, Equation (4.6) translates into an identity $\;Q \circ \iota_{\mathcal{H}\mathcal{H}^\delta} \;=\; \iota_{\mathcal{W}_2\mathcal{W}_2^\delta} \circ \bar{Q}\,.\;$ This proves our claim that $Q(\mathcal{H}^\delta) \subset \mathcal{W}_2^\delta$ $\square$

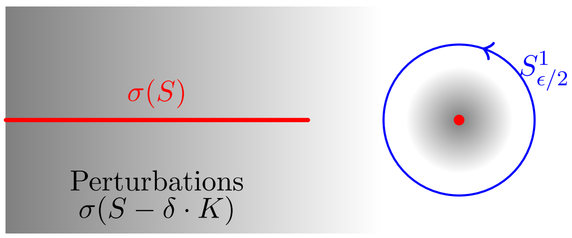


Figure 4.1: Spectral perturbation theory for an unbounded perturbation $K \in \mathcal{L}(\mathcal{H}, \mathcal{W}_2)$ as required by Step 1 of Theorem 4.15. Since the contour $S^1_{\epsilon/2}$ is a compact subset of $\rho(S)$, it will continue to be contained in $\rho(S - \delta \cdot K)$, provided that our tuning parameter δ is small enough.

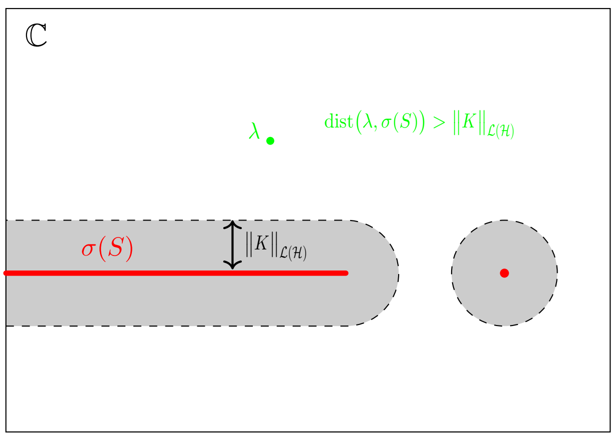


Figure 4.2: Spectral perturbation theory with bounded perturbations $K \in \mathcal{L}(\mathcal{H})$: Given a self-adjoint operator $S : \mathcal{D}(S) \longrightarrow \mathcal{H}$ and $\lambda \in \rho(S)$, the calculation

$$S - \lambda - K = \big[\mathrm{id}_{\mathcal{H}} - K R_\lambda(S)\big](S - \lambda)$$

shows that demanding $\big\|R_\lambda(S)\big\|^{-1}_{\mathcal{L}(\mathcal{H})} = \operatorname{dist}\big(\lambda, \sigma(S)\big) > \big\|K\big\|_{\mathcal{L}(\mathcal{H})}$ is sufficient to guarantee invertibility of $(S - K - \lambda) : \mathcal{D}(S) \longrightarrow \mathcal{H}$. Thus, the perturbed spectrum $\sigma(S - K)$ will be contained in a $\big\|K\big\|_{\mathcal{L}(\mathcal{H})}-$thickening of the original $\sigma(S)$.

Step 2. There exists $\bar{\delta} > 0$ such that $\ker D_A \subset \mathcal{W}_1^{\delta}$ for all $\delta \in \big[0, \bar{\delta}\,\big)$

Proof. As a preparation recall that $A : \mathbb{R} \longrightarrow \mathcal{L}(W_1, H)$ is continuous with invertible endpoints $A_\pm \in \mathcal{L}_{inv}(W_1, H)$ such that $\lim\limits_{t\to\pm\infty} \big\|A(t) - A_\pm\big\|_{\mathcal{L}(W_1,H)} = 0$. In particular, for $|t|$ large enough $A(t)$ will be contained in the open subset $\mathcal{L}_{inv}(W_1, H) \subset \mathcal{L}(W_1, H)$, so according to Auxiliary Lemma 4.16 we have a composition of continuous maps

$$\begin{tikzcd}
 & \mathcal{L}(W_1,H) & & \\
\mathbb{R} \arrow[ur, "A"] \arrow[r, dashed, "\text{large } |t|"'] & \mathcal{L}_{inv}(W_1,H) \arrow[u, hook, "\text{open}"'] \arrow[r, "(\,\cdot\,)^{-1}"] & \mathcal{L}(H,W_1) \arrow[r, "\|\cdot\|_{\mathcal{L}(H,W_1)}"] & \mathbb{R}
\end{tikzcd}$$

Hence, given our favourite constant $c > 1$ we can find T_0 such that for $|t| > T_0$ the operator $A(t) : W_1 \longrightarrow H$ is invertible and satisfies

$$\big\|A(t)^{-1}\big\|_{\mathcal{L}(H,W_1)} \;\leq\; c\,\big\|A_\pm^{-1}\big\|_{\mathcal{L}(H,W_1)}$$

with " $-$ " for $t < -T_0$ and " $+$ " for $t > T_0$. Recall from Auxiliary Lemma 4.16 that $r(A_\pm) = 1/\big\|A_\pm^{-1}\big\|_{\mathcal{L}(H,W_1)}$ is exactly the radius below which convergence of the Neumann series at $A_\pm$ is guaranteed.

Turning to our main business, assume that $v \in \mathcal{W}_1$ satisfies $D_A v = 0$, so with D_A being a regularizing operator on $(\mathcal{W}_n)_{n\geq 0}$ we obtain

$$v \in \mathcal{W}_\infty \subset \mathcal{W}_4 \subset W^{2,2}(\mathbb{R}, W_2)$$

Now Sobolev embedding $W^{2,2}(\mathbb{R}, W_2) \hookrightarrow C^1(\mathbb{R}, W_2)$ implies that the pointwise value $v(t)$ is well-defined and v belongs to $C^1(\mathbb{R}, W_k)$ for all $k \leq 2$.

By using $v \in C^1(\mathbb{R}, W_1)$ and $A \in C^1\big(\mathbb{R}, \mathcal{L}(W_1, H)\big)$ we see that the function

$$g(t) = \frac{1}{2}\big\langle\, A(t)v(t), A(t)v(t)\,\big\rangle_H \;=\; \frac{1}{2}\,\|Av\|_H^2 \;\geq 0$$

is differentiable with continuous derivative

$$\dot{g}(t) \;=\; \operatorname{Re}\Big[\langle \dot{A}v, Av\rangle + \langle A\dot{v}, Av\rangle\Big]$$

Since the pair $\dot{v}(t) \in W_1, v(t) \in W_2$ satisfies $\dot{v}(t) = A(t)v(t)$, this can be rewritten as

$$\dot{g}(t) \;=\; \operatorname{Re}\Big[\langle \dot{A}v, Av\rangle + \langle AAv, Av\rangle\Big]$$

Thus, by invoking $A \in C^2\big(\mathbb{R}, \mathcal{L}(W_1, H)\big) \cap C^1\big(\mathbb{R}, \mathcal{L}(W_2, W_1)\big)$ and $v \in C^1(\mathbb{R}, W_2)$ we conclude that $\dot{g}$ is differentiable with continuous derivative

$$\begin{aligned}\ddot{g}(t) \;=\; \operatorname{Re}\Big[\, &\underbrace{\langle \ddot{A}v, Av\rangle}_{(I)} + \underbrace{\langle \dot{A}\dot{v}, Av\rangle}_{(II)} + \langle \dot{A}v, \dot{A}v\rangle + \underbrace{\langle \dot{A}v, A\dot{v}\rangle}_{(III)} \\ &+ \underbrace{\langle \dot{A}Av, Av\rangle}_{(II)} + \underbrace{\langle A\dot{A}v, Av\rangle}_{(III)} + \langle AA\dot{v}, Av\rangle + \underbrace{\langle AAv, \dot{A}v\rangle}_{(III)} + \langle AAv, A\dot{v}\rangle \,\Big]\end{aligned} \tag{4.7}$$

Using the symmetry of $A(t) : W_1 \longrightarrow H$ to rewrite $\langle A\dot{A}v, Av\rangle = \langle \dot{A}v, AAv\rangle$, $\langle AA\dot{v}, Av\rangle = \langle A\dot{v}, AAv\rangle$ and once again substituting $\dot{v} = Av$ with $\dot{v} \in W_1, v \in W_2$, Equation (4.7) can be brought into the form

$$\ddot{g}(t) \;=\; 2\,\|AAv\|_H^2 \;+\; \big\|\dot{A}v\big\|_H^2 \;+\; \operatorname{Re}\Big[\,\underbrace{\langle \ddot{A}v, Av\rangle}_{(I)} \;+\; 2\,\underbrace{\langle \dot{A}Av, Av\rangle}_{(II)} \;+\; 3\,\underbrace{\langle \dot{A}v, AAv\rangle}_{(III)}\,\Big] \tag{4.8}$$

In the following we would like to establish a lower bound $\ddot{g}(t) \geq ?$
In doing so, the term $\|\dot{A}v\|_H^2 \geq 0$ can be neglected. Note that $\|AAv\|_H^2$ is the only term in (4.8) that does not involve a derivative of A, whereas we will see that the terms containing $\dot{A}$ and $\ddot{A}$ decay in the limit $t \longrightarrow \pm\infty$. To obtain a clean estimate, let us bound $(I), (II), (III)$ in relation to the leading term $\|AAv\|_H^2$. From now on we will work at $|t| > T_0$ such that $A(t) : W_1 \longrightarrow H$ is invertible with

$$\|A(t)^{-1}\|_{\mathcal{L}(H,W_1)} \leq c\,\|A_\pm^{-1}\|_{\mathcal{L}(H,W_1)}$$

for our favourite constant $1 < c < 2$. For $v \in W_1$ we have

$$\|v\|_{W_1} \leq \|A(t)^{-1}\|_{\mathcal{L}(H,W_1)}\,\|A(t)v\|_H \leq c\,\|A_\pm^{-1}\|_{\mathcal{L}(H,W_1)}\,\|Av\|_H$$

so for $v \in W_2$ we obtain

$$\|Av\|_H \leq \|Av\|_{W_1} \leq c\,\|A_\pm^{-1}\|_{\mathcal{L}(H,W_1)}\,\|AAv\|_H$$

Turning to the terms $(I), (II), (III)$ encountered above, we get upper estimates

$$\begin{aligned}
\left|\langle \ddot{A}v, Av\rangle\right| &\leq \|\ddot{A}\|_{\mathcal{L}(W_1,H)}\,\|v\|_{W_1}\,\|Av\|_H &&\leq \|\ddot{A}\|_{\mathcal{L}(W_1,H)}\,c^3\,\|A_\pm^{-1}\|^3_{\mathcal{L}(H,W_1)}\cdot\|AAv\|_H^2\\
\left|\langle \dot{A}Av, Av\rangle\right| &\leq \|\dot{A}\|_{\mathcal{L}(W_1,H)}\,\|Av\|_{W_1}\,\|Av\|_H &&\leq \|\dot{A}\|_{\mathcal{L}(W_1,H)}\,c^2\,\|A_\pm^{-1}\|^2_{\mathcal{L}(H,W_1)}\cdot\|AAv\|_H^2\\
\left|\langle \dot{A}v, AAv\rangle\right| &\leq \|\dot{A}\|_{\mathcal{L}(W_1,H)}\,\|v\|_{W_1}\,\|AAv\|_H &&\leq \|\dot{A}\|_{\mathcal{L}(W_1,H)}\,c^2\,\|A_\pm^{-1}\|^2_{\mathcal{L}(H,W_1)}\cdot\|AAv\|_H^2
\end{aligned}$$

Note that in our setup we consider $A(t) = A_0 + B(t)$ with a constant $A_0 \in \mathcal{L}(W_1, H)$ and potentially varying $B(t) \in \mathcal{L}(H)$, so for derivatives of A we have the simplifications

$$\begin{aligned}
\|\dot{A}\|_{\mathcal{L}(W_1,H)} &\leq \|\dot{B}\|_{\mathcal{L}(H)}\\
\|\ddot{A}\|_{\mathcal{L}(W_1,H)} &\leq \|\ddot{B}\|_{\mathcal{L}(H)}
\end{aligned}$$

The above ingredients show that for $|t| \geq T_0$ the second derivative of $g(t) = \frac{1}{2}\|Av\|_H^2$ obeys

$$\ddot{g}(t) \geq \|AAv\|_H^2 \cdot \Big[2 - c^3\,\|A_\pm^{-1}\|^3_{\mathcal{L}(H,W_1)}\,\|\ddot{B}(t)\|_{\mathcal{L}(H)} - 5\,c^2\,\|A_\pm^{-1}\|^2_{\mathcal{L}(H,W_1)}\,\|\dot{B}(t)\|_{\mathcal{L}(H)}\Big]$$

With our assumption that $\|\dot{B}(t)\|_{\mathcal{L}(H)}, \|\ddot{B}(t)\|_{\mathcal{L}(H)} \longrightarrow 0$ for $t \longrightarrow \pm\infty$ we can find $T_1 \geq T_0$ such that for $|t| \geq T_1$ one has

$$c^3\,\|A_\pm^{-1}\|^3_{\mathcal{L}(H,W_1)}\,\|\ddot{B}(t)\|_{\mathcal{L}(H)} + 5\,c^2\,\|A_\pm^{-1}\|^2_{\mathcal{L}(H,W_1)}\,\|\dot{B}(t)\|_{\mathcal{L}(H)} \leq 2 - \frac{c^2}{2}$$

and therefore

$$\ddot{g}(t) \geq \frac{c^2}{2}\,\|AAv\|_H^2 \geq \bar{\delta}_\pm^2\,g(t) \tag{4.9}$$

with

$$\bar{\delta}_\pm := 1\Big/\|A_\pm^{-1}\|_{\mathcal{L}(H,W_1)} = r(A_\pm)$$

As shown in the proof of [Sa] Lem. 2.11 the differential inequality (4.9) implies a bound

$$g(t) \leq \text{const.} \times e^{-\bar{\delta}_\pm |t|}$$

with decay rates $\bar{\delta}_-$ for $t < -T_1$ and $\bar{\delta}_+$ for $t > +T_1$. Thus, we have proven that the quantity $2\,g(t) = \|A(t)v(t)\|_H^2$ decays exponentially, with decay rates set by the convergence radius of the Neumann series at A_- and A_+, respectively.

Finally, let us unveil the reason for considering $\left\|A(t)v(t)\right\|_H$ instead of the much simpler $\left\|v(t)\right\|_H$ that has been treated in [RS] Prop. 3.14 for instance:
Exponential decay of $\left\|A(t)v(t)\right\|_H$ guarantees that the quantities

$$\begin{aligned} \left\|\dot{v}(t)\right\|_H &= \left\|A(t)v(t)\right\|_H \\ \text{and} \quad \left\|v(t)\right\|_{W_1} &\leq c\left\|A_\pm^{-1}\right\|_{\mathcal{L}(H,W_1)} \left\|A(t)v(t)\right\|_H \end{aligned}$$

decay exponentially as well. Hence, choosing $\bar{\delta} := \min(\delta_\pm)$ we have arranged for

$$\left\|\gamma_\delta v\right\|_{\mathcal{W}_1} = \left\|\gamma_\delta v\right\|_{L^2(\mathbb{R},W_1)} + \left\|\gamma_\delta v\right\|_{L^2(\mathbb{R},H)} + \left\|\gamma_\delta\left[\delta\dot{\eta}\, v + \dot{v}\right]\right\|_{L^2(\mathbb{R},H)} < \infty$$

and thus $v = \gamma_{\text{-}\delta}\left(\gamma_\delta v\right) \in \mathcal{W}_1^\delta$ at every $\delta \in [0,\bar{\delta})$. $\square$

Auxiliary Lemma 4.16 (Topological properties of the inversion map)
Let W, H be Banach spaces. Then the invertible operators $\mathcal{L}_{inv}(W,H)$ form an open subset of $\mathcal{L}(W,H)$ and the map $\mathcal{L}_{inv}(W,H) \xrightarrow{(\cdot)^{-1}} \mathcal{L}(H,W)$ is continuous.

Proof. Given an invertible $L_0 \in \mathcal{L}(W,H)$ the Inverse Mapping Theorem ensures that $L_0^{-1} \in \mathcal{L}(H,W)$ is bounded as well. For perturbations $L \in \mathcal{L}(W,H)$ we can rewrite

$$L_0 + L = \left[\mathrm{id}_H + \overbrace{L \underbrace{L_0^{-1}\,]\, L_0}_{\mathrm{id}_W}}^{\mathcal{L}(H)}\right.$$

where $\left\|LL_0^{-1}\right\|_{\mathcal{L}(H)} \leq \|L\|_{\mathcal{L}(W,H)} \left\|L_0^{-1}\right\|_{\mathcal{L}(H,W)}$ motivates the definition

$$r(L_0) := 1/\left\|L_0^{-1}\right\|_{\mathcal{L}(H,W)}$$

Choosing $\|L\|_{\mathcal{L}(W,H)} < r(L_0)$ guarantees that $\left\|LL_0^{-1}\right\|_{\mathcal{L}(H)} < 1$, so $L_0 + L \in \mathcal{L}(W,H)$ is invertible with inverse

$$(L_0 + L)^{-1} = \underbrace{L_0^{-1}}_{\mathcal{L}(H,W)} \sum_{n=0}^{\infty} \big[\text{-}\underbrace{LL_0^{-1}}_{\mathcal{L}(H)}\big]^n \in \mathcal{L}(H,W) \tag{4.10}$$

Note that on any ball $B_{qr(L_0)}(L_0) \subset \mathcal{L}(W,H)$ with $0 < q < 1$ the expression (4.10) is the uniform limit $N \longrightarrow \infty$ of continuous functions

$$L \longmapsto L_0^{-1} \sum_{n=0}^{N} \left[\text{-}LL_0^{-1}\right]^n$$

Hence, the map $L \in B_{qr(L_0)}(L_0) \longmapsto (L_0 + L)^{-1} \in \mathcal{L}(H,W)$ is continuous itself. $\square$

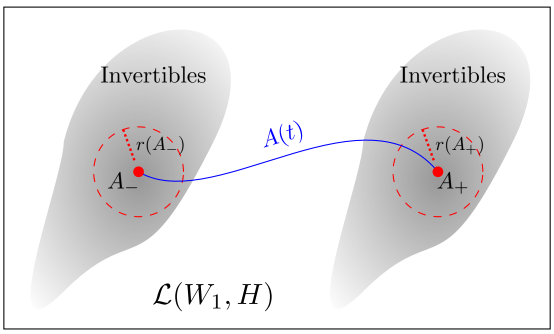


Figure 4.3: Preparation for our proof of Theorem 4.15 Step 2. The invertible operators $\mathcal{L}_{inv}(W_1,H) \subset \mathcal{L}(W_1,H)$ form an open subset. Since our operator family, while describing a continuous path in $\mathcal{L}(W_1,H)$, approaches invertible endpoints $A_\pm \in \mathcal{L}_{inv}(W_1,H)$, it is possible to find $T > 0$ such that $A(t)$ itself is invertible for all $|t| > T$.

Having completed the proof of Theorem 4.15, we are ready to address our main result:

Theorem 4.17 (D_A as an sc-Fredholm operator)
Given a baseline operator $A_0 : W_1 \longrightarrow H$ on an honest sc-Hilbert space $H \supset W_1 \supset \dots$ let us assume that $B(t) \in \mathcal{L}(H)$ is a very good perturbation and consider the operator family $A(t) = A_0 + B(t)$. Moreover, let $\delta_0 = 0 < \delta_1 < \dots$ be a weight sequence bounded by δ_∞ as determined in the proof of Theorem 4.15.
Then by restriction of $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ we obtain a sc-Fredholm operator

$$D_A : (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_n)_{n\geq 0}$$

between honest sc-Banach spaces.

Proof. Let us first comment on the spaces $\mathcal{W}^{\delta_n}_{n+k}$: In Lemma 4.6 we have identified $\beta_i = \gamma_{-\delta_i}$ as a cumulated twisting sequence on the almost sc-Banach space $(\mathcal{W}_n)_{n\geq 0}$. Thus, Lemma 4.8 ensures that by using spaces of type $\mathcal{W}^\delta_m := \gamma_{\text{-}\delta}(\mathcal{W}_m) \subset \mathcal{W}_m$ we obtain an honest sc-Banach space $(\mathcal{W}^{\delta_n}_{n+k})_{n\geq 0}$ for every $k \geq 0$.
Now let us turn to the operator D_A: In Lemma 4.14 we have seen that $D_A : \mathcal{W}_1 \longrightarrow \mathcal{H}$ is strongly twistable w.r.t. $\beta_i = \gamma_{\text{-}\delta_i}$, whereas the more difficult Theorem 4.15 confirms that D_A is also twist-regularizing. Note that all assumptions remain true when "A" is replaced by "-A". Thus, we can apply Proposition 4.13 to obtain regularizing sc-operators $D_{\pm A} : (\mathcal{W}^{\delta_n}_{n+k+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_{n+k})_{n\geq 0}$ at every $k \geq 0$.

With this information we are in a position to suitably adapt the proof of Theorem 2.38: Since $D_{\pm A} : (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_n)_{n\geq 0}$ is regularizing, we have

$$\ker D_{\pm A} \subset \bigcap_{n\geq 0} \mathcal{W}^{\delta_n}_{n+1} \subset \bigcap_{n\geq 0} \mathcal{W}^{\delta_n}_n$$

and as before $D_{\pm A} : \mathcal{W}_1 \longrightarrow \mathcal{H}$ being Fredholm guarantees that $\ker D_{\pm A}$ is finite-dimensional whereas $D_{\pm A}(\mathcal{W}_1) \subset \mathcal{H}$ is a closed subspace.
Now comes the only tricky part: With $D_{\text{-}A} : (\mathcal{W}^{\delta_n}_{n+2})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0}$ being a regularizing sc-operator, the decomposition $\mathcal{W}_1 = D_{\text{-}A}(\mathcal{W}_2) \oplus \ker D_A$ can be augmented to

$$\mathcal{W}^{\delta_n}_{n+1} = \underbrace{\mathcal{W}^{\delta_n}_{n+1} \cap D_{\text{-}A}\big(\mathcal{W}^{\delta_0}_{0+2}\big)}_{D_{\text{-}A}(\mathcal{W}^{\delta_n}_{n+2})} \oplus \ker D_A$$

whereas the regularizing sc-operator $D_A : (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_n)_{n\geq 0}$ turns $\mathcal{H} = D_A(\mathcal{W}_1) \oplus \ker D_{-A}$ into

$$\mathcal{W}^{\delta_n}_n = \underbrace{\mathcal{W}^{\delta_n}_n \cap D_A\big(\mathcal{W}^{\delta_0}_{0+1}\big)}_{D_A(\mathcal{W}^{\delta_n}_{n+1})} \oplus \ker D_{\text{-}A}$$

It remains to summarize our findings in the language of Definition 2.7: The sc-Banach spaces $U_n = \mathcal{W}^{\delta_n}_{n+1}$ and $V_n = \mathcal{W}^{\delta_n}_n$ admit finite-dimensional subspaces $\ker D_A \subset U_\infty$ and $\ker D_{\text{-}A} \subset V_\infty$. Moreover, since the regularizing sc-operator $D_A : (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_n)_{n\geq 0}$ has closed range $D_A(\mathcal{W}_1) \subset \mathcal{H}$ while the regularizing sc-operator $D_{\text{-}A} : (\mathcal{W}^{\delta_n}_{n+2})_{n\geq 0} \longrightarrow (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0}$ has closed range $D_{\text{-}A}(\mathcal{W}_2) = \mathcal{W}_1 \cap D_{\text{-}A}(\mathcal{W}_1) \subset \mathcal{W}_1$, Corollary 2.6 confirms

$$X_n = D_{\text{-}A}\big(\mathcal{W}^{\delta_n}_{n+2}\big) \quad \text{and} \quad Y_n = D_A\big(\mathcal{W}^{\delta_n}_{n+1}\big)$$

as honest sc-subspaces of $U = (\mathcal{W}^{\delta_n}_{n+1})_{n\geq 0}$ and $V = (\mathcal{W}^{\delta_n}_n)_{n\geq 0}$, respectively.
Clearly,

$$D_A : \mathcal{W}^{\delta_n}_{n+1} = \ker D_A \oplus \underbrace{D_{\text{-}A}\big(\mathcal{W}^{\delta_n}_{n+2}\big)}_{X_n} \longrightarrow \underbrace{D_A\big(\mathcal{W}^{\delta_n}_{n+1}\big)}_{Y_n} \oplus \ker D_{\text{-}A} = \mathcal{W}^{\delta_n}_n$$

restricts to an isomorphism $D_A : X_n \xrightarrow{\sim} Y_n$ at every $n \geq 0$. ☐

Chapter 5

Applicability to Floer theory

5.1 Construction of the Banach scale and baseline operator

5.1.1 ... in the general case of non-local Lagrangian boundary conditions

It is now time to provide examples for the baseline operator and admissible perturbations that were postulated in Chapter 2.

Since our arguments do not involve any reference to finite dimension, we will consider the infinite-dimensional case right away. Let $\big(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}}\big)$ be a real Hilbert space. Given an "almost complex structure" $J_0 \in \mathcal{L}(\mathbb{H})$ with $J_0^2 = -\mathrm{id}$ and $\langle J_0\cdot, J_0\cdot\rangle_{\mathbb{H}} = \langle\cdot,\cdot\rangle_{\mathbb{H}}$, the "symplectic form"

$$\omega(u,v) := \langle J_0 u, v\rangle_{\mathbb{H}} \in \mathbb{R}$$

is non-degenerate and satisfies $|\omega(u,v)| \leq \|J_0\|_{\mathcal{L}(\mathbb{H})} \|u\| \, \|v\|$. The antisymmetry of ω relies on the symmetry of $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, thus requiring us to work with a real Hilbert space $\mathbb{H}$.

Now let us consider the bounded open interval $I = (0,1)$. The operator

$$A_0 = J_0\partial_s : W^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$$

is defined on all of $W^{1,2}(I,\mathbb{H})$. However, Sobolev embedding yields bounded linear maps

$$W^{1,2}(I,\mathbb{H}) \xrightarrow[\text{unique representative}]{} C^0(\bar{I},\mathbb{H}) \begin{array}{l} \xrightarrow{\mathrm{ev}_0} \mathbb{H} \\ \xrightarrow[\mathrm{ev}_1]{} \mathbb{H} \end{array}$$

so given any (closed) subspace $\Lambda \subset \mathbb{H}\oplus\mathbb{H}$, we can restrict $A_0 = J_0\partial_s$ to the (closed) subspace

$$W^{1,2}_\Lambda(I,\mathbb{H}) := \big(\mathrm{ev}_0 \times \mathrm{ev}_1\big)^{-1}\big(\Lambda\big) \subset W^{1,2}(I,\mathbb{H})$$

consisting of functions $u \in W^{1,2}(I,\mathbb{H}) \subset C^0(\bar{I},\mathbb{H})$ such that $\big(u(0), u(1)\big) \in \Lambda$.

In Corollary 5.5 we will give a full classification for which spaces $\Lambda \subset \mathbb{H}\oplus\mathbb{H}$ the operator $A_0 : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ is symmetric (resp. self-adjoint).
As a first step, we observe the following:

Lemma 5.1 (Regularity of the adjoint)
Let $\Lambda \subset \mathbb{H}\oplus\mathbb{H}$ be any subspace.
Then the adjoint domain of $A_0 = J_0\partial_s : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ satisfies $\mathcal{D}(A_0^*) \subset W^{1,2}(I,\mathbb{H})$ and A_0^* can be obtained by restricting $J_0\partial_s : W^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ to $\mathcal{D}(A_0^*)$.

Proof. Let us abbreviate $H = L^2(I,\mathbb{H})$ and consider an element $u \in L^2(I,\mathbb{H})$. The Riesz Representation Theorem shows that $u \in \mathcal{D}(A_0^*)$ implies the existence of another $u_1 \in L^2(I,\mathbb{H})$ such that $\langle u, A_0 f\rangle_H + \langle u_1, f\rangle_H = 0$ for all $f \in W^{1,2}_\Lambda(I,\mathbb{H})$.
We claim that consequently the expression

$$\delta_\phi(u, J_0u_1) = \int_I u\cdot\partial\phi + \int_I J_0u_1\cdot\phi \ \in \mathbb{H}$$

vanishes for all test functions $\phi \in C_0^\infty(I)$, exhibiting $J_0u_1 \in L^2(I,\mathbb{H})$ as the weak derivative of u so that the adjoint can be expressed as $A_0^*u = -u_1 = J_0\partial u$.

Indeed, given any test function $\phi \in C_0^\infty(I)$ and constant vector $v \in \mathbb{H}$, the product "f" $= \phi v$ belongs to $W^{1,2}_\Lambda(I, \mathbb{H})$. As a bounded linear map, $\langle \cdot, J_0 v\rangle_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{R}$ commutes with the Bochner integral and with $\langle \cdot,\cdot\rangle_{\mathbb{H}}$ being non-degenerate, having

$$\big\langle \delta_\phi(u, J_0 u_1), J_0 v\big\rangle_{\mathbb{H}} = \int_I \langle u, J_0\, \partial[\phi v]\,\rangle_{\mathbb{H}} + \int_I \langle\, J_0 u_1, J_0\, \phi v\,\rangle_{\mathbb{H}} = \langle u, A_0 f\rangle_H + \langle u_1, f\rangle_H = 0$$

for all $v \in \mathbb{H}$ implies $\delta_\phi(u, J_0 u_1) = 0$. □

With the Fundamental Theorem of Calculus in mind, the boundary behaviour of our operator $J_0 \partial_s$ seems rather unsurprising. Yet, since our Sobolev spaces are defined on a bounded open interval, a little argument involving the non-trivial Meyers-Serrin theorem is required:

Lemma 5.2 (Partial integration for $A_0 = J_0\partial_s$ on $I = (0,1)$)
For $u, v \in W^{1,2}(I, \mathbb{H}) \subset C^0(\bar{I}, \mathbb{H})$ we have the formulae

a) $\langle u', v\rangle_H + \langle u, v'\rangle_H = \big\langle u(1), v(1)\big\rangle_{\mathbb{H}} - \big\langle u(0), v(0)\big\rangle_{\mathbb{H}}$

b) $\langle A_0 u, v\rangle_H - \langle u, A_0 v\rangle_H = \omega\big(u(1), v(1)\big) - \omega\big(u(0), v(0)\big)$

Proof. a) By the Meyers-Serrin theorem there exist sequences $u_n, v_n \in C^\infty \cap W^{1,2}(I, \mathbb{H})$ such that $u_n \longrightarrow u$ and $v_n \longrightarrow v$ in $W^{1,2}(I, \mathbb{H})$. Continuity of the evaluation maps $\mathrm{ev}_s : W^{1,2}(I, \mathbb{H}) \subset C^0(\bar{I}, \mathbb{H}) \longrightarrow \mathbb{H}$, $s \in \bar{I}$ allows us to use

$$\lim_{n\to\infty} \|u_n(s) - u(s)\|_{\mathbb{H}} = 0 \ \text{ and } \ \lim_{n\to\infty} \|v_n(s) - v(s)\|_{\mathbb{H}} = 0 \quad \text{ at } s = 0, 1.$$

Thus, formula (a) follows from the calculation

$$\begin{aligned}
\langle u', v\rangle_H + \langle u, v'\rangle_H &= \lim_{n\to\infty} \Big[\underbrace{\langle u_n', v_n\rangle_H + \langle u_n, v_n'\rangle_H}_{\int_0^1 \mathrm{d}s\, \frac{\mathrm{d}}{\mathrm{d}s} \langle u_n, v_n\rangle_{\mathbb{H}}} \Big] \\
&= \lim_{n\to\infty} \Big[\big\langle u_n(1), v_n(1)\big\rangle_{\mathbb{H}} - \big\langle u_n(0), v_n(0)\big\rangle_{\mathbb{H}} \Big] = \big\langle u(1), v(1)\big\rangle_{\mathbb{H}} - \big\langle u(0), v(0)\big\rangle_{\mathbb{H}}
\end{aligned}$$

b) Given $u \in W^{1,2}(I, \mathbb{H})$ we have $J_0\, \delta_\phi(u, \partial u) = \delta_\phi(J_0 u, J_0 \partial u)$ and therefore $J_0 u \in W^{1,2}(I, \mathbb{H})$ with $\partial(J_0 u) = J_0 \partial u \in L^2(I, \mathbb{H})$. Using formula (a) we obtain

$$\begin{aligned}
\langle A_0 u, v\rangle_H - \langle u, A_0 v\rangle_H &= \int_I \langle\, \underbrace{J_0 \partial_s u}_{\partial_s\, J_0 u}, v\rangle_{\mathbb{H}} - \int_I \underbrace{\langle u, J_0 \partial_s v\rangle_{\mathbb{H}}}_{-\langle J_0 u,\, \partial_s v\rangle_{\mathbb{H}}} \\
&= \big\langle \partial_s[J_0 u], v\big\rangle_H + \big\langle J_0 u, \partial_s v\big\rangle_H \;=\; \big\langle J_0 u(1), v(1)\big\rangle_{\mathbb{H}} - \big\langle J_0 u(0), v(0)\big\rangle_{\mathbb{H}} \\
&= \omega\big(u(1), v(1)\big) - \omega\big(u(0), v(0)\big)
\end{aligned}$$

□

As promised in the introduction, there is a nice characterization of the adjoint $(J_0\partial_s|_\Lambda)^*$ in terms of Ω-orthogonal complements:

Proposition 5.3 (Calculating the adjoint of $A_0 = J_0\partial_s$)
Given a subspace $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$, denote by $\Lambda^\Omega \subset \mathbb{H} \oplus \mathbb{H}$ the Ω-orthogonal complement of Λ under the symplectic form $\Omega = (-\omega) \oplus \omega$.
Then the adjoint of $A_0 = J_0\partial_s : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ is simply

$$J_0\partial_s : W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$$

Proof. From Lemma 5.1 we know that any $u \in \mathcal{D}(A_0^*)$ belongs to $W^{1,2}(I,\mathbb{H})$ and A_0^* arises as the restriction of $A_0 = J_0\partial_s : W^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$. To verify that the adjoint domain is exactly $\mathcal{D}(A_0^*) = W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H})$, recall from Lemma 5.2 that any pair $u, v \in W^{1,2}(I,\mathbb{H})$ obeys

$$\langle A_0 u, v\rangle_H - \langle u, A_0 v\rangle_H = \omega\big(u(1), v(1)\big) - \omega\big(u(0), v(0)\big) \tag{5.1}$$

Having $u \in \mathcal{D}(A_0^*)$ ensures that the l.h.s. of (5.1) vanishes for all $v \in W^{1,2}_\Lambda(I,\mathbb{H})$.
Since we can find a suitable $v \in W^{1,2}_\Lambda(I,\mathbb{H})$ for any combination $\big(v(0), v(1)\big) \in \Lambda$, we conclude that $\big(u(0), u(1)\big) \in \Lambda^\Omega$ and therefore $\mathcal{D}(A_0^*) \subset W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H})$.
Conversely, $u \in W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H})$ guarantees vanishing of the r.h.s. so vanishing of the l.h.s. implies $u \in \mathcal{D}(A_0^*)$ and we have proven $W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H}) \subset \mathcal{D}(A_0^*)$. $\square$

The following auxiliary result shows that we have implications $W^{1,2}_A = W^{1,2}_B \implies A = B$:

Auxiliary Lemma 5.4 (Faithful boundary detection by $W^{1,2}$)
Given subspaces $A, B \subset \mathbb{H} \oplus \mathbb{H}$ we have an equivalence

$$A \subset B \qquad \Longleftrightarrow \qquad W^{1,2}_A(I,\mathbb{H}) \subset W^{1,2}_B(I,\mathbb{H})$$

Proof. Let us focus on the non-trivial direction and assume that in spite of $W^{1,2}_A(I,\mathbb{H}) \subset W^{1,2}_B(I,\mathbb{H})$ we can find a tuple $(v_0, v_1) \in A \cap B^c$. Using bumps supported in the vicinity of 0 and 1, we can construct $v \in W^{1,2}(I,\mathbb{H})$ with $\big(v(0), v(1)\big) = \big(v_0, v_1\big)$. This produces a contradiction between $v \in W^{1,2}_A(I,\mathbb{H}) \subset W^{1,2}_B(I,\mathbb{H})$ and $v \notin W^{1,2}_B(I,\mathbb{H})$. $\square$

Now we are ready to describe the symmetric (resp. self-adjoint) restrictions of $J_0\partial_s$:

Corollary 5.5 (Criteria for symmetry and self-adjointness)
Given a subspace $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$, the operator $A_0 = J_0\partial_s : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ is ...

i) symmetric if and only if Λ is isotropic $\big($i.e. $\Lambda \subset \Lambda^\Omega\big)$

ii) self-adjoint if and only if Λ is a Lagrangian subspace $\big($i.e. $\Lambda = \Lambda^\Omega\big)$

Proof. Auxiliary Lemma 5.4 shows that $\Lambda \subset \Lambda^\Omega$ is equivalent to the statement

$$W^{1,2}_\Lambda(I,\mathbb{H}) \subset W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H})$$

which by Proposition 5.3 is reinterpreted as the statement $A_0 \subset A_0^*$.
Similarly, $\Lambda = \Lambda^\Omega$ is equivalent to $W^{1,2}_\Lambda(I,\mathbb{H}) = W^{1,2}_{\Lambda^\Omega}(I,\mathbb{H})$ which can be rephrased as $A_0 = A_0^*$. $\square$

Next, let us proceed to higher Sobolev spaces. As in section 2.4.1 every element $u \in W^{n+1,2}(I,\mathbb{H})$ can be identified with a tuple

$$u = \big(u_0, u_1, ..., u_{n+1}\big) \in L^2(I,\mathbb{H})^{\oplus n+2}$$

such that $(u_k, u_{k+1}) \in W^{1,2}(I,\mathbb{H})$ for all $k = 0, ..., n$. Thus, for all $k = 0, ..., n$ and $s \in \bar{I}$ Sobolev embedding yields bounded linear maps

$$W^{n+1,2}(I,\mathbb{H}) \xrightarrow{\ \mathrm{pr}_k\ } W^{1,2}(I,\mathbb{H}) \subset C^0(\bar{I},\mathbb{H}) \xrightarrow{\ \mathrm{ev}_s\ } \mathbb{H}\ , \qquad u \longmapsto u_k \longmapsto u_k(s)$$

and to any closed subspace $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$ we can associate the closed subspace

$$W^{n+1,2}_\Lambda(I,\mathbb{H}) := \bigcap_{k\,=\,0,\,...,\,n} \big[\,(\mathrm{ev}_0 \times \mathrm{ev}_1) \circ \mathrm{pr}_k\,\big]^{-1}\big(\mathbb{I}^k(\Lambda)\big) \ \subset W^{n+1,2}(I,\mathbb{H}) \tag{5.2}$$

where $\mathbb{I} := J_0 \oplus J_0 \in \mathcal{L}(\mathbb{H} \oplus \mathbb{H})$. More explicitly, this space can be described as

$$W^{n+1,2}_\Lambda(I,\mathbb{H}) \,=\, \Big\{(u_0, ..., u_{n+1}) \in W^{n+1,2}(I,\mathbb{H}) \,\Big|\, \big(u_k(0), u_k(1)\big) \in \mathbb{I}^k(\Lambda) \text{ for all } k = 0, ..., n\Big\}$$

which is reminiscent of the space encountered in section 7 of [FW].

Note that $A_0 = J_0\partial_s : W^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ restricts to a composition of bounded linear maps

$$\begin{array}{ccccc} W^{n+1,2}_\Lambda(I,\mathbb{H}) & \xrightarrow{\quad \partial_s \quad} & W^{n,2}_{\mathbb{I}(\Lambda)}(I,\mathbb{H}) & \xrightarrow{\quad J_0 \quad} & W^{n,2}_\Lambda(I,\mathbb{H}) \\ (u_0, u_1, ..., u_{n+1}) & \longmapsto & (u_1, ..., u_{n+1}) & \longmapsto & (J_0u_1, ..., J_0u_{n+1}) \end{array}$$

so we can regard A_0 as a **sc-operator**

$$A_0 : \big(W^{n+1,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0} \longrightarrow \big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$$

Our definition of $W^{n+1,2}_\Lambda(I,\mathbb{H})$ is motivated by the following property:

Lemma 5.6 (Regularization imposes boundary conditions on derivatives)
The sc-operator $A_0 : \big(W^{n+1,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0} \longrightarrow \big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$ is regularizing, i.e. $A_0 : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ satisfies $A_0^{-1}\big(W^{n,2}_\Lambda(I,\mathbb{H})\big) = W^{n+1,2}_\Lambda(I,\mathbb{H})$.

Proof. By Auxiliary Lemma 2.12 it suffices to show that $A_0 = J_0\partial_s : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ is an escalator for $\big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$. Hence, let us consider $u \in W^{n+1,2}(I,\mathbb{H})$ such that $J_0\partial u \in W^{n+1,2}(I,\mathbb{H})$ as well. Then u can be represented as a tuple

$$(u_0, u_1, ..., u_{n+1}) \in L^2(I,\mathbb{H})^{\oplus n+2}$$

with $\big(u_k(0), u_k(1)\big) \in \mathbb{I}^k(\Lambda)$ for $k = 0, ..., n$ whereas from $\partial u = -J_0J_0\partial u \in W^{n+1,2}_{\mathbb{I}(\Lambda)}(I,\mathbb{H})$ we obtain a tuple

$$(u_1, ..., u_{n+1}, u_{n+2}) \in L^2(I,\mathbb{H})^{\oplus n+2}$$

with $\big(u_k(0), u_k(1)\big) \in \mathbb{I}^k(\Lambda)$ for $k = 1, ..., n+1$. Gluing these tuples shows that $\big(u_k(0), u_k(1)\big) \in \mathbb{I}^k(\Lambda)$ for $k = 0, ..., n+1$ and therefore $u \in W^{n+2,2}_\Lambda(I,\mathbb{H})$. $\square$

Up to this point, our discussion can be summarized by the following result:

Theorem 5.7 ($A_0 = J_0\partial_s$ as a baseline operator)
Assume that $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$ is a Lagrangian subspace, i.e. $\Lambda^\Omega = \Lambda$. Then $A_0 = J_0\partial_s : W^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ restricts to a baseline operator

$$A_0 : \big(W^{n+1,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0} \longrightarrow \big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$$

in the sense that $A_0 : W^{1,2}_\Lambda(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ is self-adjoint as an unbounded operator on $L^2(I,\mathbb{H})$ and $A_0 : \big(W^{n+1,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0} \longrightarrow \big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$ is a regularizing sc-operator.

Proof. Combine Corollary 5.5 and Lemma 5.6. $\square$

So far we have not verified that $\left(W^{n,2}_\Lambda(I,\mathbb{H})\right)_{n\ge 0}$ is a sc-Banach space.
In fact, $W_\infty := \bigcap_{m\ge 0} W^{m,2}_\Lambda(I,\mathbb{H})$ being dense in every $W_n := W^{n,2}_\Lambda(I,\mathbb{H})$ is far from obvious: It might be tempting to regard $C_0^\infty(I,\mathbb{H}) \subset \bigcap_{m\ge 0} W^{m,2}_\Lambda(I,\mathbb{H})$ as a candidate for being dense in every $W^{n,2}_\Lambda(I,\mathbb{H})$. However, [Ad] Thm. 3.37 shows that for example $C_0^\infty(I)$ is not dense in $W^{1,2}(I)$, the reason being that $I^c = (-\infty,0]\cup[1,\infty)$ has non-vanishing measure and therefore cannot be "$(1,2)$-polar". In a more modest attempt we observe that $I=(0,1)$ clearly obeys the segment condition from [Ad] Def. 3.21 so [Ad] Thm. 3.22 ensures that the restriction map $C_0^\infty(\mathbb{R}) \longrightarrow W^{1,2}(I)$ has dense image. Now however we have lost control over the boundary values $u_\epsilon(0), u_\epsilon(1)$ of any approximating function $u_\epsilon \in C_0^\infty(\mathbb{R})$.

We can circumvent these issues in a surprisingly simple way, namely by the mere presence of a baseline operator:

Proposition 5.8 (Banach scales generated by a baseline operator)
Let $W_0 \supset W_1 \supset \dots$ be a filtration of Banach spaces with bounded inclusions such that the norm $\|\cdot\|_{W_0}$ arises from a Hilbert space structure on $H := W_0$.
Assume that $W_\infty := \bigcap_{n\ge 0} W_n$ is dense in H and there exists a baseline operator

$$A_0 : (W_{n+1})_{n\ge 0} \longrightarrow (W_n)_{n\ge 0}$$

Then $(W_n)_{n\ge 0}$ is an almost sc-Banach space.
It is an honest sc-Banach space if $A_0 : W_1 \longrightarrow H$ has compact resolvent.

Proof. Let us first prove our claim in the case of complex Banach spaces:
Self-adjointness of the operator $A_0 : W_1 \longrightarrow H$ guarantees that $A_0 - i : W_1 \longrightarrow H$ is invertible. By consulting Auxiliary Lemma 2.13 we know that $A_0 - i : (W_{n+1})_{n\ge 0} \longrightarrow (W_n)_{n\ge 0}$ is a regularizing sc-operator as well, so we get $(A_0-i)^{-1}(W_n) \subset W_{n+1}$ and the Inverse Mapping Theorem implies $(A_0-i)^{-1} \in \mathcal{L}(W_n, W_{n+1})$.
Pick any element $x \in W_n$. Since by assumption $W_\infty = \bigcap_{m\ge 0} W_m$ is dense in H, we can find an approximating sequence $y_k \in W_\infty$ such that

$$\lim_{k\to\infty} \left\|(A_0-i)^n x - y_k\right\|_H = 0$$

Now $A_0 - i$ being a regularizing operator ensures that $(A_0-i)^{-1}(W_\infty) \subset W_\infty$ and with

$$\left\|x - (A_0-i)^{-n} y_k\right\|_{W_n} \le \left\|(A_0-i)^{-n}\right\|_{\mathcal{L}(H,W_n)} \cdot \left\|(A_0-i)^n x - y_k\right\|_H$$

we conclude that $x_k := (A_0-i)^{-n} y_k \in W_\infty$ is an approximating sequence for x in W_n.
This proves that $(W_n)_{n\ge 0}$ is an almost sc-Banach space.
For the statement about honest sc-Banach spaces note that at any $\lambda \in \rho(A_0)$ from the resolvent set,

$$\begin{tikzcd} & H & \\ W_1 \arrow[ur, "\iota"] \arrow[rr, shift right, "A_0-\lambda"'] & & H \arrow[ll, shift right, "(A_0-\lambda)^{-1}"'] \arrow[ul, "R_\lambda(A_0)"'] \end{tikzcd}$$

is a commutative diagram of bounded linear maps between Banach spaces. Since the class of compact operators forms an ideal among bounded linear maps, we conclude that the resolvent $R_\lambda(A_0) \in \mathcal{L}(H)$ being compact is synonymous with compactness of the inclusion operator $\iota \in \mathcal{L}(W_1, H)$. Moreover, for every $n \ge 0$ we have a commutative diagram of bounded linear maps

$$\begin{tikzcd} H \arrow[d, "(A_0-i)^{-n}"'] & W_1 \arrow[l, "\iota"'] \\ W_n & W_{n+1} \arrow[l, "\iota"] \arrow[u, "(A_0-i)^n"'] \end{tikzcd}$$

showing that compactness of $\iota : W_1 \longrightarrow H$ implies compactness of $\iota \in \mathcal{L}(W_{n+1}, W_n)$ at all higher orders. In summary, $(W_n)_{n\ge 0}$ is an honest sc-Banach space if and only if $A_0 : W_1 \longrightarrow H$ has compact resolvent.

Finally, let us explain how our results can be transferred to the case of real Banach spaces: Assume that $W_0 \supset W_1 \supset \ldots$ is a filtration of real Banach spaces with bounded inclusions such that $\|\cdot\|_{W_0}$ originates from a real Hilbert space $H = W_0$. With details given in Appendix A we can regard the complexification

$$W_0^{\mathbb{C}} \supset W_1^{\mathbb{C}} \supset \ldots \supset W_n^{\mathbb{C}} \supset \ldots$$

as a filtration of complex Banach spaces with bounded inclusions such that $\|\cdot\|_{W_0^{\mathbb{C}}}$ originates from the Hilbert space $H^{\mathbb{C}} = H \otimes \mathbb{C}$. The complexified operator $A_0 \otimes \mathrm{id}_{\mathbb{C}} = A_0 \oplus A_0$ now being a baseline operator on $(W_n^{\mathbb{C}})_{n\geq 0}$, we can apply the above results to deduce that the set

$$\bigcap_{m\geq 0} W_m^{\mathbb{C}} = \Big[\bigcap_{m\geq 0} W_m\Big]^{\mathbb{C}} = [W_\infty]^{\mathbb{C}}$$

is dense in every $W_n^{\mathbb{C}}$ and therefore W_∞ is dense in every W_n.
In the case of honest sc-Banach spaces $W_{n+1}^{\mathbb{C}} \hookrightarrow W_n^{\mathbb{C}}$ being a compact inclusion is equivalent to compactness of the inclusion $W_{n+1} \hookrightarrow W_n$. $\square$

Having identified $J_0\partial_s$ as a baseline operator on $\big(W_\Lambda^{n,2}(I,\mathbb{H})\big)_{n\geq 0}$, we obtain an honest sc-Hilbert space for every Lagrangian subspace $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$:

Corollary 5.9 (Honest sc-Hilbert spaces generated by restrictions of $J_0\partial_s$)
Let $\Lambda = \Lambda^\Omega$ be a Lagrangian subspace of $\mathbb{H} \oplus \mathbb{H}$.
Then $\big(W_\Lambda^{n,2}(I,\mathbb{H})\big)_{n\geq 0}$ is an honest sc-Hilbert space.

Proof. Approximate $f \in L^2(I,\mathbb{H})$ by $f \cdot \chi_{[\epsilon,1-\epsilon]} \in L^2(I,\mathbb{H})$ and use mollification to conclude that the set

$$C_0^\infty(I,\mathbb{H}) \subset \bigcap_{n\geq 0} W_\Lambda^{n,2}(I,\mathbb{H})$$

is dense in $H := L^2(I,\mathbb{H})$. From Theorem 5.7 we know that

$$A_0 = J_0\partial_s : \big[W_\Lambda^{n+1,2}(I,\mathbb{H})\big]_{n\geq 0} \longrightarrow \big[W_\Lambda^{n,2}(I,\mathbb{H})\big]_{n\geq 0}$$

is a baseline operator, so Proposition 5.8 guarantees that $\big[W_\Lambda^{n,2}(I,\mathbb{H})\big]_{n\geq 0}$ is an almost sc-Banach space. By combining Sobolev embedding with the Arzelà-Ascoli theorem, we observe that the inclusion

$$\begin{tikzcd}
& W_\Lambda^{1,2}(I,\mathbb{H}) \arrow[rr, dashed] \arrow[dl, hook] & & L^2(I,\mathbb{H}) & \\
W^{1,2}(I,\mathbb{H}) \arrow[drr, hook] & & & & C^0(\bar{I},\mathbb{H}) \arrow[ul] \\
& & C^{0,1/2}(I,\mathbb{H}) = C^{0,1/2}(\bar{I},\mathbb{H}) \arrow[urr, hook] & &
\end{tikzcd}$$

is compact, so $A_0 : W_\Lambda^{1,2}(I,\mathbb{H}) \longrightarrow L^2(I,\mathbb{H})$ has compact resolvent and Proposition 5.8 confirms that $\big[W_\Lambda^{n,2}(I,\mathbb{H})\big]_{n\geq 0}$ is an honest sc-Banach space. The lowest level $H = L^2(I,\mathbb{H})$ being a Hilbert space, we are in fact dealing with an honest sc-Hilbert space. $\square$

5.1.2 ... in the special cases of local Lagrangian or periodic boundary conditions

Our treatment of non-local Lagrangian boundary conditions $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$ automatically incorporates two important special cases:

- Local Lagrangian boundary conditions:
  Choose $\Lambda = \Lambda_0 \oplus \Lambda_1$ where Λ_0 and Λ_1 are Lagrangian subspaces of $(\mathbb{H}, \omega)$. Then the Banach scale from subsection 5.1.1 can be described as
  $$W^{n+1,2}_{\Lambda_0 \oplus \Lambda_1}(I, \mathbb{H}) = \Big\{ u \in W^{n+1,2}(I, \mathbb{H}) \,\Big|\, \partial^k u(0) \in J^k(\Lambda_0) \text{ and } \partial^k u(1) \in J^k(\Lambda_1) \text{ for all } k = 0, ..., n \Big\}$$
  which can be seen as a generalization of the Lagrangian boundary conditions considered in section 7 of [FW].

- Periodic boundary conditions:
  Let Λ be the diagonal $\Delta \subset \mathbb{H} \oplus \mathbb{H}$. Note that $\Delta = \mathbb{I}(\Delta)$ is invariant under application of $\mathbb{I} = J \oplus J$, so the Banach scale is simply
  $$W^{n+1,2}_{\Delta}(I, \mathbb{H}) = \Big\{ u \in W^{n+1,2}(I, \mathbb{H}) \,\Big|\, \partial^k u(0) = \partial^k u(1) \text{ for all } k = 0, ..., n \Big\}$$

By Part II Auxiliary Lemma 7.8 we will be able to define $W^{n,2}(S^1, \mathbb{H}) \subset W^{n,2}\big((0,1), \mathbb{H}\big)$ as the projection of

$$\ker \Big[\; W^{n,2}\big((0,1), \mathbb{H}\big) \oplus W^{n,2}\big((\text{-}\epsilon, \epsilon), \mathbb{H}\big) \xrightarrow{\begin{bmatrix} \tau_{+1} & -1 \\ 1 & -1 \end{bmatrix}} W^{n,2}\big((\text{-}\epsilon, 0), \mathbb{H}\big) \oplus W^{n,2}\big((0, \epsilon), \mathbb{H}\big) \;\Big]$$

We use the remainder of this subsection to justify why $W^{n+1,2}_{\Delta}(I, \mathbb{H})$ is the same closed subspace of $W^{n+1,2}(I, \mathbb{H})$ as $W^{n+1,2}(S^1, \mathbb{H})$:

Lemma 5.10 (Pointwise gluing of Sobolev spaces $W^{1,2}$)
Let us split $I_\epsilon = (\text{-}\epsilon, \epsilon)$ into adjacent intervals $I_- = (\text{-}\epsilon, 0)$ and $I_+ = (0, \epsilon)$ and denote by $W^{1,2}(I_-, \mathbb{H}) \underset{glue}{\oplus} W^{1,2}(I_+, \mathbb{H})$ the kernel of

$$W^{1,2}(I_-, \mathbb{H}) \oplus W^{1,2}(I_+, \mathbb{H}) \longrightarrow C^0(\bar{I}_-, \mathbb{H}) \oplus C^0(\bar{I}_+, \mathbb{H}) \xrightarrow{ev_0|_{I_-} - ev_0|_{I_+}} \mathbb{H}\,, \quad (u, v) \longmapsto u(0) - v(0)\,.$$

Then the restriction map

$$W^{1,2}(I_\epsilon, \mathbb{H}) \xrightarrow{res} W^{1,2}(I_-, \mathbb{H}) \oplus W^{1,2}(I_+, \mathbb{H})\,, \qquad w \longmapsto (w|_{I_-}, w|_{I_+})$$

induces an isomorphism of Banach spaces $W^{1,2}(I_\epsilon, \mathbb{H}) \xrightarrow{\sim} W^{1,2}(I_-, \mathbb{H}) \underset{glue}{\oplus} W^{1,2}(I_+, \mathbb{H})$.

Proof. To verify that the map

$$W^{1,2}(I_\epsilon, \mathbb{H}) \longrightarrow W^{1,2}(I_-, \mathbb{H}) \underset{glue}{\oplus} W^{1,2}(I_+, \mathbb{H})$$

is surjective, let us consider $u \in W^{1,2}(I_-, \mathbb{H})$ and $v \in W^{1,2}(I_+, \mathbb{H})$ with $u(0) = v(0)$.
By the Meyers-Serrin theorem we can find approximating sequences $u_n \in C^\infty \cap W^{1,2}(I_-, \mathbb{H})$ and $v_n \in C^\infty \cap W^{1,2}(I_+, \mathbb{H})$ such that $\lim_{n\to\infty} \|u - u_n\|_{W^{1,2}(I_-, \mathbb{H})} = 0$ and $\lim_{n\to\infty} \|v - v_n\|_{W^{1,2}(I_+, \mathbb{H})} = 0$.
Taking into account that

$$W^{1,2}(I_\pm, \mathbb{H}) \longrightarrow C^0(\bar{I}_\pm, \mathbb{H}) \xrightarrow{\text{ev}_0} \mathbb{H}$$

are bounded linear maps, we conclude that $u_n(0) \to u(0)$ and $v_n(0) \to v(0)$ and therefore

$$\lim_{n\to\infty} \|u_n(0) - v_n(0)\|_{\mathbb{H}} = 0$$

For any test function $\phi \in C_0^\infty(I_\epsilon)$ we find

$$\begin{aligned}
\delta_\phi(u \sqcup v, \dot{u} \sqcup \dot{v}) &= \int_{I_-} [u\partial\phi + \dot{u}\phi] \; + \; \int_{I_+} [v\partial\phi + \dot{v}\phi] \\
&= \lim_{n\to\infty} \int_{I_-} \underbrace{[u_n\partial\phi + \dot{u}_n\phi]}_{\partial(u_n\phi)} \; + \; \lim_{n\to\infty} \int_{I_+} \underbrace{[v_n\partial\phi + \dot{v}_n\phi]}_{\partial(v_n\phi)} \\
&= \lim_{n\to\infty} \phi(0) \cdot [u_n(0) - v_n(0)] \qquad = 0
\end{aligned}$$

showing that $u \sqcup v \in W^{1,2}(I_\epsilon, \mathbb{H})$ is the desired preimage. □

Remark 5.11 (Pointwise gluing of higher Sobolev spaces $W^{n+1,2}$)
The result of Lemma 5.10 extends to higher Sobolev spaces:
For $k = 0, ..., n$ we have bounded linear maps

$$W^{n+1,2}(I_\pm, \mathbb{H}) \xrightarrow{\mathrm{pr}_k} W^{1,2}(I_\pm, \mathbb{H}), \qquad (w_0, w_1, ..., w_{n+1}) \longmapsto (w_k, w_{k+1})$$

exhibiting

$$\begin{aligned}
W^{n+1,2}(I_-, \mathbb{H}) \underset{glue}{\oplus} W^{n+1,2}(I_+, \mathbb{H}) &:= \bigcap_{k=0,\dots,n} (\mathrm{pr}_k \oplus \mathrm{pr}_k)^{-1} \big[W^{1,2}(I_-, \mathbb{H}) \underset{glue}{\oplus} W^{1,2}(I_+, \mathbb{H}) \big] \\
&= \big\{ (u, v) \in W^{1,2}(I_-, \mathbb{H}) \oplus W^{1,2}(I_+, \mathbb{H}) \;\big|\; \partial^k u(0) = \partial^k v(0) \;\; \forall\, k = 0, ..., n \big\}
\end{aligned}$$

as a closed subspace of $W^{1,2}(I_-, \mathbb{H}) \oplus W^{1,2}(I_+, \mathbb{H})$.
To verify that the restriction map

$$W^{n+1,2}(I_\epsilon, \mathbb{H}) \longrightarrow W^{n+1,2}(I_-, \mathbb{H}) \underset{glue}{\oplus} W^{n+1,2}(I_+, \mathbb{H})$$

is surjective, let us consider $u \in W^{n+1,2}(I_-, \mathbb{H})$ and $v \in W^{n+1,2}(I_+, \mathbb{H})$ with $u_k(0) = v_k(0)$ for all $k = 0, ..., n$. Since (u_k, u_{k+1}) and (v_k, v_{k+1}) belong to $W^{1,2}(I_-)$ and $W^{1,2}(I_+)$ respectively, we can apply Lemma 5.10 to conclude that $(u_k \sqcup v_k, u_{k+1} \sqcup v_{k+1}) \in W^{1,2}(I_\epsilon, \mathbb{H})$ and therefore $(u_0 \sqcup v_0, u_1 \sqcup v_1, \dots, u_{n+1} \sqcup v_{n+1}) \in W^{n+1,2}(I_\epsilon, \mathbb{H})$.

5.2 Criteria for moderate and localized perturbations

5.2.1 Differentiation of maps valued in $C^n_{\text{bounded}}(I,\mathbb{B})$

As perturbations to our baseline operator $J_0\partial_s : \big(W^{n+1,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0} \longrightarrow \big(W^{n,2}_\Lambda(I,\mathbb{H})\big)_{n\geq 0}$ we study maps $\Gamma : \mathbb{R}\times I \longrightarrow \mathcal{L}(\mathbb{H})$ which are admissible in a sense to be determined below. When cutting such a map into time-slices $\Gamma_t : I \longrightarrow \mathcal{L}(\mathbb{H})$, we want each Γ_t to be contained in $C^n_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)$ so that it can operate by multiplication on $W^{n,2}(I,\mathbb{H})$. Moreover, just as in Lemma 2.28 or Corollary 2.29 the map

$$\Gamma : \mathbb{R} \longrightarrow C^n_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)$$

has to be of class C^r with bounded derivatives in order to operate on $W^{r,2}\big(\mathbb{R},W^{n,2}(I,\mathbb{H})\big)$. We verify this kind of differentiability from scratch, by using completeness of the spaces $C^n_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)$. This will require bounds on the second time derivatives $\partial_t^2\partial_x^k\Gamma(t,x)$ to ensure uniform convergence of the difference quotients

$$\frac{\partial_x^k\Gamma(t+\delta t,\cdot)-\partial_x^k\Gamma(t,\cdot)}{\delta t}\ ,\quad k=0,...,n$$

As indicated in Figure 5.1, we will be able to keep the number of x-derivatives constant, while arbitrarily increasing the number of t-derivatives.
The following arguments work for a general open interval $I\subset\mathbb{R}$ and Banach space $\mathbb{B}$.

Lemma 5.12 (First derivative of $C^0_{\text{bounded}}(I,\mathbb{B})$-valued maps)
Given a map $\Gamma : \mathbb{R}\times I \longrightarrow \mathbb{B}$ *such that*

- $\Gamma(t,\cdot)\in C^0_{\mathit{bounded}}(I,\mathbb{B})$ *at every* $t\in\mathbb{R}$
- $\Gamma(\cdot,x)\in C^2(\mathbb{R},\mathbb{B})$ *at every* $x\in I$

let us assume $\sup\limits_{(t,x)\,\in\,\mathbb{R}\times I}\big\|\partial_t^2\Gamma\big\|_{\mathbb{B}}<\infty$.

Then $\Gamma : \mathbb{R}\longrightarrow C^0_{\mathit{bounded}}(I,\mathbb{B})$ *is differentiable with derivative* $t\longmapsto\partial_t\Gamma(t,\cdot)\in C^0_{\mathit{bounded}}(I,\mathbb{B})$.

Proof. We apply the Fundamental Theorem of Calculus twice:
Using $\Gamma(\cdot,x)\in C^1(\mathbb{R},\mathbb{B})$ one has

$$\frac{\Gamma(t+\delta t,x)-\Gamma(t,x)}{\delta t}\ =\ \int_0^1 \mathrm{d}s\ \partial_t\Gamma\big|_{(t+s\delta t,x)}$$

and with $\partial_t\Gamma(\cdot,x)\in C^1(\mathbb{R},\mathbb{B})$ we get

$$\big\|\partial_t\Gamma(t+s\delta t,x)-\partial_t\Gamma(t,x)\big\|_{\mathbb{B}}\ \leq\ |s|\,|\delta t|\int_0^1\mathrm{d}r\ \big\|\partial_t^2\Gamma\big|_{(t+r\cdot s\delta t,x)}\big\|_{\mathbb{B}}\ \leq\ |s|\,|\delta t|\sup_{(t,x)\in\mathbb{R}\times I}\big\|\partial_t^2\Gamma\big\|_{\mathbb{B}}$$

Thus, calculating pointwise at $x\in I$, we obtain

$$\left\|\frac{\Gamma(t+\delta t,x)-\Gamma(t,x)}{\delta t}-\partial_t\Gamma(t,x)\right\|_{\mathbb{B}}\ \leq\ \int_0^1\mathrm{d}s\ \big\|\partial_t\Gamma(t+s\delta t,x)-\partial_t\Gamma(t,x)\big\|_{\mathbb{B}}\ \leq\ |\delta t|\sup_{(t,x)\in\mathbb{R}\times I}\big\|\partial_t^2\Gamma\big\|_{\mathbb{B}}$$

so the difference quotient $\frac{\Gamma(t+\delta t,\cdot)-\Gamma(t,\cdot)}{\delta t}\in C^0_{\text{bounded}}(I,\mathbb{B})$ converges not only pointwise but in fact uniformly. By completeness of $C^0_{\text{bounded}}(I,\mathbb{B})$ we conclude that $\partial_t\Gamma(t,\cdot)$ belongs to $C^0_{\text{bounded}}(I,\mathbb{B})$ as well and serves as the derivative of $\Gamma : \mathbb{R}\longrightarrow C^0_{\text{bounded}}(I,\mathbb{B})$. □

Remark 5.13 (The Banach space $C^n_{\text{bounded}}(I,\mathbb{B})$)
Choose $n\geq 1$. Then

$$C^n_{\text{bounded}}(I,\mathbb{B}) := \Big\{(u_0,u_1,...,u_n)\in C^0_{\text{bounded}}(I,\mathbb{B})^{\oplus n+1}\ \Big|\ u_k \text{ is differentiable with derivative } u_k'=u_{k+1}\Big\}$$

is a closed subspace of $C^0_{\text{bounded}}(I,\mathbb{B})^{\oplus n+1}$ and therefore a Banach space itself.
By abuse of notation, we can identify $C^n_{\text{bounded}}(I,\mathbb{B})$ with its image under the projection

$$(u_0,u_1,...,u_n)\longmapsto u_0\in C^0_{\text{bounded}}(I,\mathbb{B})$$

Lemma 5.14 (First derivative of $C^n_{\text{bounded}}(I,\mathbb{B})$-valued maps)
Given a map $\Gamma : \mathbb{R}\times I \longrightarrow \mathbb{B}$ *with*

- $\Gamma(t,\cdot) \in C^n_{bounded}(I,\mathbb{B})$ *at every* $t\in\mathbb{R}$
- $\Gamma(\cdot,x),\ \ldots.,\partial^n_x\Gamma(\cdot,x) \in C^2(\mathbb{R},\mathbb{B})$ *at every* $x\in I$

assume that $\sup\limits_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial^2_t\,\partial^k_x\,\Gamma\big\|_{\mathbb{B}} < \infty$ *for all* $k=0,\ldots,n$.

Then we have $\partial_t\Gamma \in C^n_{bounded}(I,\mathbb{B})$ *with derivatives* $\partial^k_x[\partial_t\Gamma] = \partial_t[\partial^k_x\Gamma]$.
Moreover, $\Gamma : \mathbb{R}\longrightarrow C^n_{bounded}(I,\mathbb{B})$ *is differentiable with derivative* $\partial_t\Gamma : \mathbb{R}\longrightarrow C^n_{bounded}(I,\mathbb{B})$.

Proof. As in Lemma 5.12 we have pointwise estimates

$$\left\|\frac{\partial^k_x\Gamma(t+\delta t,x)-\partial^k_x\Gamma(t,x)}{\delta t} - \partial_t\partial^k_x\Gamma(t,x)\right\|_{\mathbb{B}} \;\le\; |\delta t| \sup_{(t,x)\in\mathbb{R}\times I}\big\|\partial^2_t\partial^k_x\,\Gamma\big\|_{\mathbb{B}}$$

showing that

$$\partial_t\partial^k_x\Gamma(t,\cdot) \;=\; \lim_{\delta t\to 0}\frac{\partial^k_x\Gamma(t+\delta t,\cdot)-\partial^k_x\Gamma(t,\cdot)}{\delta t} \;\in C^0_{\text{bounded}}(I,\mathbb{B})$$

Since $C^n_{\text{bounded}}(I,\mathbb{B}) \subset C^0_{\text{bounded}}(I,\mathbb{B})^{\oplus n+1}$ is a closed subspace, we conclude that $\partial_t\Gamma(t,\cdot)\in C^n_{\text{bounded}}(I,\mathbb{B})$ with x-derivatives $\partial^k_x[\partial_t\Gamma] = \partial_t[\partial^k_x\Gamma]$ is the t-derivative of $\Gamma : \mathbb{R}\longrightarrow C^n_{\text{bounded}}(I,\mathbb{B})$. $\square$

Proposition 5.15 (Smooth $C^n_{\text{bounded}}(I,\mathbb{B})$-valued maps)

i) Given a map $\Gamma : \mathbb{R}\times I \longrightarrow \mathbb{B}$ *with*

- $\Gamma(t,\cdot) \in C^n_{bounded}(I,\mathbb{B})$ *at every* $t\in\mathbb{R}$
- $\Gamma(\cdot,x),\ \ldots.,\partial^n_x\Gamma(\cdot,x) \in C^\infty(\mathbb{R},\mathbb{B})$ *at every* $x\in I$

assume that $\sup\limits_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial^l_t\,\partial^k_x\,\Gamma\big\|_{\mathbb{B}} < \infty$ *for all* $k=0,\ldots,n$ *and* $l\ge 2$.
Then we have $\Gamma \in C^\infty\big(\mathbb{R}, C^n_{bounded}(I,\mathbb{B})\big)$ *with derivatives* $t\longmapsto \partial^l_t\Gamma(t,\cdot)\in C^n_{bounded}(I,\mathbb{B})$.

ii) If in addition $\sup\limits_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial^l_t\,\partial^k_x\,\Gamma\big\|_{\mathbb{B}} < \infty$ *for all* $k=0,\ldots,n$ *and* $l=0,1$
then $\Gamma : \mathbb{R}\longrightarrow C^n_{bounded}(I,\mathbb{B})$ *is smooth with bounded derivatives.*

Proof. i) All we have to do is iterate Lemma 5.14:
With the functions $\partial^k_x\Gamma \in C^0_{\text{bounded}}(I,\mathbb{B})$ satisfying $\partial^k_x\Gamma(\cdot,x)\in C^\infty(\mathbb{R},\mathbb{B})$ at every $x\in I$, existence of the expressions $\partial^l_t\partial^k_x\Gamma : \mathbb{R}\times I\longrightarrow\mathbb{B}$ is guaranteed from the outset.
Now assume that at every $t\in\mathbb{R}$ the map $\partial^l_t\Gamma : \mathbb{R}\times I\longrightarrow\mathbb{B}$ obeys $\partial^l_t\Gamma(t,\cdot)\in C^n_{\text{bounded}}(I,\mathbb{B})$ with x-derivatives given by

$$\partial^k_x[\partial^l_t\Gamma] \;=\; \partial^l_t[\partial^k_x\Gamma] \in C^0_{\text{bounded}}(I,\mathbb{B})$$

Then we have $\sup\limits_{(t,x)\,\in\,\mathbb{R}\times I}\big\|\partial^2_t\partial^k_x[\partial^l_t\Gamma]\big\|_{\mathbb{B}} < \infty$, so Lemma 5.14 shows that $\partial^{l+1}_t\Gamma(t,\cdot)\in C^n_{\text{bounded}}(I,\mathbb{B})$ with x-derivatives

$$\partial^k_x[\partial^{l+1}_t\Gamma] \;=\; \partial_t\partial^k_x[\partial^l_t\Gamma] \;=\; \partial^{l+1}_t[\partial^k_x\Gamma] \in C^0_{\text{bounded}}(I,\mathbb{B})$$

is the t-derivative of $\partial^l_t\Gamma : \mathbb{R}\longrightarrow C^n_{\text{bounded}}(I,\mathbb{B})$ and our claim follows by induction.
ii) By part (i) we have a sequence of derivatives $\partial^l_t\,\Gamma : \mathbb{R}\longrightarrow C^n_{\text{bounded}}(I,\mathbb{B})$, $l\ge 0$ with norm

$$\big\|\partial^l_t\Gamma(t,\cdot)\big\|_{C^n_{\text{bounded}}(I,\mathbb{B})} \;=\; \sum_{k=0}^n \big\|\partial^l_t\partial^k_x\Gamma(t,\cdot)\big\|_{C^0_{\text{bounded}}(I,\mathbb{B})} \;\le\; \sum_{k=0}^n \sup_{(t,x)\,\in\,\mathbb{R}\times I}\big\|\partial^l_t\partial^k_x\Gamma(t,x)\big\|_{\mathbb{B}}$$

For $\Gamma : \mathbb{R}\longrightarrow C^n_{\text{bounded}}(I,\mathbb{B})$ to be smooth with bounded derivatives, all of these norms have to be uniformly bounded in $t\in\mathbb{R}$, as can be achieved by assuming

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I}\big\|\partial^l_t\,\partial^k_x\,\Gamma(t,x)\big\|_{\mathbb{B}} < \infty \qquad \text{for all } k=0,..,n \text{ and } l\ge 0$$

$\square$

For practical situations, there is no cost in discarding our constraint on the number of x-derivatives since Γ (for example being the Hessian of a Hamiltonian function) comes from smooth data anyway. Thus, we will from now on continue with the much more generous assumption of Γ being a C^∞-function on $\Sigma = \mathbb{R} \times I$.

Corollary 5.16 (Simplifying the conditions of Proposition 5.15)
Given a map $\Gamma \in C^\infty(\mathbb{R} \times I, \mathbb{B})$ assume that

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I} \left\|\partial_t^l\,\partial_x^k\,\Gamma\right\|_{\mathbb{B}} < \infty \quad \textit{for all } l, k \geq 0$$

Then Γ satisfies the conditions of Proposition 5.15,
so $\Gamma : \mathbb{R} \longrightarrow C^n_{bounded}(I, \mathbb{B})$ is smooth with bounded derivatives for all $n \geq 0$.

Proof. Let us begin by an alternative definition that clarifies the "computational complexity" of belonging to $C^\infty(\mathbb{R}\times I, \mathbb{B})$: Having $\Gamma \in C^\infty(\mathbb{R}\times I, \mathbb{B})$ means that to every binary sequence

$$A = a_1 a_2 .... a_n \quad \text{with } a_i \in \{t, x\}$$

we can assign a map $\Gamma_A \in C^1(\mathbb{R} \times I, \mathbb{B})$ such that

$$\partial_a \Gamma_A \,=\, \Gamma_{aA} \qquad \text{for all binary sequences } A \text{ and } a \in \{t, x\}.$$

Note that by Schwarz's theorem the terms $\Gamma_{a_1 .... a_n}$ are necessarily permutation invariant, so every Γ_A can be written in the form $\Gamma_{t....tx....x}$ and our assumption $\sup_{(t,x)\,\in\,\mathbb{R}\times I} \left\|\partial_t^l \partial_x^k \Gamma\right\|_{\mathbb{B}} < \infty$ translates to

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I} \left\|\Gamma_A\right\|_{\mathbb{B}} < \infty \qquad \text{for all binary sequences } A$$

The conditions of Proposition 5.15 are immediate because $\Gamma(t, \cdot) \in C^n_{\text{bounded}}(I, \mathbb{B})$ follows from

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I} \left\|\partial_x^k\,\Gamma\right\|_{\mathbb{B}} < \infty \quad \text{for all } k = 0, ..., n$$

and $\Gamma(\cdot, x)$, ... , $\partial_x^n \Gamma(\cdot, x) \in C^\infty(\mathbb{R}, \mathbb{B})$ is due to the possibility of successively applying ∂_t. $\square$

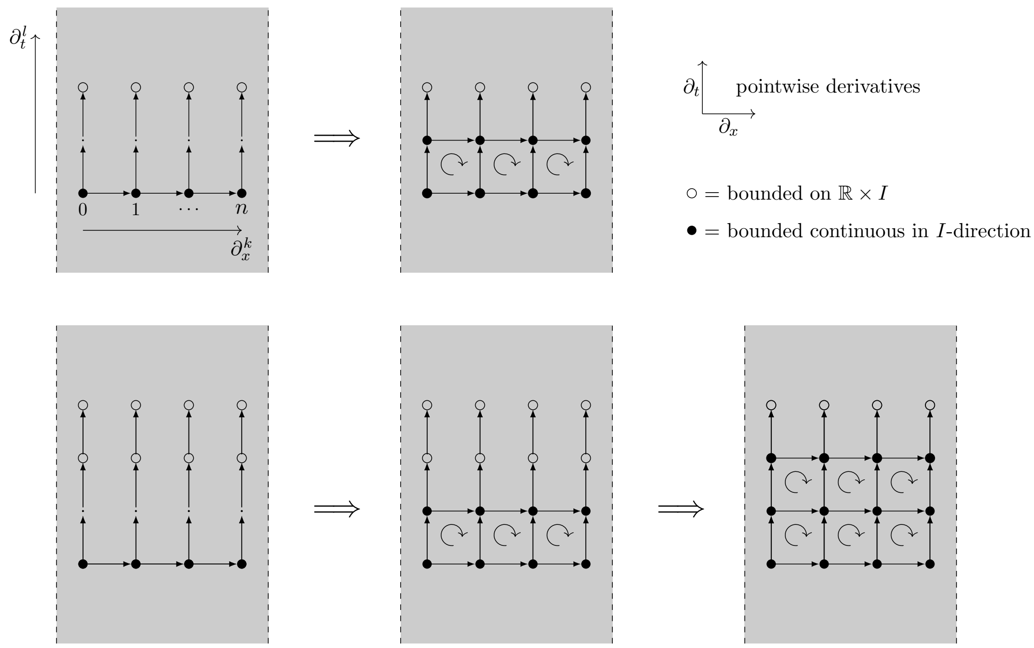


Figure 5.1: Schematic illustration of the inductive argument proving Proposition 5.15. Each point in the diagram represents an $\mathbb{R}$-family of maps $I \longrightarrow \mathbb{B}$. An arrow indicates that two such families $\mathbb{R} \times I \longrightarrow \mathbb{B}$ are related by taking a pointwise derivative. Note that by the assumptions of Proposition 5.15, we can arbitrarily increase the number of t-derivatives, while keeping the number of x-derivatives constant. As indicated by the "commutative squares", we obtain an alternative to the classical Schwarz theorem.

5.2.2 Compatibility with the boundary conditions

From now on, we will only consider the interval $I = (0, 1)$.

Returning to the setting of section 5.1.1, let us choose $\mathbb{B} = \mathcal{L}(\mathbb{H})$, where $\big(\mathbb{H}, \langle\cdot,\cdot\rangle_{\mathbb{H}}\big)$ is a Hilbert space equipped with an "almost complex structure" $J_0 \in \mathcal{L}(\mathbb{H})$ and "symplectic form" $\omega = \langle J_0\cdot, \cdot\rangle_{\mathbb{H}}$.

The proof of Lemma 2.28 provides us with inequalities

$$\|fu\|_{W^{n,2}(I,\mathbb{H})} \leq \text{const.} \times \|f\|_{C^n_{\text{bounded}}(I,\mathcal{L}(\mathbb{H}))} \|u\|_{W^{n,2}(I,\mathbb{H})}$$

giving rise to bounded linear inclusions

$$C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big) \hookrightarrow \mathcal{L}\big(W^{n,2}(I,\mathbb{H})\big)$$

Thus, any $\Gamma \in C^\infty\big(\mathbb{R} \times I, \mathcal{L}(\mathbb{H})\big)$ satisfying the requirements of Corollary 5.16 will induce a moderate family of sc-operators on the filtration $\big(W^{n,2}(I,\mathbb{H})\big)_{n\geq 0}$. Note however that, according to section 5.1.1, we have to impose boundary conditions by choosing a Lagrangian subspace $\Lambda \subset (\mathbb{H}\oplus\mathbb{H}, \Omega = (-\omega)\oplus\omega)$ and that scalar multiplication by $f \in C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)$ does not necessarily preserve the subspace $W^{n,2}_\Lambda(I,\mathbb{H}) \subset W^{n,2}(I,\mathbb{H})$.

We will overcome this issue by going to the subspace $C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda$ to be introduced in Remark 5.19. To properly define this space, some preparations are in order:

Remark 5.17 (The space of Λ-compatible operators $\mathcal{L}_\Lambda \subset \mathcal{L}(\mathbb{H}\oplus\mathbb{H})$)
Let us consider $\mathbb{E} := \mathbb{H}\oplus\mathbb{H}$ as a Hilbert space with block-diagonal inner product

$$\langle u_0 \oplus u_1, v_0 \oplus v_1\rangle_{\mathbb{H}\oplus\mathbb{H}} = \langle u_0, v_0\rangle_{\mathbb{H}} + \langle u_1, v_1\rangle_{\mathbb{H}}$$

By defining $\tilde{\mathbb{I}} := (-J_0) \oplus J_0 \in \mathcal{L}(\mathbb{H}) \oplus \mathcal{L}(\mathbb{H}) \subset \mathcal{L}(\mathbb{H}\oplus\mathbb{H})$ in contrast to $\mathbb{I} = J_0 \oplus J_0$, the symplectic form $\Omega = (-\omega)\oplus\omega$ can be written as $\Omega = \langle\tilde{\mathbb{I}}\cdot, \cdot\rangle_{\mathbb{H}\oplus\mathbb{H}}$. Since for any $u \in \mathbb{H}\oplus\mathbb{H}$ we have $\Omega(u,\cdot) \in \mathcal{L}(\mathbb{H}\oplus\mathbb{H}, \mathbb{R})$, the Ω-orthogonal complement of any subset $\Lambda \subset \mathbb{H}\oplus\mathbb{H}$ is automatically a closed subspace $\Lambda^\Omega \subset \mathbb{H}\oplus\mathbb{H}$. In particular, every Lagrangian subspace $\Lambda = \Lambda^\Omega$ is closed.

Now given a closed subspace $\Lambda \subset \mathbb{E}$, we have an orthogonal decomposition $\mathbb{E} = \Lambda \oplus \Lambda^\perp$ with projectors $p_\Lambda, p_{\Lambda^\perp} \in \mathcal{L}(\mathbb{E})$, allowing us to exhibit

$$\mathcal{L}_\Lambda := \big\{\alpha \in \mathcal{L}(\mathbb{E}) \,\big|\, \alpha(\Lambda) \subset \Lambda \text{ and } \alpha\big(\mathbb{I}(\Lambda)\big) \subset \mathbb{I}(\Lambda)\big\} = \bigcap_{u\in\Lambda} \ker(\mathrm{ev}_u \circ p_{\Lambda^\perp} \circ \_) \cap \ker(\mathrm{ev}_u \circ p_{\Lambda^\perp} \circ \mathbb{I} \circ \_ \circ \mathbb{I})$$

$$\text{and} \quad \mathcal{L}_\Lambda \cdot \mathbb{I} = \big\{\alpha \in \mathcal{L}(\mathbb{E}) \,\big|\, \alpha\big(\mathbb{I}(\Lambda)\big) \subset \Lambda \text{ and } \alpha(\Lambda) \subset \mathbb{I}(\Lambda)\big\}$$

as closed subspaces of $\mathcal{L}(\mathbb{E}) = \mathcal{L}(\mathbb{H}\oplus\mathbb{H})$.

Remark 5.18 (Extracting the boundary behaviour of $f \in C^{n+1}_{\text{bounded}}(I, \mathbb{B})$)
Consider any two points x_0, x_1 from our interval $I = (0, 1)$.
Then any $f \in C^1_{\text{bounded}}(I, \mathbb{B})$ obeys

$$f(x_1) - f(x_0) = (x_1 - x_0) \int_0^1 \mathrm{d}s\, f'\big|_{x_0 + s\cdot(x_1 - x_0)}$$

so with
$$\|f(x_1) - f(x_0)\|_{\mathbb{B}} \leq |x_1 - x_0| \cdot \|f'\|_{C^0_{\text{bounded}}(I,\mathbb{B})}$$

we conclude that $f(x) \in \mathbb{B}$ becomes a Cauchy sequence whenever x approaches the boundary. By completeness of $\mathbb{B}$, we receive well-defined boundary values $f(0)$ and $f(1)$, putting us into a position to consider bounded linear maps

$$\begin{array}{ccccccc} C^{n+1}_{\text{bounded}}(I,\mathbb{B}) & \xrightarrow{\mathrm{pr}_k} & C^1_{\text{bounded}}(I,\mathbb{B}) & \hookrightarrow & C^0(\bar{I},\mathbb{B}) & \xrightarrow{\mathrm{ev}_0 \times \mathrm{ev}_1} & \mathbb{B}\oplus\mathbb{B} \\ (f_0, f_1, ..., f_{n+1}) & \longmapsto & (f_k, f_{k+1}) & \longmapsto & f_k & \longmapsto & \big[f_k(0), f_k(1)\big] \end{array}$$

Note that in our situation $\mathbb{B}\oplus\mathbb{B} = \mathcal{L}(\mathbb{H}) \oplus \mathcal{L}(\mathbb{H}) \subset \mathcal{L}(\mathbb{H}\oplus\mathbb{H})$ is simply the subspace of block-diagonal maps.

Remark 5.19 (Making $C^{n+1}_{\text{bounded}}(I,\mathbb{B})$ respect the boundary conditions)
Let $\Lambda \subset \mathbb{H} \oplus \mathbb{H}$ be a closed subspace.
By combining Remarks 5.17 and 5.18 we observe that

$$\begin{aligned} C^{n+1}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)_\Lambda &:= \bigcap_{k=0}^{n} \big[[\mathrm{ev}_0 \times \mathrm{ev}_1] \circ \mathrm{pr}_k\big]^{-1}\big(\mathcal{L}_\Lambda \cdot \mathbb{I}^k\big) \\ &= \Big\{ f \in C^{n+1}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big) \;\Big|\; [f_k(0), f_k(1)] \in \mathcal{L}_\Lambda \cdot \mathbb{I}^k \;\text{ for all } k = 0,...,n \Big\} \end{aligned}$$

is a closed subspace of $C^{n+1}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)$. To understand the motivation for this definition, let us multiply $u \in W^{n+1,2}_\Lambda(I,\mathbb{H})$ by $f \in C^{n+1}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)$:
Having $u \in W^{n+1,2}_\Lambda(I,\mathbb{H})$ means that $\big[u_k(0), u_k(1)\big] \in \mathbb{I}^k(\Lambda)$ for all $k = 0,...,n$.
Recall that maps from $\mathcal{L}_\Lambda$ preserve the subspaces Λ and $\mathbb{I}(\Lambda)$, so for $m = 0,...,n$ we have

$$[fu]_m(0) \oplus [fu]_m(1) \;=\; \sum_{k=0}^{m} \binom{m}{k} \underbrace{\big[f_{m-k}(0) \oplus f_{m-k}(1)\big]}_{\mathcal{L}_\Lambda \cdot \mathbb{I}^{m-k}} \cdot \underbrace{\big[u_k(0) \oplus u_k(1)\big]}_{\mathbb{I}^k(\Lambda)} \in \mathbb{I}^m(\Lambda)$$

and therefore $fu \in W^{n+1,2}_\Lambda(I,\mathbb{H})$.
Writing $\mathcal{L}\big(W^{n,2}(I,\mathbb{H})\big)_\Lambda$ for the space of all $\alpha \in \mathcal{L}\big(W^{n,2}(I,\mathbb{H})\big)$ that preserve the subspace $W^{n,2}_\Lambda(I,\mathbb{H}) \subset W^{n,2}(I,\mathbb{H})$, we obtain a commutative diagram

$$\begin{array}{ccccc} C^{n}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big) & \longrightarrow & \mathcal{L}\big(W^{n,2}(I,\mathbb{H})\big) & \xrightarrow{\text{restr.}} & \mathcal{L}\big(W^{n,2}_\Lambda(I,\mathbb{H}), W^{n,2}(I,\mathbb{H})\big) \\ \bigcup & & \bigcup & & \bigcup \\ C^{n}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)_\Lambda & \longrightarrow & \mathcal{L}\big(W^{n,2}(I,\mathbb{H})\big)_\Lambda & \xrightarrow{\text{restr.}} & \mathcal{L}\big(W^{n,2}_\Lambda(I,\mathbb{H})\big) \end{array}$$

improving on the map $C^{n}_{\text{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big) \longrightarrow \mathcal{L}\big(W^{n,2}(I,\mathbb{H})\big)$ mentioned in the beginning of this section.

With the extra ingredient of compatible boundary conditions, we are now ready to refine Corollary 5.16 such as to provide the desired information about moderate and localized perturbations:

Proposition 5.20 (Criterium for moderate and localized perturbations)
Let $\Lambda = \Lambda^\Omega$ be a Lagrangian subspace of $\mathbb{H} \oplus \mathbb{H}$.

i) Assume we are given a map $\Gamma \in C^\infty\big(\mathbb{R} \times I, \mathcal{L}(\mathbb{H})\big)$ that in addition to

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial_t^l\, \partial_x^k\, \Gamma\big\|_{\mathbb{B}} < \infty \quad \textit{for all } l,k \geq 0$$

satisfies $\big[\partial_x^k\,\Gamma(t,0), \partial_x^k\,\Gamma(t,1)\big] \in \mathcal{L}_\Lambda \cdot \mathbb{I}^k$ for all $k \geq 0$ and $t \in \mathbb{R}$.
Then

$$\Gamma : \mathbb{R} \longrightarrow C^{n}_{\textit{bounded}}\big(I,\mathcal{L}(\mathbb{H})\big)_\Lambda \longrightarrow \mathcal{L}\big(W^{n,2}_\Lambda(I,\mathbb{H})\big)$$

is smooth with bounded derivatives for all $n \geq 0$,
exhibiting $\Gamma(t)$ as a moderate perturbation on the Banach scale $\big(W^{n,2}_\Lambda(I,\mathbb{I})\big)_{n\geq 0}$.

ii) If in addition

$$\lim_{t\to\pm\infty}\, \sup_{x\in I} \big\|\partial_t^l\,\Gamma(t,x)\big\|_{\mathcal{L}(\mathbb{H})} \;=\; 0 \quad \textit{for } l = 1,2$$

then $\Gamma(t)$ is not only moderate, but also a localized perturbation.

Proof. i) From Corollary 5.16 we already know that $\Gamma : \mathbb{R} \longrightarrow C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)$ is smooth with bounded derivatives. The additional requirement

$$\big[\partial_x^k \Gamma(t,0), \partial_x^k \Gamma(t,1)\big] \in \mathcal{L}_\Lambda \cdot \mathbb{I}^k$$

ensures that $\Gamma(t) \in C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda$ for all $n \geq 0$. Since $C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda$ is a closed subspace of $C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)$, we observe that having $\Gamma(t) \in C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda$ at every $t \in \mathbb{R}$ automatically implies $\Gamma^{(l)}(t) \in C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda$ for all higher derivatives $l \geq 1$ and therefore $\Gamma \in C^\infty\big(\mathbb{R}, C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda\big)$ instead of just $\Gamma \in C^\infty\big(\mathbb{R}, C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)\big)$.
By applying the bounded linear map

$$C^n_{\text{bounded}}\big(I, \mathcal{L}(\mathbb{H})\big)_\Lambda \longrightarrow \mathcal{L}\big(W^{n,2}(I, \mathbb{H})\big)_\Lambda \longrightarrow \mathcal{L}\big(W^{n,2}_\Lambda(I, \mathbb{H})\big)$$

from Remark 5.19, we conclude that $\Gamma : \mathbb{R} \longrightarrow \mathcal{L}\big(W^{n,2}_\Lambda(I, \mathbb{H})\big)$ is smooth with bounded derivatives at every $n \geq 0$.
ii) For $H = L^2(I, \mathbb{H})$ we have

$$\big\|\partial_t^l \Gamma(t,\cdot)\big\|_{\mathcal{L}(H)} \;\leq\; \big\|\partial_t^l \Gamma(t,\cdot)\big\|_{C^0_{\text{bounded}}(I,\mathcal{L}(\mathbb{H}))} \;=\; \sup_{x\in I} \big\|\partial_t^l \Gamma(t,x)\big\|_{\mathcal{L}(\mathbb{H})}$$

so localized perturbations can be realized by demanding

$$\lim_{t\to\pm\infty} \sup_{x\in I} \big\|\partial_t^l \Gamma(t,x)\big\|_{\mathcal{L}(\mathbb{H})} \;=\; 0 \quad \text{for } l = 1, 2.$$

□

While motivated by abstract consistency arguments, our boundary conditions deliver reasonable output in the two special cases of interest:

Example 5.21 (Periodic boundary conditions)
Let Λ be the diagonal $\Delta \subset \mathbb{H} \oplus \mathbb{H}$.
Then Λ is invariant under application of $\mathbb{I} = J_0 \oplus J_0$, i.e. we have $\mathbb{I}(\Lambda) = \Lambda$.
As a result, we obtain $\mathcal{L}_\Lambda \cdot \mathbb{I} = \mathcal{L}_\Lambda$ and

$$\mathcal{L}_\Lambda \cap \big[\mathcal{L}(\mathbb{H}) \oplus \mathcal{L}(\mathbb{H})\big] \;=\; \big\{\alpha \in \mathcal{L}(\mathbb{H}) \oplus \mathcal{L}(\mathbb{H}) \,\big|\, \alpha(\Delta) \subset \Delta\big\}$$

is simply the diagonal in $\mathcal{L}(\mathbb{H}) \oplus \mathcal{L}(\mathbb{H})$. Thus, having $\big[\partial_x^k \Gamma(t,0), \partial_x^k \Gamma(t,1)\big] \in \mathcal{L}_\Lambda \cdot \mathbb{I}^k$ amounts to demanding $\partial_x^k \Gamma(t,0) \;=\; \partial_x^k \Gamma(t,1)$.

Example 5.22 (Local Lagrangian boundary conditions)
Assume that Λ is of the form $\Lambda = \Lambda_0 \oplus \Lambda_1$ with Lagrangian subspaces $\Lambda_0, \Lambda_1 \subset (\mathbb{H}, \omega)$.
Since the symplectic form arises as $\omega = \langle J_0 \cdot, \cdot \rangle_{\mathbb{H}}$, we have identifications

$$\Lambda_i^\perp \;=\; J_0(\Lambda_i^\omega) \;=\; J_0(\Lambda_i)$$

showing that $\mathbb{I}(\Lambda) \;=\; \Lambda_0^\perp \oplus \Lambda_1^\perp$. Thus, when restricted to $\mathcal{L}(\mathbb{H}) \oplus \mathcal{L}(\mathbb{H}) \subset \mathcal{L}(\mathbb{H} \oplus \mathbb{H})$, the spaces $\mathcal{L}_\Lambda$ and $\mathcal{L}_\Lambda \cdot \mathbb{I}$ consist of matrices $M_0 \oplus M_1$ where M_i is diagonal (resp. off-diagonal) w.r.t. the decomposition $\mathbb{H} = \Lambda_i \oplus \Lambda_i^\perp$.

5.2.3 Simplified criteria in the case of linear sigma models $\Phi : \mathbb{R} \times S^1 \longrightarrow \mathbb{H}$

Up to now, we allowed our "target" $\mathbb{H}$ to be a possibly infinite-dimensional Hilbert space. In realistic situations, outlined for instance in [Sa], one may consider maps $\Phi : \mathbb{R} \times I \longrightarrow M$ into a symplectic manifold (M, ω, J) together with an I-family of Hamiltonians $H_x : M \longrightarrow \mathbb{R}$ such that Φ is of finite energy

$$E(\Phi) \;=\; \frac{1}{2}\int_{\mathbb{R}\times I} |\partial_t \Phi|^2 + |\partial_x \Phi - \nabla H_x|^2 \quad \overset{!}{<} \infty$$

and subject to the constraint that

$$\partial_t \Phi + J(\Phi)\partial_x \Phi - [\nabla H_x](\Phi) \;\in \Phi^* TM$$

vanishes at every $(t, x) \in \mathbb{R} \times I$. As explained in [Sa] Prop. 1.21, these conditions lead to a-priori estimates bounding the derivatives of Φ.
Regarding the 'linearized Floer equation'

$$\partial_t \, \delta\Phi + J_0 \partial_x \, \delta\Phi + \Gamma \cdot \delta\Phi \;=\; 0$$

for example encoutered in [Sa] and [RS], it is the Φ-part in $\Gamma =$ "$F \circ \Phi$" that is responsible for Γ being a 'localized moderate perturbation' in the sense of Definition 2.21.
Proposition 5.23 illustrates this point in the simplified setting where Γ arises as a composition

$$\mathbb{R} \times I \xrightarrow{\;\;\Phi\;\;} \mathbb{H} \xrightarrow{\;\;F\;\;} \mathcal{L}(\mathbb{H})$$

with linear target space $\mathbb{H} = \mathbb{R}^{2n}$. This situation is of interest for Landau-Ginzburg models where our Hamiltonian H is replaced by a holomorphic 'superpotential' $W : M \longrightarrow \mathbb{C}$ and holomorphy of W requires M to be non-compact, so one typically takes $M = \mathbb{C}^n$.

Proposition 5.23 (Moderate localized perturbations in the closed string setting)
Let us focus on the situation $\mathbb{H} = \mathbb{R}^{2n}$, $I = (0, 1)$.

i) Assume we are given maps $\Phi \in C^\infty(\mathbb{R} \times I, \mathbb{H})$ and $F \in C^\infty\big(\mathbb{H}, \mathcal{L}(\mathbb{H})\big)$ such that Φ satisfies

- $\sup_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial_t^l \partial_x^k \Phi\big\|_{\mathbb{H}} < \infty \qquad$ *for all $l, k \geq 0$*
- $\partial_x^k \Phi(t, 0) = \partial_x^k \Phi(t, 1) \qquad$ *for all $k \geq 0$ and $t \in \mathbb{R}$*

Then $\Gamma = F \circ \Phi : \mathbb{R} \times I \longrightarrow \mathcal{L}(\mathbb{H})$ defines a moderate perturbation on $\big(W^{n,2}(S^1, \mathbb{H})\big)_{n \geq 0}$.

ii) If in addition

$$\lim_{t\to\pm\infty} \sup_{x\in I} \big\|\partial_t^l \Phi(t, x)\big\|_{\mathbb{H}} \;=\; 0 \quad \textit{for } l = 1, 2$$

then Γ is not only moderate, but also a localized perturbation.

Proof. i) The set

$$R_\Phi := \left\{ \sum_{\substack{\text{finite}\\ \text{sum}}} a_i \cdot [B_i \circ \Phi] \;\middle|\; \begin{array}{c} B_i : \mathbb{H} \longrightarrow \mathcal{L}(\mathbb{H}) \;\; \text{smooth} \\ a_i : \mathbb{R} \times I \to \mathbb{R} \text{ smooth w/ bounded derivatives} \end{array} \right\}$$

is a subring of $C^\infty\big(\mathbb{R}\times I, \mathcal{L}(\mathbb{H})\big)$. Since for simplicity we are working with a finite-dimensional target space $\mathbb{H} = \mathbb{R}^{2n}$, the condition $\sup \big\|\Phi\big\|_{\mathbb{H}} < \infty$ implies that $\overline{\Phi(\mathbb{R} \times I, \mathbb{H})}$ is compact. As a result, all terms of the form $B \circ \Phi$ with $B \in C^\infty\big(\mathbb{H}, \mathcal{L}(\mathbb{H})\big)$ and therefore all elements of R_Φ are bounded functions on our "world-sheet" $\Sigma = \mathbb{R} \times I$.
Note that $R_\Phi \subset C^\infty\big(\mathbb{R} \times I, \mathcal{L}(\mathbb{H})\big)$ is closed under application of ∂_t and ∂_x:
For example, the t-derivative of an element $\sum_i a_i \cdot [B_i \circ \Phi] \in R_\Phi$ consists of terms

$$\partial_t a_i \cdot [B_i \circ \Phi] \,+\, a_i \partial_t \Phi^\mu \, [\partial_\mu B_i] \circ \Phi$$

where our assumption

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial_t^l\, \partial_x^k\, \Phi\big\|_{\mathbb{H}} < \infty \quad \text{for all } l \geq 1,\, k \geq 0$$

ensures that $\partial_t \Phi^\mu : \mathbb{R} \times I \longrightarrow \mathbb{R}$ is smooth with bounded derivatives.
The above discussion shows that all derivatives of $\Gamma = F \circ \Phi$ belong to R_Φ and therefore

$$\sup_{(t,x)\,\in\,\mathbb{R}\times I} \big\|\partial_t^l\, \partial_x^k\, \Gamma\big\|_{L(\mathbb{H})} < \infty \quad \text{for all } l, k \geq 0$$

Next remark that our assumption $\Phi(t,0) = \Phi(t,1)$ guarantees $B \circ \Phi(t,0) = B \circ \Phi(t,1)$ for all $B \in C^\infty\big(\mathbb{H}, \mathcal{L}(\mathbb{H})\big)$. Since $\partial_x^k \Gamma \in R_\Phi$ is of the form $\sum_i a_i \cdot [B_i \circ \Phi]$ where B_i is a derivative of F and a_i contains only x-derivatives $\partial_x^r \Phi^\mu$, we observe that our assumption

$$\partial_x^k \Phi(t,0) = \partial_x^k \Phi(t,1) \quad \text{for all } k \geq 0$$

translates to the statement $\partial_x^k \Gamma(t,0) = \partial_x^k \Gamma(t,1)$ encountered in Example 5.21.
Thus, we can apply Proposition 5.20 to conclude that $\Gamma = F \circ \Phi : \mathbb{R} \times I \longrightarrow \mathcal{L}(\mathbb{H})$ is a moderate perturbation.
ii) With the lowest order t-derivatives given by

$$\begin{aligned} \partial_t [F \circ \Phi] \,&=\, \partial_t \Phi^\mu [\partial_\mu F] \circ \Phi \\ \partial_t^2 [F \circ \Phi] \,&=\, \partial_t^2 \Phi^\mu [\partial_\mu F] \circ \Phi \,+\, \partial_t \Phi^\mu \partial_t \Phi^\nu [\partial_\nu \partial_\mu F] \circ \Phi \end{aligned}$$

our assumptions

$$\overline{\Phi(\mathbb{R} \times I, \mathbb{H})} \text{ compact} \qquad \text{and} \qquad \lim_{t\to\pm\infty} \sup_{x\in I} \big\|\partial_t^l \Phi(t,x)\big\|_{\mathbb{H}} \,=\, 0 \;\text{ for } l = 1, 2$$

suffice to ensure

$$\lim_{t\to\pm\infty} \sup_{x\in I} \big\|\partial_t^l\, \Gamma(t,x)\big\|_{\mathcal{L}(\mathbb{H})} \,=\, 0 \quad \text{for } l = 1, 2.$$

as required for Proposition 5.20(ii). □

Part II

An M-polyfold chart assembling the topology-changing time slices of a pair-of-pants worldsheet

Chapter 6

Contravariant Sobolev Spaces

6.1 The Sobolev space associated to a vector field

Before being able to formulate the 'crossover retraction' in Chapter 7 and prove its sc-smoothness in Chapter 8, we have to introduce vector-field-dependent Sobolev spaces as a framework for calculations.

Throughout this section, we will consider open subsets $\Omega \subset \mathbb{R}^n$ equipped with the datum of a metric g and a distinguished vector field V.
Moreover, we will work with test functions $\phi \in C_0^1(\Omega)$ instead of $C_0^\infty(\Omega)$.

Generalizing our approach from Remark 2.25, we will use the following construction:

Remark 6.1 (Sobolev space associated to a vector field)
Given a fixed Banach space $\mathbb{B}$ like for instance $\mathbb{B} = \mathbb{R}^m$ we consider functions $u : \Omega \longrightarrow \mathbb{B}$ and define $L_g^2(\Omega)$ by demanding $\int_\Omega \sqrt{g}\, \|u\|^2 < \infty$.
With Young's inequality we verify that

$$\delta^V_{\sqrt{g}\phi}(u_0, u_1) := \int \sqrt{g}\, \underbrace{\frac{1}{\sqrt{g}} \partial_\mu\big(\sqrt{g}\phi V^\mu\big)}_{\in C_0^1(\Omega)}\, u_0 + \int \sqrt{g}\phi\, u_1$$

defines a bounded linear map $L_g^2(\Omega) \oplus L_g^2(\Omega) \longrightarrow \mathbb{B}$.
This makes

$$W^{1,2}_{V,g}(\Omega) := \bigcap_{\phi \in C_0^1(\Omega)} \ker \delta^V_\phi$$

a closed subspace of $L_g^2(\Omega) \oplus L_g^2(\Omega)$ and thus a Banach space itself.

Higher Sobolev spaces will be defined as

$$W^{n+1,2}_{V,g}(\Omega) := \left\{(u_0, ..., u_{n+1}) \in L_g^2(\Omega)^{\oplus n+2} \,\middle|\, (u_k, u_{k+1}) \in W^{1,2}_{V,g}(\Omega) \text{ for all } k = 0, ..., n\right\}$$

so all relevant properties can be derived from $W^{1,2}_{V,g}(\Omega)$.

Our specific choice of the 'differentiation constraints' $\delta_\phi^V(u_0,u_1)$ guarantees the following simple transformation behaviour under diffeomorphisms of the domain Ω:

Proposition 6.2 (Contravariant transformation behaviour of the Sobolev spaces $W^{1,2}_{V,g}$)
Let $\Phi : \Omega' \longrightarrow \Omega$ be a diffeomorphism between open subsets of $\mathbb{R}^n$
Then Φ induces an isometry

$$\begin{aligned} W^{1,2}_{V,g}(\Omega) &\longrightarrow W^{1,2}_{\Phi^*V,\Phi^*g}(\Omega') \\ (u_0,u_1) &\longmapsto (u_0\circ\Phi\,,\,u_1\circ\Phi) \end{aligned}$$

Proof. By the Transformation Theorem for L^1-functions one has

$$\int_\Omega \mathrm{d}^n y\,\sqrt{g}\,\|u\|^2 = \int_{\Omega'} \mathrm{d}^n x\,\underbrace{\left|\det\left(\tfrac{\partial y}{\partial x}\right)\right|\sqrt{g\circ\Phi}}_{\sqrt{\Phi^* g}}\,\|u\circ\Phi\|^2$$

so the componentwise map $L^2_g(\Omega)^{\oplus 2} \longrightarrow L^2_{\Phi^*g}(\Omega')^{\oplus 2}$, $(u_0,u_1)\longmapsto(u_0\circ\Phi, u_1\circ\Phi)$
is an isometry. Using the Levi-Civita connection of g we observe that for every vector field V the quantity

$$\frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{g}V^\mu\big) \;=\; \nabla_\mu V^\mu \;=\; \operatorname{trace}\big[X\longmapsto\nabla_X V\big] \tag{6.1}$$

transforms as a scalar, in the sense that under a diffeomorphism Φ we have

$$\left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial y^\mu}\big[\sqrt{g}V^\mu\big]\right]\circ\Phi \;=\; \frac{1}{\sqrt{\Phi^*g}}\frac{\partial}{\partial x^\mu}\Big[\sqrt{\Phi^*g}\;[\Phi^*V]^\mu\Big]$$

Rescaling V by $\phi\in C^1_0(\Omega)$ we get $\Phi^*\big[\phi V\big]=(\phi\circ\Phi)\cdot\Phi^*V$,
so with the Transformation Theorem we find

$$\begin{aligned} \delta^V_{\sqrt{g}\phi}(u_0,u_1) &= \int_\Omega \mathrm{d}^n y\,\sqrt{g}\,\frac{1}{\sqrt{g}}\partial_\mu\big[\sqrt{g}\phi V^\mu\big]\,u_0 + \int_\Omega \mathrm{d}^n y\,\sqrt{g}\phi\,u_1 \\ &= \int_{\Omega'} \mathrm{d}^n x\,\sqrt{\Phi^*g}\,\frac{1}{\sqrt{\Phi^*g}}\,\partial_\mu\Big[\sqrt{\Phi^*g}\,(\phi\circ\Phi)\cdot[\Phi^*V]^\mu\Big]\,u_0\circ\Phi \;+\; \int_{\Omega'} \mathrm{d}^n x\,\sqrt{\Phi^*g}\,(\phi\circ\Phi)\cdot u_1\circ\Phi \\ &= \delta^{\Phi^*V}_{\sqrt{\Phi^*g}\,\phi\,\circ\,\Phi}(u_0\circ\Phi, u_1\circ\Phi) \end{aligned}$$

Given $\rho\in C^\infty(\Omega',\mathbb{R}_{>0})$ and a diffeomorphism $\Phi:\Omega'\longrightarrow\Omega$ one has bijections

$$\begin{aligned} C^1_0(\Omega) &\xrightarrow{\;\sim\;} C^1_0(\Omega') \\ \phi &\longmapsto \phi\circ\Phi \end{aligned} \qquad\qquad \begin{aligned} C^1_0(\Omega') &\xrightarrow{\;\sim\;} C^1_0(\Omega') \\ \xi &\longmapsto \rho\cdot\xi \end{aligned}$$

Thus, every $\varphi\in C^1_0(\Omega')$ can be written as $\varphi=\sqrt{\Phi^*g}\,\phi\circ\Phi$ for some $\phi\in C^1_0(\Omega)$ and we obtain

$$\delta^{\Phi^*V}_{\varphi}(u_0\circ\Phi,u_1\circ\Phi) \;=\; \delta^V_{\sqrt{g}\phi}(u_0,u_1)$$

As a result, the isometry $L^2_g(\Omega)^{\oplus 2} \xrightarrow{-\circ\Phi} L^2_g(\Omega')^{\oplus 2}$ maps $W^{1,2}_{V,g}(\Omega)$ to $W^{1,2}_{\Phi^*V,\Phi^*g}(\Omega')$. □

Remark. To obtain formula 6.1 compare $\dfrac{\partial_\nu\sqrt{\det(g)}}{\sqrt{\det(g)}} = \dfrac{1}{2}\operatorname{tr}(g^{-1}\partial_\nu g)$ with the $\rho\mu$-contraction of

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\,g^{\rho\kappa}\left(\partial_\nu g_{\kappa\mu} + \partial_\mu g_{\kappa\nu} - \partial_\kappa g_{\mu\nu}\right)$$

6.2 Criteria for compactness

From now on we will focus on the case $n = 1$ and replace Ω by open intervals $I \subset \mathbb{R}$.
As a useful tool, we will consider the following 1-dimensional version of a flow:

Remark 6.3 (Straightening diffeomorphism)
Let $I_x \subset \mathbb{R}$ be an open interval and fix a basepoint $x_0 \in I_x$.
Given a vector field $V = V(x)\partial_x$ with $V(x) > 0$, the map

$$\varphi : I_x \longrightarrow I_y \subset \mathbb{R} \qquad x \longmapsto y(x) = \int_{x_0}^{x} \frac{\mathrm{d}x'}{V(x')}$$

defines a diffeomorphism between I_x and another open interval $I_y \subset \mathbb{R}$.
Its inverse $\Phi = \varphi^{-1} : I_y \longrightarrow I_x$ will be called the *straightening diffeomorphism* for V because it trivialises our vector field in the sense that $\Phi^* V = \partial_y$. Note, however, that the pullback of a given metric g on I_x will be an a priori unspecified metric $g_{str} := \Phi^* g$ on I_y.

Luckily, some information about g_{str} can be recovered directly from the data (I_x, V, g).
For instance, the divergence of our vector field V can be used to control the logarithmic variation $\frac{\partial\sqrt{g_{str}}}{\sqrt{g_{str}}}$. This allows us to derive the following criteria for (non-)compactness of the inclusion $W^{1,2}_{V,g}(I) \hookrightarrow L^2_g(I)$ depending on whether the flow of V exists for all times:

Lemma 6.4 (Criteria for/against Compactness)
Let $I \subset \mathbb{R}$ be an open interval with metric g and distinguished vector field $V = V(x)\partial_x$, $V(x) > 0$

i) Assume that $div(V) = \frac{1}{\sqrt{g}}\partial\big[\sqrt{g}V\big]$ is bounded, whereas $\displaystyle\int_I \frac{dx}{V(x)} = \infty$.
Then the inclusion $W^{1,2}_{V,g}(I) \hookrightarrow L^2_g(I)$ is non-compact.

ii) Assume there exist constants $c, C > 0$ such that $c < \sqrt{g}V < C$.
Then the condition $\displaystyle\int_I \frac{dx}{V(x)} < \infty$ ensures that there are compact inclusions

$$W^{1,2}_{V,g}(I) \hookrightarrow L^2_g(I) \quad \textit{and} \quad W^{1,2}_{V,g}(I) \hookrightarrow C^0_{bounded}(I)$$

Proof. Let $\Phi : I_{str} \longrightarrow I$ be the straightening diffeomorphism from Remark 6.3 and write $g_{str} = \Phi^* g$
Part (i). The condition $\displaystyle\int_I \frac{\mathrm{d}x}{V(x)} = \infty$ means that $I_{str} \subset \mathbb{R}$ is an unbounded interval.
So let us treat the case $[0, \infty) \subset I_{str}$: Given a bump $\phi \in C_0^\infty(-a, a)$ with $a < 1$ let us write $w_n := \max\limits_{[n-a,\, n+a]} \sqrt{g_{str}}$ and consider the sequence $\phi_n := \frac{1}{w_n^{1/2}}\, \tau_{-n}\phi$.
Note that the action of $\Phi^* V = \partial$ on ϕ_n is just $\partial\phi_n = \big[\partial\phi\big]_n$, so the estimate

$$\int_{I_{str}} \sqrt{g_{str}}\, \|\phi_n\|^2 \leq \int_{\mathbb{R}} \|\tau_{-n}\phi\|^2 = \int_{\mathbb{R}} \|\phi\|^2 = \text{const.}$$

shows that $\phi_n \in W^{1,2}_{\partial, g_{str}}(I_{str})$ is a bounded sequence.
To argue that $\phi_n \in L^2_{g_{str}}(I_{str})$ does not admit a convergent subsequence,
we first establish a lower bound on $\frac{\sqrt{g_{str}}}{w_n}$: By assumption, the expression

$$\frac{\partial\sqrt{g_{str}}}{\sqrt{g_{str}}} = \frac{1}{\sqrt{\Phi^* g}}\partial\big[\sqrt{\Phi^* g} \cdot 1\big] = \Big[\frac{1}{\sqrt{g}}\partial\big[\sqrt{g}V\big]\Big] \circ \Phi$$

is bounded by some constant $C > 0$. Now choose $y_0 \in [n-a, n+a]$ such that $\sqrt{g_{str}}\,(y_0) = \max\limits_{[n-a,\, n+a]} \sqrt{g_{str}}$.
With the Fundamental Theorem of Calculus we find

$$0 \geq \log\frac{\sqrt{g_{str}}}{w_n} = \int_{y_0}^{y} \frac{\partial\sqrt{g_{str}}}{\sqrt{g_{str}}} = -\left|\int_{y_0}^{y} \frac{\partial\sqrt{g_{str}}}{\sqrt{g_{str}}}\right| \geq -\int\limits_{[n-a, n+a]} \left|\frac{\partial\sqrt{g_{str}}}{\sqrt{g_{str}}}\right| \geq -2a \cdot C$$

and therefore $$\frac{\sqrt{g_{str}}}{w_n} \geq e^{-2a\cdot C} = \text{const.} > 0$$

Since any two different ϕ_n, ϕ_m have disjoint supports, we observe that

$$\|\phi_n - \phi_m\|^2_{L^2_{g_{str}}(I_{str})} = \int\limits_{[n-a,n+a]} \sqrt{g_{str}}\ \|\phi_n\|^2 \ + \int\limits_{[m-a,m+a]} \sqrt{g_{str}}\ \|\phi_m\|^2 \ \geq\ 2 \cdot e^{-2a\cdot C} \int_{\mathbb{R}} \|\phi\|^2$$

so $\phi_n \in L^2_{g_{str}}(I_{str})$ cannot contain a Cauchy sequence.

Part (ii). By assumption $\sqrt{g_{str}} = \left[\sqrt{g}V\right] \circ \Phi_{str}$ is bounded above and below. As a result, we have isomorphisms of Banach spaces $L^2_g(I_{str}) \cong L^2(I_{str})$ and $W^{1,2}_{\partial,g_{str}}(I_{str}) \cong W^{1,2}(I_{str})$. Sobolev embedding yields a commutative diagram

$$\begin{tikzcd}
W^{1,2}_{V,g}(I) \arrow[r, "\text{canonical projection}"] \arrow[d, "\wr"', "-\circ\Phi"] & L^2_g(I) & C^0_{bounded}(I) \arrow[l, hook'] \\
W^{1,2}_{\partial,g_{str}}(I_{str}) \arrow[d, phantom, sloped, "\cong"] & L^2_{g_{str}}(I_{str}) \arrow[u, "-\circ\Phi^{-1}", "\wr"'] \arrow[d, phantom, sloped, "\cong"] & \\
W^{1,2}(I_{str}) \arrow[r, hook] \arrow[d, hook] & L^2(I_{str}) & C^0_{bounded}(I_{str}) \arrow[l, hook'] \arrow[uu, "-\circ\Phi^{-1}", "\wr"'] \\
C^{0,1/2}(I_{str}) = C^{0,1/2}(\bar{I}_{str}) \arrow[r, hook, "\alpha"] & C^0(\bar{I}_{str}) \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Note that the condition $\displaystyle\int_I \frac{\mathrm{d}x}{V(x)} < \infty$ makes $I_{str} \subset \mathbb{R}$ a bounded interval, so by the Arzelà-Ascoli Theorem the embedding $\alpha : C^{0,1/2}(\bar{I}_{str}) \hookrightarrow C^0(\bar{I}_{str})$ is compact. Our claims follow because the compact operators form an ideal among bounded linear maps.

$\square$

Example. The inclusion $W^{1,2}_{x\partial_x,1}(\delta, 1) \hookrightarrow L^2(\delta, 1)$ is compact for $\delta > 0$ and non-compact for $\delta = 0$. As a result, $W^{1,2}_{x\partial_x,1}(0, 1)$ cannot continuously embed into the standard Sobolev space $W^{1,2}(0, 1)$. The reverse inclusion $W^{1,2}(0, 1) \hookrightarrow W^{1,2}_{x\partial_x,1}(0, 1)$, however, does exist, as can be seen by the methods of the next section.

6.3 Algebraic structures

6.3.1 Rules for changing the vector field

In this interlude section, we establish a rule by which Sobolev spaces associated to different vector fields can be related. For this purpose, we continue to work over a general open interval $I \subset \mathbb{R}$. Our discussion will exploit the interplay between iteratively defined 'expansion coefficients' $\tilde{C}^V_{n,k}[f] \in C^\infty(I)$ and a certain subring $\mathcal{R}(I,V) \subset C^\infty(I)$ from which they cannot escape.

Remark 6.5 (Ring of smooth functions with bounded powers of V)

The set $\mathcal{R}(I,V) := \left\{ f \in C^\infty(I) \,\middle|\, \sup_{x\in I} \left\| V^k[f] \right\| < \infty \text{ for all } k \geq 0 \right\}$ is a subring of $C^\infty(I)$.

It is closed under the action of V in the sense that

$$f \in \mathcal{R}(I,V) \quad \Longrightarrow \quad V[f] \in \mathcal{R}(I,V)$$

Definition 6.6 (Expansion Coefficients)

Given vector fields V, W and a function $f \in C^\infty(I)$ we define the coefficients $\tilde{C}^W_{n,k}[f]$, $C^V_{n,k}[f] \in C^\infty(I)$ iteratively by

$$\begin{aligned} \tilde{C}_{0,k} &= \delta_{0,k} & C_{0,k} &= \delta_{0,k} \\ \tilde{C}_{n+1,k} &= W[\tilde{C}_{n,k}] + f \cdot \tilde{C}_{n,k-1} & C_{n+1,k} &= k \cdot V[f] C_{n,k} + f \cdot V[C_{n,k}] + C_{n,k-1} \end{aligned}$$

Remark 6.7 (Immediate Properties of the Expansion Coefficients)

- Induction in $n \geq 1$ shows that $C_{n,n} = 1$ and $C_{n,k} = 0$ unless $1 \leq k \leq n$.
  Thus, $\left[C_{n,k}\right]^{k\,=\,1,\ldots,N}_{n\,=\,1,\ldots,N} \in \mathrm{SL}\big(N, C^\infty(I)\big)$ is a lower triangular matrix with unit diagonal.
- Having $f \in \mathcal{R}(I,V)$ ensures $C_{n,k} \in \mathcal{R}(I,V)$ for all n, k.
- Having $f \in \mathcal{R}(I,W)$ ensures $\tilde{C}_{n,k} \in \mathcal{R}(I,W)$ for all n, k.

Example 6.8 (Integer coefficients)

In the case $I = \mathbb{R}$, $V = \partial_x$ and $f = x$ one has $V[f] = 1$, so induction in n shows that $V[C_{n,k}] = 0$. Again by induction we conclude that $C_{n,k} \in \mathbb{N}$ for all n, k.

Remark 6.9 (Relating the powers of different vector fields)

Induction in n shows that $\tilde{C}^{fV}_{n,k}[f] = C^V_{n,k}[f] \cdot f^k$

Thus, if W and V are related by $W = f \cdot V$, we have expansions

$$W^n[g] = \sum_{k=0}^{n} \tilde{C}^W_{n,k} \, V^k[g] = \sum_{k=0}^{n} C^V_{n,k} \, f^k \, V^k[g]$$

relating the powers of W and V on any $g \in C^\infty(I)$.

Lemma 6.10 (Rescaling the vector field gives ring extensions)

Given vector fields V, W such that $W = f \cdot V$ for some $f \in C^\infty(I)$ assume that

a) $f \in \mathcal{R}(I,V)$ or b) $f \in \mathcal{R}(I,W)$

Then we have an inclusion $\mathcal{R}(I,V) \subset R(I,W)$.

Proof. Consider $g \in C^\infty(I)$ such that all powers $V^k[g]$ are bounded functions on I. Provided that $f \in \mathcal{R}(I,W)$ or $f \in \mathcal{R}(I,V)$, the expansions

$$W^n[g] = \sum_{k=0}^{n} \underbrace{\tilde{C}_{n,k}[f]}_{\substack{\mathcal{R}(I,W) \\ \text{if } f\in\mathcal{R}(I,W)}} V^k[g] \qquad W^n[g] = \sum_{k=0}^{n} \underbrace{C_{n,k}[f]\, f^k}_{\substack{\mathcal{R}(I,V) \\ \text{if } f\in\mathcal{R}(I,V)}} V^k[g]$$

show that the powers $W^n[g]$ are bounded as well. □

Example 6.11 (Strict inclusion)

For $I = (0,1)$ one has $x \in \mathcal{R}(I,\partial_x) \subset \mathcal{R}(I, x\partial_x)$.

This inclusion is strict because $\sqrt{x} \in \mathcal{R}(I, x\partial_x)$ but $\sqrt{x} \notin \mathcal{R}(I,\partial_x)$.

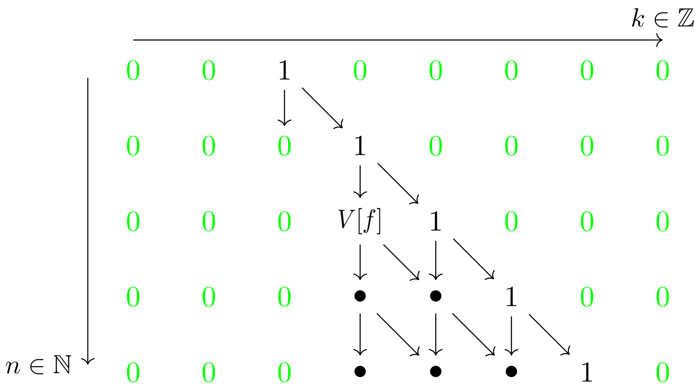


Figure 6.1: Iterative construction of the 'expansion coefficients' $C^V_{n,k}[f] \in C^\infty(I)$ associated to $f \in C^\infty(I)$. Observe that for every $N \geq 1$,

$$\left(C_{n,k}\right)_{1\leq n,k\leq N}$$

will be an invertible $N \times N$-matrix over the ring $C^\infty(I)$, so the expansion

$$[fV]^n = \sum_{k=1}^{n} C_{n,k} \cdot f^k\, V^k$$

can be inverted to give

$$f^n\, V^n = \sum_{k=1}^{n} C^{-1}_{n,k} \cdot [fV]^k$$

In the case $I = \mathbb{R}_a, V = \frac{\partial}{\partial a}, f = a$, this inverted expansion reads

$$a^n \left[\frac{\partial}{\partial a}\right]^n = \sum_{k=1}^{n} C^{-1}_{n,k} \cdot \left[a\,\frac{\partial}{\partial a}\right]^k$$

with integer coefficients $C^{-1}_{n,k} \in \mathbb{Z}$.

Coming back to our 'contravariant Sobolev spaces', we will adopt the perspective that $W^{n;2}_{V,g}(I)$ is a module over the ring $\mathcal{R}(I,V)$.
In the following we assume that a particular metric g has been singled out, so we can identify "$W^{n,2}_V(I)$" $= W^{n,2}_{V,g}(I)$ with a subspace of $L^2_g(I)$ just like in Lemma 2.26.
Using only minimal assumptions, we obtain the following basic result:

Auxiliary Lemma 6.12 (Rescaling of the vector field vs. rescaling of the argument)
Pick $f \in C^1(I)$.

1) *Assume $\sup\limits_{x\in I} \|f\| < \infty$. Then $u \in W^{1,2}_V(I)$ implies $u \in W^{1,2}_{f\cdot V}(I)$.*
2) *Assume $\sup\limits_{x\in I} \|f\| < \infty$ and $\sup\limits_{x\in I} \| V[f] \| < \infty$.*
   Then $u \in W^{1,2}_V(I)$ implies $f \cdot u \in W^{1,2}_V(I)$ with $V[f\cdot u] = V[f]u + fV[u]$

Proof. Write $u = (u_0, u_1) \in W^{1,2}_V(I) \subset L^2_g(I)^{\oplus 2}$.
The assumptions ensure $fu_1 \in L^2_g(I)$ and $fu_1 + V[f]u_0 \in L^2_g(I)$, respectively.
Multiplying $\phi \in C^1_0(I)$ by $f \in C^1(I)$ we have $f \cdot \phi \in C^1_0(I)$. Thus, our claims follow from

$$\delta^{fV}_\phi\big(u_0,\, fu_1\big) = \int u_0\, \partial(f\, V\, \phi) + \int f\, u_1\, \phi = \delta^V_{f\phi}(u_0, u_1) = 0$$

$$\delta^V_\phi\big(fu_0, V[f]u_0 + fu_1\big) = \int u_0\, \underbrace{[f\partial(V\phi) + V[f]\phi]}_{\partial(f\,V\,\phi)} + \int f\, u_1\, \phi = \delta^V_{f\phi}(u_0, u_1) = 0$$

□

By combining parts 1) and 2), we arrive at the following extension criterion for our vector-field-dependent Sobolev spaces:

Lemma 6.13 (Rescaling the vector field extends the Sobolev space)
Given vector fields related as $W = f \cdot V$ with $f \in \mathcal{R}(I,W)$ there is a (unique) bounded linear map $W^{n,2}_V(I) \hookrightarrow W^{n,2}_W(I)$ covering $\mathrm{id}_{L^2_g(I)}$.

Proof. Having $f \in \mathcal{R}(I,W)$ ensures that all $\tilde{C}^W_{m,k}[f]$ belong to $\mathcal{R}(I,W)$, so in particular they are bounded functions. We show that, when restricted to $W^{n,2}_V(I)$, the bounded linear map

$$L^2_g(I)^{\oplus n+1} \longrightarrow L^2_g(I)^{\oplus n+1}, \quad [u_m]_{m=0,\dots,n} \longmapsto \left[\sum_{k=0}^{m} \tilde{C}_{m,k}\, u_k\right]_{m=0,\dots,n}$$

takes values in $W^{n,2}_W(I)$: Pick $u \in W^{n,2}_V(I)$. For $k \le n-1$ we have $V^k u \in W^{1,2}_V(I)$ and since part 1) of Auxiliary Lemma 6.12 does not rely on information about "$V[f]$" we get $V^k u \in W^{1,2}_W(I)$. Now apply part 2) with W instead of V to verify that for $m = 0, \dots, n-1$

$$\sum_k \tilde{C}_{m,k} V^k u \;\in W^{1,2}_W(I)$$

$$\text{with} \quad W\Big[\sum_k \tilde{C}_{m,k} V^k u\Big] = \sum_k W[\tilde{C}_{m,k}] \cdot V^k u + \tilde{C}_{m,k}\, f \cdot V^{k+1} u$$

$$= \sum_k \Big[W[\tilde{C}_{m,k}] + \tilde{C}_{m,k-1} f\Big]\, V^k u \;=\; \sum_k \tilde{C}_{m+1,k} V^k u$$

□

Corollary 6.14 *Given $f \in \mathcal{R}(I,V)^\times$ there is an isomorphism of Banach spaces $W^{n,2}_V(I) \cong W^{n,2}_{f\cdot V}(I)$*

Proof. Having $W = f\cdot V$ with $f \in \mathcal{R}(I,V) \subset \mathcal{R}(I,W)$ provides a morphism $W^{n,2}_V(I) \xrightarrow{\alpha} W^{n,2}_W(I)$ and having $V = f^{-1}W$ with $f^{-1} \in \mathcal{R}(I,V)$ provides a morphism $W^{n,2}_W(I) \xrightarrow{\beta} W^{n,2}_V(I)$. These morphisms as well as their compositions cover the identity $\mathrm{id}_{L^2_g(I)}$. Since morphisms covering the identity are unique, we conclude that α and β are mutually inverse. □

6.3.2 The category of displacement and pointwise superposition

In this slightly technical section, we introduce rules by which the superposition of functions from different Sobolev spaces can be evaluated in the Sobolev space of yet another vector field. This allows us to regard matrices over the rings $\mathcal{R}_\mathcal{P} = C^\infty(\mathbb{R})$ and $\mathcal{R}_{\mathcal{P}\mathcal{Q}} = \bigoplus_{\phi \in \mathrm{Diff}(\mathbb{R})} C^\infty(\mathbb{R}) \cdot \phi$ as morphisms between the correct Sobolev spaces. As an application, we will be able to rigorously identify the space $\mathcal{N}_n(\text{-}a, a) = W^{n,2}_{V^a_{int}; \rho^a_{int}}(\text{-}a, a)$ involved in our formulation of the "anti-gluing map" in Proposition 7.3.

The following notation will also be used in subsequent chapters:

Notation ($W^{n,2}_{V;\rho}$ vs. $W^{n,2}_{V,g}$)

Instead of demanding $\displaystyle\int_I \sqrt{g}\, \|u\|^2 < \infty$ it will be more convenient to write $\displaystyle\int_I \|\rho \cdot u\|^2 < \infty$.

To highlight the difference we denote the respective spaces by $L^2_g(I)$ and $L^2(I)_\rho$.

At higher orders $n \geq 1$ we will adopt the notations $W^{n,2}_{V,g}(I) \subset L^2_g(I)^{\oplus n+1}$ and $W^{n,2}_{V;\rho}(I) \subset L^2(I)^{\oplus n+1}_\rho$.

Note that ρ plays the role of $g^{1/4}$, i.e. under diffeomorphisms $\Phi : I_x \longrightarrow I_y$ it transforms as

$$\Phi^* \rho \,=\, \left|\frac{\partial y}{\partial x}\right|^{1/2} \rho \circ \Phi$$

In Chapters 7 and 8 we will be dealing with different powers ρ^m of the same function ρ. When working with $W^{n,2}_{V;\rho^m}$ the transformation behaviour is still

$$\Phi^*(\rho^m) \,=\, \left|\frac{\partial y}{\partial x}\right|^{1/2} \rho^m \circ \Phi \,\neq\, (\Phi^* \rho)^m \,=\, \left|\frac{\partial y}{\partial x}\right|^{m/2} \rho^m \circ \Phi$$

so $W^{n,2}_{V;\rho^m}(I_y) \cong W^{n,2}_{\Phi^* V;\, \Phi^*(\rho^m)}(I_x) \ncong W^{n,2}_{\Phi^* V;\, (\Phi^* \rho)^m}(I_x)$ unless $|\partial y / \partial x| = 1$.

The matrix $\begin{bmatrix} R_{1/a}\alpha & R_{1/a}\gamma \\ -R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}$ from Proposition 7.3 can be regarded as a morphism in the following category:

Definition 6.15 (The category of pointwise superposition $\mathcal{P}$)

Denote by $\mathcal{P}$ the "category of pointwise superposition" consisting of the following data:

0) The objects (I, V, ρ) are ordered tuples $\big([I_i, V_i, \rho_i]\big)_{i=1,\dots,m}$
   with open intervals $I_i \subset \mathbb{R}$ and $V_i \,, \rho_i \in C^\infty(I_i, \mathbb{R}_{>0})$

1) Morphisms $(J, W, \theta) \longrightarrow (I, V, \rho)$ are matrices $(\alpha_{ij})^{j=1,\dots,n}_{i=1,\dots,m}$ with entries in $C^\infty(\mathbb{R})$ satisfying the following three constraints:

   - $I_i \cap \overline{\{\alpha_{ij} \neq 0\}} \subset J_j$
   - $\alpha_{ij} \in \mathcal{R}(I_i, V_i)$
   - there exist open sets S_{ij} with $I_i \cap \overline{\{\alpha_{ij} \neq 0\}} \subset S_{ij} \subset I_i \cap J_j$
     so that $V_i / W_j \in \mathcal{R}(S_{ij}, V_i)$ and $\rho_i / \theta_j\big|_{S_{ij}}$ is bounded

Lemma 6.16 (Composition in $\mathcal{P}$)
Multiplication of matrices equips $\mathcal{P}$ with a well-defined composition.

Proof. Consider a pair of composable morphisms $(K,Z,\kappa) \xrightarrow{\beta} (J,W,\theta) \xrightarrow{\alpha} (I,V,\rho)$.
Bullet point 3 of the definition yields open neighbourhoods S_{ij} and T_{jk} satisfying

$$I_i \cap \overline{\{\alpha_{ij} \neq 0\}} \subset S_{ij} \subset I_i \cap J_j \qquad \text{and} \qquad J_j \cap \overline{\{\beta_{jk} \neq 0\}} \subset T_{jk} \subset J_j \cap K_k$$

One has

$$I_i \cap \overline{\{\alpha_{ij}\beta_{jk} \neq 0\}} \;=\; \underbrace{I_i \cap \overline{\{\alpha_{ij} \neq 0\}}}_{\subset J_j} \cap \overline{\{\beta_{jk} \neq 0\}} \;\subset\; S_{ij} \cap T_{jk} \;\subset\; I_i \cap J_j \cap K_k$$

so choosing $R_{ik} = \bigcup_j S_{ij} \cap T_{jk}$ we get

$$I_i \cap \overline{\{\textstyle\sum_j \alpha_{ij}\beta_{jk} \neq 0\}} \;\subset\; \textstyle\bigcup_j I_i \cap \overline{\{\alpha_{ij}\beta_{jk} \neq 0\}} \;\subset\; R_{ik} \;\subset\; I_i \cap K_k$$

Lemma 6.10 with "f" $= \frac{V_i}{W_j} \in \mathcal{R}(S_{ij}, V_i)$ ensures that $\mathcal{R}(S_{ij}, W_j) \subset \mathcal{R}(S_{ij}, V_i)$.
This has the following consequences:

- $\beta_{jk} \in \mathcal{R}(J_j, W_j)$ restricts to $\beta_{jk} \in \mathcal{R}(S_{ij}, V_i)$, so the product $\alpha_{ij}\beta_{jk}$ belongs to $\mathcal{R}(S_{ij}, V_i)$.
  Since α_{ij} vanishes on an open neighbourhood of $I_i \cap \partial S_{ij}$,
  extension by zero shows $\alpha_{ij}\beta_{jk} \in \mathcal{R}(I_i, V_i)$ and therefore $\big(\alpha\beta\big)_{ik} = \sum_j \alpha_{ij}\beta_{jk} \in \mathcal{R}(I_i, V_i)$.
- $W_j/Z_k \in \mathcal{R}(T_{jk}, W_j)$ restricts to $W_j/Z_k \in \mathcal{R}(S_{ij} \cap T_{jk}, V_i)$,
  so $V_i/Z_k = V_i/W_j \cdot W_j/Z_k \in \mathcal{R}(S_{ij} \cap T_{jk}, V_i)$.

Since $R_{ik} = \bigcup_j S_{ij} \cap T_{jk}$ is a finite cover, we conclude that $V_i/Z_k \in \mathcal{R}(R_{ik}, V_i)$.
Similarly $\rho_i/\kappa_k = \rho_i/\theta_j \cdot \theta_j/\kappa_k$ is bounded on each component $S_{ij} \cap T_{jk}$ and therefore bounded on the entire R_{ik}. □

Lemma 6.17 (Constructing a family of functors $\mathcal{F} : \mathcal{P} \longrightarrow$ Banach spaces)
Denote by $\mathcal{B}$ the category of Banach spaces and bounded linear maps.
For every $n \geq 0$ there exists a well-defined functor $\mathcal{F}_n : \mathcal{P} \longrightarrow \mathcal{B}$ such that

- $\big([I_i, V_i, \rho_i]\big)_{i=1,\dots,m}$ *gets mapped to* $\displaystyle\bigoplus_{i=1,\dots,m} W^{n,2}_{V_i\,;\,\rho_i}(I_i)$
- *every morphism* $(J,W,\theta) \xrightarrow{\alpha} (I,V,\rho)$ *translates into a bounded linear map*

  $$\mathcal{F}_n(\alpha) : \textstyle\bigoplus_j W^{n,2}_{W_j\,;\,\theta_j}(J_j) \longrightarrow \bigoplus_i W^{n,2}_{V_i\,;\,\rho_i}(I_i)$$

  sending $u_j \in W^{n,2}_{W_j\,;\,\theta_j}(J_j)$ *to* $\sum_j \alpha_{ij}u_j \in W^{n,2}_{V_i\,;\,\rho_i}(I_i)$.

Proof. Consider a morphism $(J,W,\theta) \xrightarrow{\alpha} (I,V,\rho)$.
As in point 3 of the definition we have open sets S_{ij} with $I_i \cap \overline{\{\alpha_{ij} \neq 0\}} \subset S_{ij} \subset I_i \cap J_j$.
By Lemma 6.13 with "f" $= V_i/W_j \in \mathcal{R}(S_{ij}, V_i)$ there is a bounded linear inclusion

$$W^{n,2}_{W_j\,;\theta_j}(S_{ij}) \hookrightarrow W^{n,2}_{V_i\,;\theta_j}(S_{ij}) \tag{6.2}$$

Taking into account that $\alpha_{ij} \in \mathcal{R}(I_i, V_i)$, the assignment $u_j \longmapsto \alpha_{ij}u_j$ can be understood as a composition of bounded linear maps

$$W^{n,2}_{W_j;\theta_j}(J_j) \xrightarrow{\text{restr.}} W^{n,2}_{W_j;\theta_j}(S_{ij}) \xrightarrow{(6.2)} W^{n,2}_{V_i;\theta_j}(S_{ij}) \xrightarrow{\alpha_{ij}\,\cdot\,\_} W^{n,2}_{V_i;\theta_j}(S_{ij}) \longrightarrow W^{n,2}_{V_i;\rho_i}(S_{ij})$$

where in the last step we have used that ρ_i/θ_j is a bounded function on S_{ij}.
With $\tilde{S}_{ij} := I_i \cap \overline{\{\alpha_{ij} \neq 0\}}^c$ we have an open cover $I_i = S_{ij} \cup \tilde{S}_{ij}$,
so by gluing $\alpha_{ij}u_j \in W^{n,2}_{V_i;\rho_i}(S_{ij})$ and $0 \in W^{n,2}_{V_i;\rho_i}(\tilde{S}_{ij})$ we obtain $\alpha_{ij}u_j \in W^{n,2}_{V_i;\rho_i}(I_i)$ with

$$\|\alpha_{ij}u_j\|_{W^{n,2}_{V_i;\rho_i}(I_i)} = \|\alpha_{ij}u_j\|_{W^{n,2}_{V_i;\rho_i}(S_{ij})} + \|0\|_{W^{n,2}_{V_i;\rho_i}(\tilde{S}_{ij})} - \|0\|_{W^{n,2}_{V_i;\rho_i}(S_{ij}\cap\tilde{S}_{ij})} = \|\alpha_{ij}u_j\|_{W^{n,2}_{V_i;\rho_i}(S_{ij})} \leq \text{const.} \times \|u_j\|_{W^{n,2}_{W_j;\theta_j}(J_j)}$$

That $\mathcal{F}_n : \mathcal{P} \longrightarrow \mathcal{B}$ respects the composition of morphisms is a special case of Lemma 6.21 □

To interpret matrices like $\begin{bmatrix} R_{1/a}(\tau_{+1}\alpha^2)\cdot \mathrm{id} & R_{1/a}(\tau_{+1}\alpha\gamma)\cdot\tau_{+2a} \\ R_{1/a}(\tau_{-1}\alpha\gamma)\cdot\tau_{-2a} & R_{1/a}(\tau_{-1}\gamma^2)\cdot \mathrm{id}\end{bmatrix}$, we have to reiterate our discussion by allowing diffeomorphisms of the domains.

As an extension of $\mathcal{R}_{\mathcal{P}} = C^\infty(\mathbb{R})$, we will use the following ring:
Definition 6.18 (The ring $\mathcal{R}_{\mathcal{PQ}}$)
In the following, we will consider $\mathcal{R}_{\mathcal{PQ}} = \bigoplus_{\phi\in \mathrm{Diff}(\mathbb{R})} C^\infty(\mathbb{R})\cdot\phi$ as a ring with multiplication

$$\big(\alpha|\phi\big)\cdot\big(\beta|\varphi\big) := \big(\alpha\cdot(\beta\circ\phi)\,|\,\varphi\circ\phi\big)$$

Thus, using the notation $(\alpha_\phi)_{\phi\in\mathrm{Diff}(\mathbb{R})}$ one has $(\alpha\beta)_\Phi = \sum_{\phi\in\mathrm{Diff}(\mathbb{R})} \alpha_\phi\cdot\big(\beta_{\Phi\circ\phi^{-1}}\circ\phi\big) \in C^\infty(\mathbb{R})$.

Definition 6.19 (The category of displacement and pointwise superposition $\mathcal{PQ}$)
Denote by $\mathcal{PQ}$ the "category of displacement and pointwise superposition" consisting of the following data:

0) The objects (I,V,ρ) are ordered tuples $\big([I_i,V_i,\rho_i]\big)_{i=1,\dots,m}$
with open intervals $I_i\subset\mathbb{R}$ and $V_i\,,\rho_i\in C^\infty\big(I_i,\mathbb{R}\setminus\{0\}\big)$

1) Morphisms $(J,W,\theta)\longrightarrow(I,V,\rho)$ are matrices $(\alpha_{ij})_{i=1,\dots,m}^{j=1,\dots,n}$ with entries $\alpha_{ij} = \sum_{\phi\in\mathrm{Diff}(\mathbb{R})}\alpha_{ij}^\phi \in\mathcal{R}_{\mathcal{PQ}}$ satisfying the following three constraints:

- $I_i\cap\overline{\big\{\alpha_{ij}^\phi\neq 0\big\}}\subset\phi^{-1}\big(J_j\big)$
- $\alpha_{ij}^\phi\in\mathcal{R}(I_i,V_i)$
- there exist open sets S_{ij}^ϕ with $I_i\cap\overline{\big\{\alpha_{ij}^\phi\neq 0\big\}}\subset S_{ij}^\phi\subset I_i\cap\phi^{-1}\big(J_j\big)$
so that $V_i/\phi^*W_j\in\mathcal{R}(S_{ij}^\phi,V_i)$ and $\rho_i/\phi^*\theta_j\big|_{S_{ij}^\phi}$ is bounded

Reminder. Under diffeomorphisms $\phi: x\longmapsto y$ we impose the transformation behaviour

$$\phi^*V := \frac{1}{\partial y/\partial x}V\circ\phi \qquad \phi^*\rho := \left|\frac{\partial y}{\partial x}\right|^{1/2}\rho\circ\phi \qquad \text{so } \rho \text{ plays the role of } g^{1/4}.$$

Convention. We choose $S_{ij}^\phi=\emptyset$ whenever $\alpha_{ij}^\phi = 0$. This is the case for all but finitely many $\phi\in\mathrm{Diff}(\mathbb{R})$.

Lemma 6.20 (Composition in $\mathcal{PQ}$)
Multiplication of matrices equips $\mathcal{PQ}$ with a well-defined composition.

Proof. Consider a pair of composable morphisms $(K, Z, \kappa) \xrightarrow{\beta} (J, W, \theta) \xrightarrow{\alpha} (I, V, \rho)$.
At fixed i, j, k and fixed $\phi, \varphi \in \mathrm{Diff}(\mathbb{R})$ bullet point 3 of the definition yields open sets S_{ij}^{ϕ} and T_{jk}^{φ} with

$$I_i \cap \overline{\{\alpha_{ij}^{\phi} \neq 0\}} \subset S_{ij}^{\phi} \subset I_i \cap \phi^{-1}\big(J_j\big) \qquad \text{and} \qquad J_j \cap \overline{\{\beta_{jk}^{\varphi} \neq 0\}} \subset T_{jk}^{\varphi} \subset J_j \cap \varphi^{-1}\big(K_k\big)$$

By applying $\phi^{-1}(\cdot)$ the second identity can be rewritten as

$$\phi^{-1}\big(J_j\big) \cap \overline{\{\beta_{jk}^{\varphi} \circ \phi \neq 0\}} \ \subset \ \phi^{-1}\big(T_{jk}^{\varphi}\big) \ \subset \ \phi^{-1}\big(J_j\big) \cap (\varphi \circ \phi)^{-1}\big(K_k\big)$$

Thus, one has

$$I_i \cap \overline{\{\alpha_{ij}^{\phi} \cdot (\beta_{jk}^{\varphi} \circ \phi) \neq 0\}} \ = \ \underbrace{I_i \cap \overline{\{\alpha_{ij}^{\phi} \neq 0\}}}_{\subset \phi^{-1}(J_j)} \cap \overline{\{\beta_{jk}^{\varphi} \circ \phi \neq 0\}} \ \subset \ S_{ij}^{\phi} \cap \phi^{-1}\big(T_{jk}^{\varphi}\big) \ \subset \ I_i \cap \phi^{-1}\big(J_j\big) \cap (\varphi \circ \phi)^{-1}\big(K_k\big)$$

and choosing $R_{ik}^{\Phi} = \bigcup_{j,\phi} S_{ij}^{\phi} \cap \phi^{-1}\big(T_{jk}^{\Phi \circ \phi^{-1}}\big)$ we get

$$I_i \cap \overline{\Big\{ \sum_{j,\phi} \alpha_{ij}^{\phi} \cdot \big(\beta_{jk}^{\Phi \circ \phi^{-1}} \circ \phi\big) \neq 0 \Big\}} \ \subset \ \bigcup_{j,\phi} I_i \cap \overline{\{\alpha_{ij}^{\phi} \cdot (\beta_{jk}^{\varphi} \circ \phi) \neq 0\}} \ \subset \ R_{ik}^{\Phi} \ \subset \ I_i \cap \Phi^{-1}\big(K_k\big)$$

Lemma 6.10 with "f" $= \frac{V_i}{\phi^* W_j} \in \mathcal{R}(S_{ij}^{\phi}, V_i)$ ensures $\mathcal{R}(L, \phi^* W_j) \subset \mathcal{R}(L, V_i)$ for all open subsets $L \subset S_{ij}^{\phi}$. This has the following consequences:

- $\beta_{ij}^{\varphi} \circ \phi \in \mathcal{R}\big(\phi^{-1}(J_j), \phi^* W_j\big)$ restricts to $\beta_{ij}^{\varphi} \circ \phi \in \mathcal{R}\big(S_{ij}^{\phi}, V_i\big)$,
  so the product $\alpha_{ij}^{\phi} \cdot (\beta_{ij}^{\varphi} \circ \phi)$ belongs to $\mathcal{R}\big(S_{ij}^{\phi}, V_i\big)$.
  Since α_{ij}^{ϕ} vanishes on an open neighbourhood of $I_i \cap \partial S_{ij}^{\phi}$, extension by zero shows $\alpha_{ij}^{\phi} \cdot (\beta_{jk}^{\varphi} \circ \phi) \in \mathcal{R}(I_i, V_i)$ and therefore $\big(\alpha\beta\big)_{ik}^{\Phi} = \sum_{j,\phi} \alpha_{ij}^{\phi} \cdot (\beta_{jk}^{\Phi\phi^{-1}} \circ \phi) \in \mathcal{R}(I_i, V_i)$.
- $V_i / \Phi^* Z_k = V_i / \phi^* W_j \cdot \phi^* W_j / \Phi^* Z_k$ belongs to $\mathcal{R}\big(S_{ij}^{\phi} \cap \phi^{-1}(T_{jk}^{\varphi}), V_i\big)$
  because we have $V_i / \phi^* W_j \in \mathcal{R}(S_{ij}^{\phi}, V_i)$ and by combining Lemma 8.6 with Remark 8.10 $W_j / \varphi^* Z_k \in \mathcal{R}(T_{jk}^{\varphi}, W_j)$ gets mapped to $\phi^* W_j / \Phi^* Z_k = \big(W_j / \varphi^* Z_k\big) \circ \phi \in \mathcal{R}\big(\phi^{-1}(T_{jk}^{\varphi}), \phi^* W_j\big)$.

Since $R_{ik}^{\Phi} = \bigcup_{j,\phi} S_{ij}^{\phi} \cap \phi^{-1}\big(T_{jk}^{\Phi \circ \phi^{-1}}\big)$ is a finite cover, we conclude that $V_i / \Phi^* Z_k \in \mathcal{R}(R_{ik}^{\Phi}, V_i)$.
Similarly $\rho_i / \Phi^* \kappa_k = \rho_i / \phi^* \theta_j \cdot \big(\theta_j / \varphi^* \kappa_k\big) \circ \phi$ is bounded on each component $S_{ij}^{\phi} \cap \phi^{-1}\big(T_{jk}^{\varphi}\big)$ and therefore bounded on the entire R_{ik}^{Φ}. $\square$

Lemma 6.21 (Constructing a family of functors $\mathcal{F} : \mathcal{PQ} \longrightarrow$ Banach spaces)
As before, denote by $\mathcal{B}$ the category of Banach spaces and bounded linear maps. For every $n \geq 0$ there exists a well-defined functor $\mathcal{F}_n : \mathcal{PQ} \longrightarrow \mathcal{B}$ such that

- $\big([I_i, V_i, \rho_i]\big)_{i=1,\dots,m}$ *gets mapped to* $\displaystyle\bigoplus_{i=1,\dots,m} W^{n,2}_{V_i\,;\,\rho_i}(I_i)$
- *every morphism* $(J, W, \theta) \xrightarrow{\alpha} (I, V, \rho)$ *translates into a bounded linear map*

$$\mathcal{F}_n(\alpha) : \bigoplus_j W^{n,2}_{W_j\,;\,\theta_j}(J_j) \longrightarrow \bigoplus_i W^{n,2}_{V_i\,;\,\rho_i}(I_i)$$

sending $u_j \in W^{n,2}_{W_j\,;\,\theta_j}(J_j)$ *to* $\sum_{j,\phi} \alpha^{\phi}_{ij}\, \phi^* u_j \in W^{n,2}_{V_i\,;\,\rho_i}(I_i)$.

Proof. Well-definedness of the map $\mathrm{Mor}(\mathcal{PQ}) \longrightarrow$ [bounded linear maps] can be seen by adapting the proof of Lemma 6.17.

Now let $(K, Z, \kappa) \xrightarrow{\beta} (J, W, \theta) \xrightarrow{\alpha} (I, V, \rho)$ be consecutive morphisms in $\mathcal{PQ}$.
With the same notation as in the proof of Lemma 6.20 we consider the following commutative diagram in $\mathcal{B}$:

$$\begin{tikzcd}
W^{n,2}_{Z_k;\kappa_k}\big(K_k\big) \arrow[r, "\varphi^*"] \arrow[rr, bend left, "\Phi^*"] \arrow[dr, dashed] \arrow[ddrr, dashed, bend right] & W^{n,2}_{\varphi^* Z_k;\varphi^*\kappa_k}\big(\varphi^{-1}(K_k)\big) \arrow[r, "\phi^*"] \arrow[d, "\iota_{W_j/\varphi^* Z_k} \circ \mathrm{res}"] & W^{n,2}_{\Phi^* Z_k;\Phi^*\kappa_k}\big(\Phi^{-1}(K_k)\big) \arrow[d, "\iota_{\phi^* W_j/\Phi^* Z_k} \circ \mathrm{res}"] \arrow[dd, bend left, "\iota_{V_i/\Phi^* Z_k} \circ \mathrm{res}"] \\
 & W^{n,2}_{W_j;\theta_j}\big(T^{\varphi}_{jk}\big) \arrow[r, "\phi^*"] \arrow[dr, dashed] & W^{n,2}_{\phi^* W_j;\phi^*\theta_j}\big(\phi^{-1}(T^{\varphi}_{jk})\big) \arrow[d, "\iota_{V_i/\phi^* W_j} \circ \mathrm{res}"] \\
 & & W^{n,2}_{V_i;\rho_i}\big(S^{\phi}_{ij} \cap \phi^{-1}(T^{\varphi}_{jk})\big)
\end{tikzcd}$$

showing that

$$\iota_{V_i/\Phi^* Z_k} \circ \mathrm{res} \circ \Phi^* \;=\; \big(\iota_{V_i/\phi^* W_j} \circ \mathrm{res} \circ \phi^*\big) \circ \big(\iota_{W_j/\varphi^* Z_k} \circ \mathrm{res} \circ \varphi^*\big)$$

Using the ring morphisms

$$\mathcal{R}\big(S^{\phi}_{ij} \cap \phi^{-1}(T^{\varphi}_{jk}), V_i\big) \xleftarrow{\text{incl.}} \mathcal{R}\big(S^{\phi}_{ij} \cap \phi^{-1}(T^{\varphi}_{jk}), \phi^* W_j\big) \qquad \mathcal{R}\big(\phi^{-1}(T^{\varphi}_{jk}), \phi^* W_j\big) \xleftarrow{\quad \phi^* \quad} \mathcal{R}\big(T^{\varphi}_{jk}, W_j\big) \xleftarrow{\quad \mathrm{res} \quad} \mathcal{R}\big(J_j, W_j\big) \ni \beta^{\varphi}_{jk}$$

$$\mathcal{R}\big(S^{\phi}_{ij} \cap \phi^{-1}(T^{\varphi}_{jk}), \phi^* W_j\big) \xleftarrow{\mathrm{res}} \mathcal{R}\big(\phi^{-1}(T^{\varphi}_{jk}), \phi^* W_j\big)$$

to restrict scalars, we can regard the maps $\iota_{V_i/\phi^* W_j}$, res, ϕ^* as morphisms of $\mathcal{R}(J_j, W_j)$-modules, meaning that they commute with $\mathrm{mult}_{\beta^{\varphi}_{jk}}$.
Thus, multiplying by $\beta^{\varphi}_{jk} \circ \phi$ we verify

$$\mathrm{mult}_{\beta^{\varphi}_{jk} \circ \phi} \circ \iota_{V_i/\Phi^* Z_k} \circ \mathrm{res} \circ \Phi^* \;=\; \big(\iota_{V_i/\phi^* W_j} \circ \mathrm{res} \circ \phi^*\big) \circ \mathrm{mult}_{\beta^{\varphi}_{jk}} \circ \big(\iota_{W_j/\varphi^* Z_k} \circ \mathrm{res} \circ \varphi^*\big)$$

When applied to $u_k \in W^{n,2}_{Z_k;\kappa_k}\big(K_k\big)$ this becomes

$$\big(\beta^{\varphi}_{jk} \circ \phi\big) \cdot \Phi^* u_k \;=\; \phi^*\big[\, \beta^{\varphi}_{jk} \cdot \varphi^* u_k \,\big] \;=\; \phi^*\big[\, (\beta^{\varphi}_{jk}|\varphi) u_k \,\big]$$

so calculating in the $\mathcal{R}(S^{\phi}_{ij} \cap \phi^{-1}(T^{\varphi}_{jk}), V_i)$-module $W^{n,2}_{V_i;\rho_i}\big(S^{\phi}_{ij} \cap \phi^{-1}(T^{\varphi}_{jk})\big)$ we have

$$\big[\, (\alpha^{\phi}_{ij}|\phi) \cdot (\beta^{\varphi}_{jk}|\varphi) \,\big] u_k \;=\; \big[\alpha^{\phi}_{ij} \cdot (\beta^{\varphi}_{jk} \circ \phi)\big] \cdot \Phi^* u_k \;=\; \alpha^{\phi}_{ij} \cdot \phi^*\big[\, (\beta^{\varphi}_{jk}|\varphi) u_k \,\big] \;=\; (\alpha^{\phi}_{ij}|\phi)\big[\, (\beta^{\varphi}_{jk}|\varphi) u_k \,\big]$$

By linearity we obtain

$$[\alpha\beta]_{ik} u_k = \sum_{j,\phi,\varphi} \big[\, (\alpha^{\phi}_{ij}|\phi) \cdot (\beta^{\varphi}_{jk}|\varphi) \,\big] u_k = \sum_j \underbrace{\textstyle\sum_\phi (\alpha^{\phi}_{ij}|\phi)}_{\alpha_{ij}} \Big[\underbrace{\textstyle\sum_\varphi (\beta^{\varphi}_{jk}|\varphi)}_{\beta_{jk}}\, u_k\Big]$$

Hence the multiplication on $\mathcal{R}_{\mathcal{PQ}}$ is defined in such a way to ensure $\mathcal{F}_n(\alpha \circ \beta) = \mathcal{F}_n(\alpha) \circ \mathcal{F}_n(\beta)$. ∎

Chapter 7

Construction of the Crossover Retraction

7.1 Geometric Setup

Though this will not be used in subsequent sections, we start our discussion by describing a distinguished Morse function on a pair-of-pants worldsheet $\Sigma = \mathbb{C}P^1 \setminus \{\pm 1, \infty\}$, allowing us to parametrize all level sets in a uniform way. This will become relevant as soon as one tries to pullback geometric data from Σ to the M-polyfold $\mathrm{im}(r_\Sigma)$ that we seek to construct.

As a first attempt to write down an explicit Morse function on $\Sigma = \mathbb{C} \setminus \{\pm 1\}$, the author considered the absolute value of

$$f(z) = \frac{1}{1-z} + \frac{1}{1+z}$$

but then realized that

$$\frac{1}{1-z} + \frac{1}{1+z} = \frac{2}{1-z^2}$$

is the $(w = z^2)$-pullback of a simpler function $\frac{2}{1-w}$ defined on the cylinder $\mathbb{C} \setminus \{+1\}$.

This inspires the following general procedure:

Proposition 7.1 (Recipe for obtaining Morse functions on a Riemann surface)
Let $\overline{\Sigma}$ be a compact Riemann surface and assume we are given a non-constant holomorphic function $f : \overline{\Sigma} \longrightarrow \mathbb{C}P^1$ such that f has no branch points $p \in \mathbb{C}P^1$ of order ≥ 3 except possibly 0 and ∞. From these ingredients we obtain the following structure:

1) *$\nu = \log|f| : \Sigma \longrightarrow \mathbb{R}$ is a Morse function on $\Sigma := \overline{\Sigma} - f^{-1}(0) - f^{-1}(\infty)$ whose critical points are exactly the ramification points of $f|_\Sigma$. All of these have Morse index 1.*
2) *Every non-critical $p \in \Sigma$ admits a neighbourhood U_p such that $\kappa = \log f : U_p \longrightarrow \mathbb{C}$ is the unique holomorphic chart with $p \mapsto \kappa(p)$ and $Re\,\kappa = \nu$.*
3) *Writing $\kappa = \nu + i\omega$ one has globally defined vector fields ∂_ν, ∂_ω on $\Sigma \setminus \{crit.\ pts.\}$ With z a local coordinate and $g(\partial_z, \partial_z) = g(\partial_{\overline{z}}, \partial_{\overline{z}}) = 0$, $g(\partial_z, \partial_{\overline{z}}) = c(z, \overline{z})$ any hermitean metric on Σ we have*

$$g(\partial_\nu, \partial_\omega) = 0$$
$$g(\partial_\nu, \partial_\nu) = g(\partial_\omega, \partial_\omega) = \frac{2c(z,\overline{z})|f|^2}{|\partial f|^2}$$

so the vector fields $\partial_\nu, \partial_\omega$ are always orthogonal and their norms blow up at critical points.

Proof. **1)** Since we are working on $\Sigma := \overline{\Sigma} - f^{-1}(\{0,\infty\})$, the identities

$$2\,\partial_z \log|f| = \partial_z \log|f|^2 = \partial_z f/f$$
$$2\,\partial_{\overline{z}} \log|f| = \partial_{\overline{z}} \log|f|^2 = \partial_{\overline{z}}\overline{f}/\overline{f}$$

show that $\qquad \mathrm{d}\nu = 0 \iff \partial_z f = 0$

Thus, critical points of ν correspond to ramification points of $f|_\Sigma$. Since by assumption all of these ramification points have order 2, we can find coordinates $z = x + iy$ such that in the vicinity of a critical point $p \in \Sigma$ we get an exact identity

$$f(z) = f(p)(1 + z^2)$$

For small $|z|$ we approximate

$$\nu(z) - \nu(p) = \log\left|1 + z^2\right| \approx \left|1 + z^2\right| - 1 \approx \operatorname{Re} z^2 = x^2 - y^2$$

which exhibits p as a saddle point of ν.
2) At a non-critical $p \in \Sigma$ we have $\partial_z f(p) \neq 0$ and therefore $\partial_z \kappa(p) = \partial_z f/f\,(p) \neq 0$, so by the Inverse Function Theorem there exist neighbourhoods $U_p \subset \Sigma$, $V_p \subset \mathbb{C}$ such that $\kappa : U_p \longrightarrow V_p$ is a diffeomorphism whose inverse is automatically holomorphic. It so happens that $\operatorname{Re}\kappa = \operatorname{Re}\log f = \log|f| = \nu$ and by Auxiliary Lemma 7.2 we observe that (after suitably shrinking U_p) $\kappa : U_p \longrightarrow \mathbb{C}$ is the *unique* holomorphic map sending p to $\kappa(p)$ with this property.
3) Under a holomorphic coordinate change $\kappa \mapsto z$ we have $\partial_z = \partial_z\kappa \cdot \partial_\kappa = \partial_z f/f \cdot \partial_\kappa$ and thus

$$\partial_\nu = \;\; 2\operatorname{Re}\partial_\kappa \;=\; \frac{f}{\partial_z f}\partial_z + \text{c.c}$$
$$\partial_\omega = -2\operatorname{Im}\partial_\kappa = i\left[\frac{f}{\partial_z f}\partial_z - \text{c.c}\right]$$

This can be understood as expressing the *global* vector fields ∂_ν, ∂_ω in any coordinate z on Σ. Note the appearance of a first order pole singularity $\frac{1}{\partial_z f} \sim \frac{1}{z-p}$ at critical points $p \in \Sigma$. $\square$

Auxiliary Lemma 7.2 (Rigidity of holomorphic maps with prescribed real part)
Let $h : \nu + i\omega \longrightarrow \tilde{\nu} + i\tilde{\omega}$ be a holomorphic map defined on an open ball $B_\epsilon(0) \subset \mathbb{C}$ and assume that $\tilde{\nu}(\nu,\omega) = \nu$ as well as $h(0) = 0$.
Then h is the identity.

Proof. This is an application of the Cauchy-Riemann equations:
Since $\frac{\partial\tilde{\omega}}{\partial\nu} = -\frac{\partial\tilde{\nu}}{\partial\omega} = 0$ implies that $\tilde{\omega}(\nu,\omega) = \tilde{\omega}(\omega)$ does not depend on ν,
we can use $\frac{\partial\tilde{\omega}}{\partial\omega} = \frac{\partial\tilde{\nu}}{\partial\nu} = 1$ to conclude that $\tilde{\omega} = \omega + \text{const}$.
As we are working on a connected domain, our assumption $h(0) = 0$ guarantees that the constant offset has to vanish and therefore $\tilde{\omega} = \omega$ in addition to $\tilde{\nu} = \nu$. $\square$

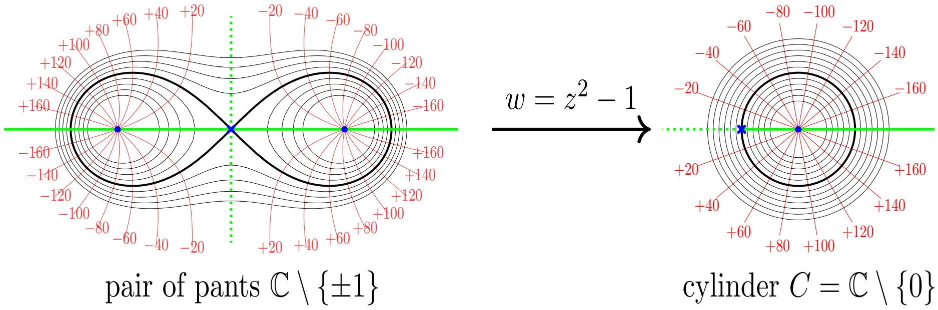


Figure 7.1: Illustration of Proposition 7.1 in the case of a pair-of-pants worldsheet. The holomorphic map

$$f : \overline{\Sigma} = \mathbb{C}P^1 \xrightarrow{w = z^2 - 1} \mathbb{C}P^1$$

restricts to a ramified cover

$$f : \Sigma = \mathbb{C}P^1 \setminus \{\pm 1, \infty\} \longrightarrow C := \mathbb{C}P^1 \setminus \{0, \infty\}$$

with ramification locus $\mathcal{R} = \{0\}$ and branch locus $\mathcal{B} = \{-1\}$.
Every injective path $\gamma : (0,1) \longrightarrow C \setminus \mathcal{B}$ has two non-intersecting lifts, so we can use the angular coordinate on C to parametrize the level sets of $|f|$.
In fact, using $\log f = \nu + i\omega$ as a *holomorphic* change of coordinates on $\Sigma \setminus \mathcal{R}$, the level sets are locally presented as $\nu =$ const. with a common parametrization by the angle ω.

7.2 Modelling Morse critical points by Dynamical Gluing

In this section, we provide an explicit formula for the 'crossover retraction', while postponing the proof of its sc-smoothness to our main effort in Chapter 8. Let us start with the most basic building block, the 'dynamical gluing' of two adjacent intervals $(\text{-}1, 0)$ and $(0, 1)$ on an overlap of size $2a$.

As mentioned in the introduction, we will replace the cutoff functions β and $1-\beta$ shown in Figure 7.2 by their normalized versions

$$\alpha := \frac{\beta}{\sqrt{\beta^2 + (1-\beta)^2}} \qquad \text{and} \qquad \gamma := \frac{1-\beta}{\sqrt{\beta^2 + (1-\beta)^2}}$$

with $\operatorname{supp}(\alpha) = \operatorname{supp}(\beta) = (\text{-}\infty, \frac{1}{2})$ and $\operatorname{supp}(\gamma) = \operatorname{supp}(1-\beta) = (\text{-}\frac{1}{2}, \infty)$.
Observe that α^2 and $\gamma^2 = 1 - \alpha^2$ have the same shape as β and $1-\beta$ respectively, whereas $\alpha\gamma \in C_0^\infty\left(\text{-}\frac{1}{2}, \frac{1}{2}\right)$ is a bump.

To calculate the retraction, we work at fixed gluing parameter $a > 0$. While $(\text{-}1, 0)$ and $(0, 1)$ are equipped with weight factors $\rho = \frac{1}{|x|^k}$ and vector fields $V_\pm = \pm x\partial_x$, their overlap $(\text{-}a, a)$ will carry the vector field

$$V^a_{int} = R_{1/a}\beta \cdot \tau_{+a}V_+ + R_{1/a}[1-\beta] \cdot \tau_{-a}V_-$$

and weight factor

$$\rho^a_{int} = R_{1/a}\beta \cdot \tau_{+a}\rho \, + \, R_{1/a}[1-\beta] \cdot \tau_{-a}\rho$$

which, as shown in Figure 7.2, interpolate between the data coming from $(\text{-}1, 0)$ and $(0, 1)$.

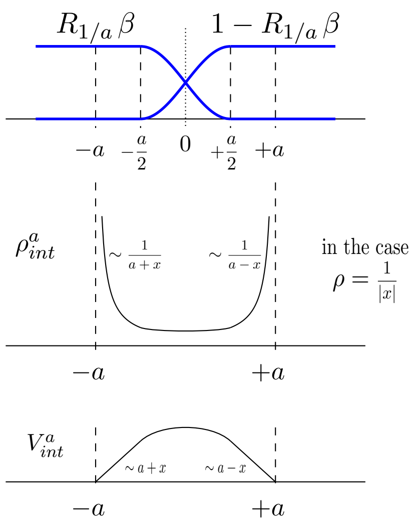


Figure 7.2: Data associated with the overlap $(\text{-}a, a)$. The cutoff functions $R_{1/a}\beta$ and $1 - R_{1/a}\beta$ can be used to interpolate between Sobolev functions from $(\text{-}1 + a, a)$ and $(\text{-}a, 1\text{-}a)$. Similarly, V^a_{int} and ρ^a_{int} are interpolated versions of the vector field and weight function, here shown in the case $V_\pm = \pm x\partial_x$ and $\rho = \frac{1}{|x|}$

The functor $\mathcal{F}_n : \mathcal{PQ} \longrightarrow \mathcal{B}$ from section 6.3.2 allows us to perform our calculations by multiplying matrices over the ring

$$\mathcal{R}_{\mathcal{PQ}} = \bigoplus_{\phi \in \mathrm{Diff}(\mathbb{R})} C^\infty(\mathbb{R}) \cdot \phi$$

while making sure that these act between the correct Sobolev spaces:

Proposition 7.3 (Gluing adjacent intervals $(-1,0)$ and $(0,1)$)

- *At fixed $a > 0$ there are mutually inverse isomorphisms of Banach spaces*

$$W^{n,2}_{\partial;1}(-1+a,1-a) \oplus W^{n,2}_{V^a_{int};\rho^a_{int}}(-a,a) \underset{\mathcal{F}_n \underbrace{\begin{bmatrix} R_{1/a}\alpha & -R_{1/a}\gamma \\ +R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}}_{R_{1/a}B}}{\overset{\mathcal{F}_n \overbrace{\begin{bmatrix} R_{1/a}\alpha & +R_{1/a}\gamma \\ -R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}}^{R_{1/a}B^T}}{\rightleftarrows}} W^{n,2}_{\tau_{-a}V_-;\tau_{-a}\rho}(-1+a,a) \oplus W^{n,2}_{\tau_{+a}V_+;\tau_{+a}\rho}(-a,1-a) \underset{\mathcal{F}_n \underbrace{\begin{bmatrix} \tau_{+a} & \\ & \tau_{-a} \end{bmatrix}}_{T^{-1}}}{\overset{\mathcal{F}_n \overbrace{\begin{bmatrix} \tau_{-a} & \\ & \tau_{+a} \end{bmatrix}}^{T}}{\rightleftarrows}} W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1)$$

- *Let us "insert" the projector* $P_\partial := \mathcal{F}_n \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \circlearrowright W^{n,2}_{\partial;1}(-1+a,1-a) \oplus W^{n,2}_{V^a_{int};\rho^a_{int}}(-a,a)$.

  When restricted to the image of the retraction

$$r_a = \mathcal{F}_n\Big[T^{-1} \cdot R_{1/a}B \cdot \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \cdot R_{1/a}B^T \cdot T\Big] = \begin{bmatrix} R_{1/a}(\tau_{+1}\alpha^2)\ id & R_{1/a}(\tau_{+1}\alpha\gamma)\ \tau_{+2a} \\ R_{1/a}(\tau_{-1}\alpha\gamma)\ \tau_{-2a} & R_{1/a}(\tau_{-1}\gamma^2)\ id \end{bmatrix} \circlearrowright W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1)\,,$$

  the gluing map

$$\mathcal{G}_a : im(r_a) \subset W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1) \xrightarrow{\big[R_{1/a}\alpha\,\tau_{-a}\ \ R_{1/a}\gamma\,\tau_{+a}\big]} W^{n,2}_{\partial;1}(-1+a,1-a) \xrightarrow{R_{1-a}} W^{n,2}_{\partial;1}(-1,1)$$

  is an isomorphism of Banach spaces.

Proof. The isomorphism

$$W^{n,2}_{\tau_{-a}V_-;\tau_{-a}\rho}(-1+a,a) \oplus W^{n,2}_{\tau_{+a}V_+;\tau_{+a}\rho}(-a,1-a) \underset{\begin{bmatrix} \tau_{+a} & \\ & \tau_{-a} \end{bmatrix}}{\overset{\begin{bmatrix} \tau_{-a} & \\ & \tau_{+a} \end{bmatrix}}{\rightleftarrows}} W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1)$$

is a direct result of Proposition 6.2.
On the other hand, note that $B^T = \begin{bmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{bmatrix}$ belongs to $O\big(2, C^\infty(\mathbb{R})\big) \subset O\big(2, \mathcal{R}_{PQ}\big)$,
so after applying the ring morphism $R_{1/a} = (1|R_{1/a}) \circ \_ \circ (1|R_{1/a})^{-1} : \mathcal{R}_{PQ} \longrightarrow \mathcal{R}_{PQ}$,

$$R_{1/a}B^T = \begin{bmatrix} R_{1/a}\alpha & R_{1/a}\gamma \\ -R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix} \quad \text{and} \quad R_{1/a}B = \begin{bmatrix} R_{1/a}\alpha & -R_{1/a}\gamma \\ R_{1/a}\gamma & R_{1/a}\alpha \end{bmatrix}$$

are still mutually inverse matrices over the ring $C^\infty(\mathbb{R}) \subset \mathcal{R}_{PQ}$.
The choice of V^a_{int} and ρ^a_{int} is such that $R_{1/a}B$ and $R_{1/a}B^T$ can be understood as $\mathcal{P}$-morphisms between $\Big([(\text{-}1{+}a, a), \tau_{\text{-}a}V_-, \tau_{\text{-}a}\rho], [(\text{-}a, 1\text{-}a), \tau_{+a}V_+, \tau_{+a}\rho]\Big)$ and $\Big([(\text{-}1{+}a, 1\text{-}a), 1, 1], [(\text{-}a, +a), V^a_{int}, \rho^a_{int}]\Big)$.
Thus, by applying the functor $\mathcal{F}_n$ of Lemma 6.17 we obtain an isomorphism

$$W^{n,2}_{\partial;1}(-1+a,1-a) \oplus W^{n,2}_{V^a_{int};\rho^a_{int}}(-a,a) \underset{\mathcal{F}_n[R_{1/a}B]}{\overset{\mathcal{F}_n[R_{1/a}B^T]}{\underset{\sim}{\rightleftarrows}}} W^{n,2}_{\tau_{-a}V_-;\tau_{-a}\rho}(-1+a,a) \oplus W^{n,2}_{\tau_{+a}V_+;\tau_{+a}\rho}(-a,1-a)$$

It remains to verify the explicit expression for our retraction $r \circlearrowright W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1)$.
According to Lemma 6.21 this can be done by multiplying matrices over the ring $\mathcal{R}_{\mathcal{PQ}} = \bigoplus_{\phi \in \mathrm{Diff}(\mathbb{R})} C^\infty(\mathbb{R})\phi$:

$$R_{1/a}B \underbrace{\begin{bmatrix}1 & \\ & 0\end{bmatrix}}_{R_{\frac{1}{a}}\begin{bmatrix}1 & \\ & 0\end{bmatrix}} R_{1/a}B^T \;=\; R_{1/a} \underbrace{\begin{bmatrix}\alpha & \text{-}\gamma \\ \gamma & \alpha\end{bmatrix}}_{B} \underbrace{\begin{bmatrix}1 & \\ & 0\end{bmatrix}}_{|\psi\rangle\langle\psi|} \underbrace{\begin{bmatrix}\alpha & \gamma \\ \text{-}\gamma & \alpha\end{bmatrix}}_{B^T} \;=\; R_{1/a} \underbrace{\begin{bmatrix}\alpha^2 & \alpha\gamma \\ \alpha\gamma & \gamma^2\end{bmatrix}}_{|B\psi\rangle\langle B\psi|} \;=\; \begin{bmatrix}R_{1/a}\,\alpha^2 & R_{1/a}\,\alpha\gamma \\ R_{1/a}\,\alpha\gamma & R_{1/a}\,\gamma^2\end{bmatrix}$$

and therefore $$r \;=\; \underbrace{\begin{bmatrix}\tau_{+a} & \\ & \tau_{\text{-}a}\end{bmatrix}}_{T^{-1}} \begin{bmatrix}R_{1/a}\,\alpha^2 & R_{1/a}\,\alpha\gamma \\ R_{1/a}\,\alpha\gamma & R_{1/a}\,\gamma^2\end{bmatrix} \underbrace{\begin{bmatrix}\tau_{\text{-}a} & \\ & \tau_{+a}\end{bmatrix}}_{T} \;=\; \begin{bmatrix}R_{1/a}(\tau_{+1}\alpha^2)\,\mathrm{id} & R_{1/a}(\tau_{+1}\alpha\gamma)\,\tau_{+2a} \\ R_{1/a}(\tau_{\text{-}1}\alpha\gamma)\,\tau_{\text{-}2a} & R_{1/a}(\tau_{\text{-}1}\gamma^2)\,\mathrm{id}\end{bmatrix}$$

where in the last step we have used that on $f \in C^\infty(\mathbb{R})$ one has $\tau_a R_\lambda(f) = R_\lambda \tau_{a\cdot\lambda}(f)$.
To see that the gluing map $\mathcal{G}_a$ is an isomorphism, observe that r is constructed as $r = C^{-1} \circ P_\partial \circ C$ with an invertible map

$$C = \mathcal{F}_n\big[R_{1/a}B^T \circ T\big] : \;\; W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1) \xrightarrow{\quad\sim\quad} W^{n,2}_{\partial;1}(-1+a, 1-a) \oplus W^{n,2}_{V^a_{int};\rho^a_{int}}(-a,a)$$

that identifies $\mathrm{im}(r) \subset W^{n,2}_{V_-;\rho}(-1,0) \oplus W^{n,2}_{V_+;\rho}(0,1)$ and $\mathrm{im}(P_\partial) \cong W^{n,2}_{\partial;1}(-1+a, 1-a)$. $\square$

Next, let us state our main result, which, by interweaving two copies of the retraction from Proposition 7.3, allows us to model the transition at a Morse critical point as a two-sided breaking process. To ensure that our Banach scales have compact inclusions, we will use increasing powers of our basic weight factor $\rho = |x|^{-1}$. As we shall see in the proof of Proposition 8.27, the weight difference between different levels of regularity will also be required to cancel pole divergences that would otherwise prevent sc-smoothness of the retraction at $a = 0$.

Theorem 7.4 (Sc-smooth retraction associated to a Morse critical point)
The 'crossover retraction' defined in Figure 7.3 is a fibre-linear sc$^\infty$*-map*

$$r_{Cross}: \; \underset{a}{(\text{-}\epsilon,\epsilon)} \oplus \underset{A,B}{\Big[W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0)\Big]^{\oplus 2}} \oplus \underset{C,D}{\Big[W^{n+1,2}_{V_+;\rho^n}(0,1)\Big]^{\oplus 2}} \longrightarrow \underset{A,B}{\Big[W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0)\Big]^{\oplus 2}} \oplus \underset{C,D}{\Big[W^{n+1,2}_{V_+;\rho^n}(0,1)\Big]^{\oplus 2}}$$

Proof. Combine Propositions 7.5 and 7.6 below. $\square$

Proposition 7.5 (Sc-Smoothness of the off-diagonal terms)
The prescription

$$(a,u) \longmapsto \begin{cases} R_{1/a}(\tau_{+1}\alpha\gamma)\,\tau_{2a}u & \text{if } a > 0 \\ 0 & \text{if } a \le 0 \end{cases}$$

defines a fibre-linear sc$^\infty$*-map* $\big(-\epsilon,\epsilon\big) \oplus W^{n+1,2}_{V_+;\rho^n}(0,1) \longrightarrow W^{n+1,2}_{V_-;\rho^n}(-1,0)$.

Proof. According to Proposition 8.27 our map satisfies all conditions of Theorem 8.26. $\square$

Proposition 7.6 (Sc-Smoothness of the diagonal terms)
The prescription

$$(a,v) \longmapsto \begin{cases} R_{1/|a|}(\tau_{+1}\alpha^2)\,v & \text{if } a \neq 0 \\ v & \text{if } a = 0 \end{cases}$$

defines a fibre-linear sc$^\infty$*-map* $\big(-\epsilon,\epsilon\big) \oplus W^{n+1,2}_{V_-;\rho^n}(-1,0) \longrightarrow W^{n+1,2}_{V_-;\rho^n}(-1,0)$.

Proof. Omitted, but similar to (and simpler than) the off-diagonal case. $\square$

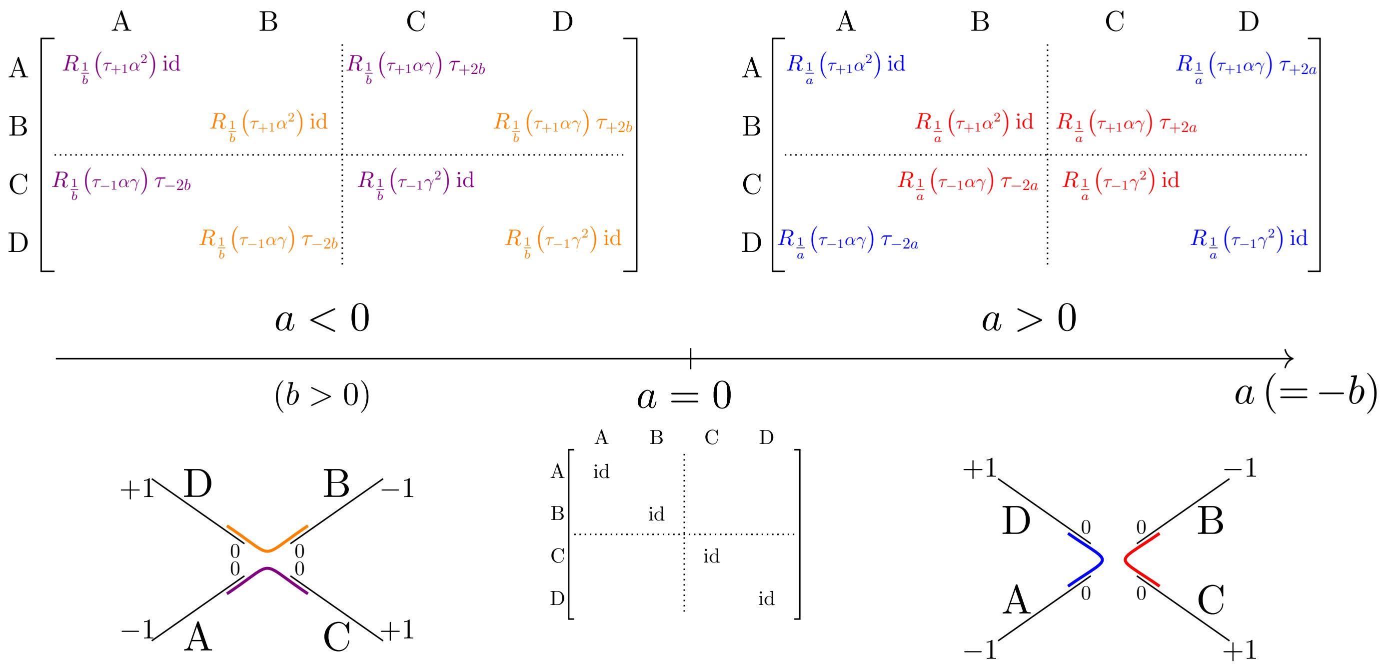


Figure 7.3: The 'crossover retraction' r_{cross} of Theorem 7.4

7.3 Transition to the global setting (Static Gluing)

To globalize our construction from the previous section, we have to connect the intervals A, D as well as B, C at a safe distance where the dynamical gluing process remains invisible. This can be achieved in a static, a-independent way by restricting each copy of

$$W^{n+1,2}_{V_-;\rho^n}(\text{-}1,0) \oplus W^{n+1,2}_{V_+;\rho^n}(0,1)$$

to the kernel of a bounded linear map called 'static gluing'.

Remark 7.7 (Static Gluing)
The combined 'static gluing'

$$\Big[\underset{\substack{\mathrm{A}\\\mathrm{B}}}{W^{n,2}_{V_-;\theta}\big(-1,0\big)} \oplus \underset{\substack{\mathrm{D}\\\mathrm{C}}}{W^{n,2}_{V_+;\theta}\big(0,1\big)} \oplus \underset{\substack{\mathrm{E}\\\mathrm{F}}}{W^{n,2}\Big(\text{-}\tfrac{1}{2},+\tfrac{1}{2}\Big)}\Big]^{\oplus 2} \xrightarrow{\ \mathcal{S}^{\oplus 2}\ } \Big[\underset{\substack{\mathrm{DE}\\\mathrm{CF}}}{W^{n,2}\Big(\text{-}\tfrac{1}{2},0\Big)} \oplus \underset{\substack{\mathrm{AE}\\\mathrm{BF}}}{W^{n,2}\Big(0,+\tfrac{1}{2}\Big)}\Big]^{\oplus 2}$$

given by the composition

$$\mathcal{S}^{\oplus 2} = \begin{array}{c|ccc:ccc}
 & \mathrm{A} & \mathrm{D} & \mathrm{E} & \mathrm{B} & \mathrm{C} & \mathrm{F} \\ \hline
\mathrm{DE} & & \tau_{+1} & -\mathrm{res}_{\left(\text{-}\frac{1}{2},0\right)} & & & \\
\mathrm{AE} & \tau_{-1} & & -\mathrm{res}_{\left(0,+\frac{1}{2}\right)} & & & \\ \hdashline
\mathrm{CF} & & & & & \tau_{+1} & -\mathrm{res}_{\left(\text{-}\frac{1}{2},0\right)} \\
\mathrm{BF} & & & & \tau_{-1} & & -\mathrm{res}_{\left(0,+\frac{1}{2}\right)}
\end{array}
\circ
\begin{array}{c|ccc:ccc}
 & \mathrm{A} & \mathrm{D} & \mathrm{E} & \mathrm{B} & \mathrm{C} & \mathrm{F} \\ \hline
\mathrm{A} & \mathrm{res}_{\left(-1,\text{-}\frac{1}{2}\right)} & & & & & \\
\mathrm{D} & & \mathrm{res}_{\left(+\frac{1}{2},+1\right)} & & & & \\
\mathrm{E} & & & 1 & & & \\ \hdashline
\mathrm{B} & & & & \mathrm{res}_{\left(-1,\text{-}\frac{1}{2}\right)} & & \\
\mathrm{C} & & & & & \mathrm{res}_{\left(+\frac{1}{2},+1\right)} & \\
\mathrm{F} & & & & & & 1
\end{array}$$

is a bounded linear map, so its kernel $\ker(\mathcal{S}^{\oplus 2}) = \ker(\mathcal{S})^{\oplus 2}$ is again a Banach space. Auxiliary Lemma 7.8 shows that, via the canonical projection, $\ker(\mathcal{S})$ can be identified with a closed subspace of $W^{n,2}_{V_-;\theta}\big(-1,0\big) \oplus W^{n,2}_{V_+;\theta}\big(0,1\big)$.

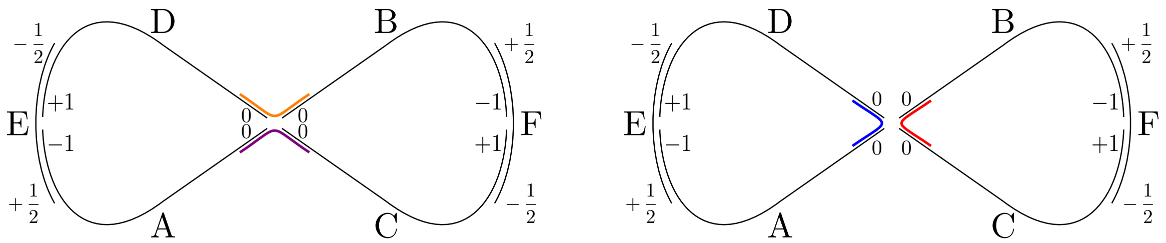


Figure 7.4: Globalized version of Figure 7.3, exhibiting the topologically distinct level sets at $a < 0$ and $a > 0$. The general recipe of Auxiliary Lemma 7.8 has been used to "plaster" the intervals A and D as well as B and C.

As already mentioned, our static gluing procedure makes use of the following general observation which allows us to discard the 'plaster' once gluing has been achieved.

Auxiliary Lemma 7.8 (Plastering of domain gaps)
Given Banach spaces X, Y, Z let $\gamma : X \oplus Y \longrightarrow Z$ be a bounded linear map such that $\|y\| \leq \|\gamma(0,y)\|$ for all $y \in Y$. Then the canonical projection $p : X \oplus Y \longrightarrow X$ restricts to a linear isomorphism

$$p : \ker\gamma \xrightarrow{\sim} p(\ker\gamma)$$

With $p(\ker\gamma) \subset X$ being a closed subspace, this becomes an isomorphism of Banach spaces.

Proof. Since γ is a bounded linear map, we can find a constant $C > 0$ such that

$$\|y\| \leq \|\gamma(0,y)\| \leq C \cdot \Big[\, \|x\| + \|\gamma(x,y)\| \,\Big]$$

For $(x,y) \in \ker\gamma$ this becomes

$$\|y\| \leq C \cdot \|x\| \tag{7.1}$$

which immediately implies that $p|_{\ker\gamma} : (x,y) \longmapsto x$ is injective.
To establish that $p(\ker\gamma) \subset X$ is a closed subspace, consider a convergent sequence $x_n \longrightarrow x \in X$ with $x_n \in p(\ker\gamma)$. Let $y_n \in Y$ be such that $(x_n, y_n) \in \ker\gamma$. Then (7.1) implies that y_n is a Cauchy sequence. By completeness of Y, the limit $(x_n, y_n) \longrightarrow (x,y)$ exists and since γ is continuous, we have $(x,y) \in \ker\gamma$ which shows that $x \in p(\ker\gamma)$. □

By using static gluing to constrain the fibres of our splicing core, we obtain the following globalized version of Theorem 7.4:

Theorem 7.9 (Globalization by restriction)
Denote by $W_n \subset W^{n+1,2}_{V_-;\rho^n}\big(-1,0\big) \oplus W^{n+1,2}_{V_+;\rho^n}\big(0,1\big)$ the closed subspace representing $\ker\mathcal{S}$.
Then r_{Cross} restricts to a sc-smooth splicing $r : (-\epsilon,\epsilon) \oplus W_n^{\oplus 2} \longrightarrow W_n^{\oplus 2}$.

Sketch of proof. Since our "static gluing" is implemented at a safe distance where deformations due to the dynamical gluing are invisible, the relations

$$\begin{array}{ll} \mathrm{res}_{\left(-1,-\frac{1}{2}\right)}\big(R_{1/a}\alpha^2\big) = 1 & \mathrm{res}_{\left(-1,-\frac{1}{2}\right)}\big(R_{1/a}\alpha\gamma\big) = 0 \\ \mathrm{res}_{\left(+\frac{1}{2},+1\right)}\big(R_{1/a}\alpha\gamma\big) = 0 & \mathrm{res}_{\left(+\frac{1}{2},+1\right)}\big(R_{1/a}\gamma^2\big) = 1 \end{array}$$

ensure that at every $a \in \mathbb{R}$ we have

$$\mathcal{S}^{\oplus 2} \circ \begin{bmatrix} r_{Cross}(a) & \\ & \mathrm{id}_{E,F} \end{bmatrix} = \mathcal{S}^{\oplus 2}$$

As a result, r_{Cross} maps $W_n^{\oplus 2}$ to $W_n^{\oplus 2}$.
It remains to show that $\big[W_n\big]_{n\in\mathbb{N}}$ indeed defines a sc-Banach space. This can be analysed by writing W_n as a space $W^{n+1,2}_{V;\rho^n}(-2,0)$ and applying the straightening diffeomorphism from Remark 6.3 to compare this space with ordinary Sobolev spaces. □

As a preparation for Proposition 7.11 below, we collect some immediate properties of the static and dynamical gluing maps:

Remark 7.10 (Compatibility between Dynamical and Static Gluing)
For simplicity we work at fixed $a > 0$. To treat the case $a < 0$ one has to put $(\cdot)^{\oplus 2}$ everywhere and modify the "combinatorics" of the maps.

a) When restricted to $\operatorname{im}\big[r_a \oplus \mathrm{id}\big] = \ker\big[(1-r_a)\oplus 0\big]$, the dynamical gluing

$$\bar{\mathcal{G}}_a:\ \Big[W^{n,2}_{V_-;\theta}(\text{-}1,0)\oplus W^{n,2}_{V_+;\theta}(0,1)\Big]\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right)\xrightarrow{\begin{bmatrix} R_{\frac{1}{a}}(\alpha)\cdot\tau_{\text{-}a} & R_{\frac{1}{a}}(\gamma)\cdot\tau_{+a} \\ & \mathrm{id}\end{bmatrix}} W^{n,2}(\text{-}1+a,1-a)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right)$$

becomes an isomorphism of Banach spaces.

b) The static gluings

$$\mathcal{S}:\ W^{n,2}_{V_-;\theta}(-1,0)\oplus W^{n,2}_{V_+;\theta}(0,1)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right)\xrightarrow{\begin{bmatrix} & \tau_{+1} & -1\\ \tau_{-1} & & -1\end{bmatrix}} W^{n,2}\left(\text{-}\tfrac{1}{2},0\right)\oplus W^{n,2}\left(0,\tfrac{1}{2}\right)$$

$$\mathcal{S}_a:\ W^{n,2}(\text{-}1+a,1-a)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right)\xrightarrow{\begin{bmatrix}\tau_{+1\text{-}a} & \text{-}1\\ \tau_{\text{-}1+a} & \text{-}1\end{bmatrix}} W^{n,2}\left(\text{-}\tfrac{1}{2},0\right)\oplus W^{n,2}\left(0,\tfrac{1}{2}\right)$$

are bounded linear operators between Banach spaces, so their kernels are Banach spaces as well. Auxiliary Lemma 7.8 shows that the canonical projection

$$W^{n,2}(\text{-}1+a,1-a)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right)\xrightarrow{\ p\ } W^{n,2}(\text{-}1+a,1-a)\xrightarrow[R_{1\text{-}a}]{\sim} W^{n,2}(\text{-}1,1)$$

restricts to an isomorphism of Banach spaces

$$\ker\mathcal{S}_a\xrightarrow{\ \sim\ } p\big(\ker\mathcal{S}_a\big)\xrightarrow[R_{1\text{-}a}]{\sim} W^{n,2}(S^1)\subset W^{n,2}(\text{-}1,1)$$

c) Compatibility between the dynamical gluing $\bar{\mathcal{G}}_a$ and the static gluings $\mathcal{S},\mathcal{S}_a$ is expressed by the commutative diagram

$$\begin{tikzcd}
 & W^{n,2}\left(\text{-}\tfrac{1}{2},0\right)\oplus W^{n,2}\left(0,\tfrac{1}{2}\right) & \\
W^{n,2}\left(\text{-}1,\text{-}\tfrac{1}{2}\right)\oplus W^{n,2}\left(\tfrac{1}{2},1\right)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right) \arrow[ur, dashed, "\begin{bmatrix} & \tau_{+1} & -1\\ \tau_{-1} & & -1\end{bmatrix}"] \arrow[rr, dashed, "\begin{bmatrix}\tau_{-a} & & \\ & \tau_{+a} & \\ & & 1\end{bmatrix}"] \arrow[u, phantom, "\text{(A)}"] & & W^{n,2}\left(\text{-}1+a,\text{-}\tfrac{1}{2}+a\right)\oplus W^{n,2}\left(\tfrac{1}{2}-a,1-a\right)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right) \arrow[ul, dashed, "\begin{bmatrix} & \tau_{1-a} & -1\\ \tau_{-1+a} & & -1\end{bmatrix}"'] \arrow[u, phantom, "\text{(A')}"] \\
W^{n,2}_{V_-;\theta}(-1,0)\oplus W^{n,2}_{V_+;\theta}(0,1)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right) \arrow[u, dashed, "\mathrm{res}"] \arrow[uur, bend left, "\mathcal{S}"] \arrow[rr, "\bar{\mathcal{G}}_a = \begin{bmatrix} R_{1/a}\alpha\cdot\tau_{-a} & R_{1/a}\gamma\cdot\tau_{+a}\\ & 1\end{bmatrix}"'] \arrow[urr, phantom, "\text{(B)}"] & & W^{n,2}(-1+a,1-a)\oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right) \arrow[u, dashed, "\mathrm{res}"'] \arrow[uul, bend right, "\mathcal{S}_a"']
\end{tikzcd}$$

The factorisations (A) and (A') can be seen as defining $\mathcal{S}$ and $\mathcal{S}_a$ respectively, whereas the commutative square (B) is due to

$$R_{1/a}\alpha = \begin{cases} 1 & \text{on } \left(\text{-}1+a,\text{-}\tfrac{1}{2}+a\right)\\ 0 & \text{on } \left(\tfrac{1}{2}-a,1-a\right)\end{cases}\qquad R_{1/a}\gamma = \begin{cases} 0 & \text{on } \left(\text{-}1+a,\text{-}\tfrac{1}{2}+a\right)\\ 1 & \text{on } \left(\tfrac{1}{2}-a,1-a\right)\end{cases}$$

Note that $\mathcal{S}_a\circ\bar{\mathcal{G}}_a = \mathcal{S}$ implies $\bar{\mathcal{G}}_a^{-1}\big(\ker\mathcal{S}_a\big) = \ker\mathcal{S}$.

After these preparations, we are ready to recover the fibre

$$r_{a>0}\big(W_n^{\oplus 2}\big) \;\cong\; \big[W^{n+1,2}(S^1)\big]^{\oplus 2}$$

of the splicing core constructed in Theorem 7.9:

Proposition 7.11 (Interpretation of the fibre at $a > 0$)
Static gluing constructs closed subspaces $\overline{W}_{n\text{-}1} \subset W^{n,2}_{V_-;\theta}(\text{-}1,0) \oplus W^{n,2}_{V_+;\theta}(0,1)$ *and* $W^{n,2}(S^1) \subset W^{n,2}(-1,1)$ *such that the dynamical gluing map*

$$\mathcal{G}_a : W^{n,2}_{V_-;\theta}(-1,0) \oplus W^{n,2}_{V_+;\theta}(0,1) \longrightarrow W^{n,2}(-1,1)$$

restricts to an isomorphism between $\overline{W}_{n\text{-}1}$ *and* $W^{n,2}(S^1)$*.*

Proof. Observations a-c from Remark 7.10 can be assembled into a commutative diagram

$$\begin{tikzcd}
& W^{n,2}\left(\text{-}\tfrac{1}{2},0\right) \oplus W^{n,2}\left(0,\tfrac{1}{2}\right) & & & \\
\Big[W^{n,2}_{V_-\,\theta}(-1,0) \oplus W^{n,2}_{V_+\,\theta}(0,1)\Big] \oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right) \arrow[ur, "\mathcal{S}"] \arrow[rr, "\bar{\mathcal{G}}_a"] & & W^{n,2}(\text{-}1+a,1-a) \oplus W^{n,2}\left(\text{-}\tfrac{1}{2},\tfrac{1}{2}\right) \arrow[ul, "\mathcal{S}_a"'] \arrow[r, "p"] & W^{n,2}(\text{-}1+a,1-a) \arrow[r, "\sim", "R_{1\text{-}a}"'] & W^{n,2}(-1,1) \\
\mathrm{im}(r_a \oplus \mathrm{id}) \arrow[u, phantom, sloped, "\subset"] \arrow[urr, "\sim"] & & \ker \mathcal{S}_a \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"] & p\big(\ker \mathcal{S}_a\big) \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"] & W^{n,2}(S^1) \arrow[u, phantom, sloped, "\subset"] \\
\begin{array}{c}\mathrm{im}(r_a \oplus \mathrm{id}) \cap \ker \mathcal{S} \\ = \bar{\mathcal{G}}_a^{\,-1}\big(\ker \mathcal{S}_a\big)\Big|_{\mathrm{im}(r_a \oplus \mathrm{id})}\end{array} \arrow[u, phantom, sloped, "\subset"] \arrow[urr, "\sim"] & & & &
\end{tikzcd}$$

Using Auxiliary Lemma 7.8 to identify $\mathrm{im}(r_a \oplus \mathrm{id}) \cap \ker \mathcal{S}$ with the closed subspace

$$\overline{W}_{n\text{-}1} := \mathrm{im}(r_a) \cap W_{n\text{-}1} \subset W^{n,2}_{V_-\,\theta}(-1,0) \oplus W^{n,2}_{V_+\,\theta}(0,1)$$

we observe that the gluing map $\mathcal{G}_a : W^{n,2}_{V_-\,\theta}(-1,0) \oplus W^{n,2}_{V_+\,\theta}(0,1) \longrightarrow W^{n,2}(-1,1)$ induces an isomorphism of Banach spaces between $\overline{W}_{n\text{-}1}$ and $W^{n,2}(S^1)$. $\square$

Chapter 8

Sc-smoothness of the retraction

8.1 Differentiation by the gluing parameter

In this final chapter we prove the sc-smoothness of our retraction r_{Cross} from Figure 7.3, focussing on its off-diagonal parts. As a first step, we calculate the a-derivatives of

$$R_{1/a}f \cdot \tau_{2a}u \in W^{1,2}_{V_-}(\text{-}1, 0)$$

with $u \in W^{n+1,2}_{V_+}(0,1)$ and a bump $f \in C^\infty_0\big(\text{-}\frac{3}{2}, \text{-}\frac{1}{2}\big)$. We give individual treatments for the 'shift' and 'rescaling' parts $\tau_{2a}u$ and $R_{1/a}f$ in sections 8.1.1 and 8.1.2 respectively, before combining these in section 8.1.3.

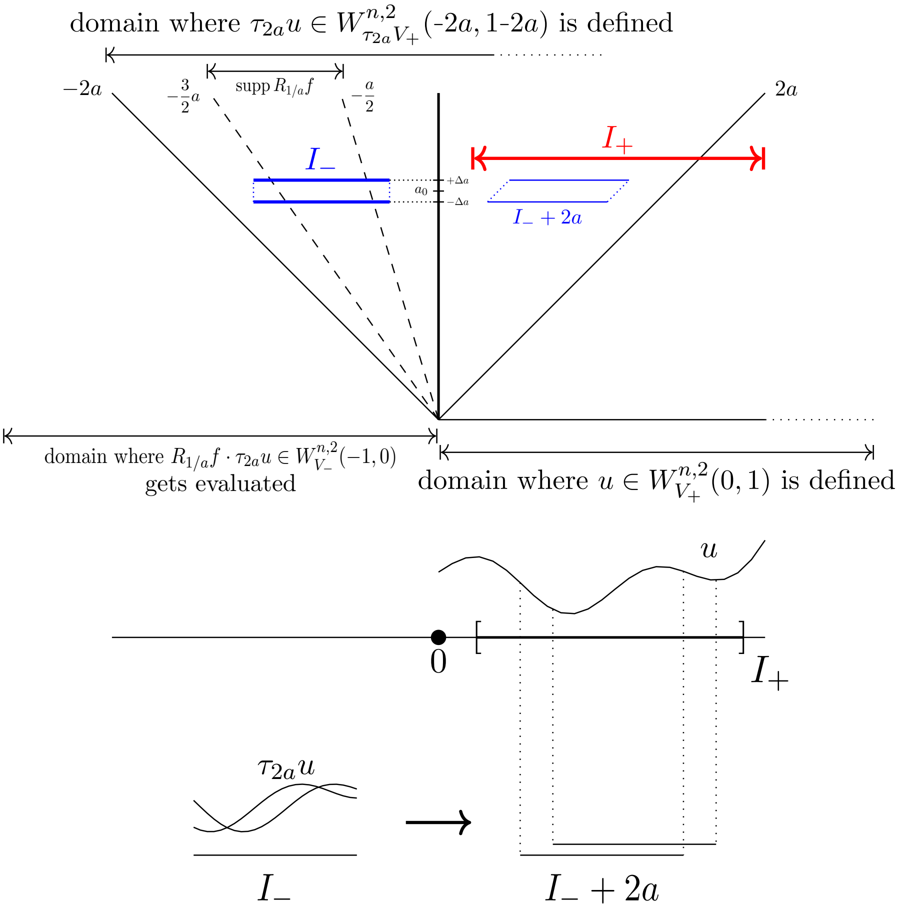


Figure 8.1: Illustration of the map $a \longmapsto R_{1/a}f \cdot \tau_{2a}u \in W^{n,2}_{V_-}(\text{-}1, 0)$ with $f \in C^\infty_0\big(\text{-}\frac{3}{2}, \text{-}\frac{1}{2}\big)$ and $u \in W^{n,2}_{V_+}(0,1)$. To analyse differentiability, we work on a sufficiently small neighbourhood $(a_0 - \Delta a, a_0 + \Delta a)$ where it is possible to treat our maps on a fixed domain I_- that due to its a_0-dependence will be called the 'comoving interval'.

8.1.1 Differentiation of the shift map

Let us work locally in the gluing parameter around a value $a_0 > 0$. According to Figure 8.1, we can find an a_0-dependent width Δa as well as a_0-dependent open intervals $I_\pm$ such that

$$\left(\text{-}\tfrac{3}{2}a, \text{-}\tfrac{a}{2}\right) \subset I_- \subset (\text{-}2a, 0) \quad \text{and} \quad I_- + 2a \subset I_+ \subset\subset (0,1)$$

for all $a \in (a_0 - \Delta a, a_0 + \Delta a)$. These conditions ensure that for any given $f \in C_0^\infty\left(\text{-}\frac{3}{2}, \text{-}\frac{1}{2}\right)$ and $u \in W^{2,2}(I_+)$, the Sobolev function $R_{1/a}f \cdot \tau_{2a}u$ will be compactly supported inside I_- as long as $|a - a_0| < \Delta a$. As illustrated in Figure 8.1, this allows us to differentiate

$$a \in (a_0 - \Delta a, a_0 + \Delta a) \longmapsto \tau_{2a}u \in W^{1,2}(I_-)$$

as a family of functions over a fixed domain I_-.
Our strategy will be to rewrite the difference quotient

$$\frac{\tau_{2(a+\delta a)}u - \tau_{2a}u}{\delta a}$$

as a $W^{1,2}(I_-)$-valued Bochner integral with continuous integrand.
In the following, it will be convenient to abbreviate $(a_0 \pm \Delta a) := (a_0 - \Delta a, a_0 + \Delta a)$.

Lemma 8.1 (Continuity of the shift map on L^2)
Given a fixed $u \in L^2(I_+)$, the map $a \in (a_0 \pm \Delta a) \longmapsto \tau_{2a}u \in L^2(I_-)$ is continuous.

Proof. Given $\epsilon > 0$ let us fix $\tilde{u} \in C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\|u - \tilde{u}\|_{L^2(I_+)} \leq \epsilon/4$.
Having $x \in I_-$ guarantees $x + 2a \in I_+$ for all $a \in (a_0 \pm \Delta a)$. Thus, going from a to $a+\delta a$ we get

$$\tilde{u}\big(x + 2(a + \delta a)\big) - \tilde{u}\big(x + 2a\big) \;=\; \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}\big(x + 2(a + t\delta a)\big) \;=\; 2\delta a \int_0^1 \mathrm{d}t\, \partial\tilde{u}\big|_{x+2(a+t\delta a)}$$

Taking the supremum over all $x \in I_-$ we observe that

$$\sup_{x \in I_-} \left\|\tau_{2(a+\delta a)}\tilde{u} - \tau_{2a}\tilde{u}\right\|_{\mathbb{B}} \;\leq\; |\delta a| \cdot \overbrace{2 \sup_{x \in I_+} \|\partial\tilde{u}\|_{\mathbb{B}}}^{K:=}$$

and therefore

$$\left\|\tau_{2(a+\delta a)}u - \tau_{2a}u\right\|_{L^2(I_-)} \;\leq\; \underbrace{\left\|u - \tilde{u}\right\|_{L^2(I_- + 2(a+\delta a))} + \left\|u - \tilde{u}\right\|_{L^2(I_- + 2a)}}_{\leq\, \epsilon/2} + \underbrace{\left\|\tau_{2(a+\delta a)}\tilde{u} - \tau_{2a}\tilde{u}\right\|_{L^2(I_-)}}_{\leq \sqrt{|I_-|}\cdot K\, |\delta a|}$$

□

Corollary 8.2 (Continuity of the shift map on $W^{n,2}$)
Given a fixed $u \in W^{n,2}(I_+)$, the map $a \in (a_0 \pm \Delta a) \longmapsto \tau_{2a}u \in W^{n,2}(I_-)$ is continuous.

Proof. In keeping with Proposition 6.2 the map

$$\tau_{2a} : (u_0, u_1, ..., u_n) \in \widehat{W}^{n,2}(I_+) \longrightarrow (\tau_{2a}u_0, \tau_{2a}u_1, ..., \tau_{2a}u_n) \in \widehat{W}^{n,2}(I_-)$$

acts componentwise, so our claim follows by componentwise application of Lemma 8.1. □

Lemma 8.3 (Difference quotient of the shift map as a $W^{1,2}$-valued Bochner integral)
Given $a_1, a_2 \in (a_0 \pm \Delta a)$ and a fixed $u \in W^{2,2}(I_+)$,
the formula $\tau_{2a_2}u - \tau_{2a_1}u \;=\; 2(a_2 - a_1) \int_0^1 dt\, \tau_{2a_1+2(a_2-a_1)t}\, \partial u$ *holds in $W^{1,2}(I_-)$.*

Proof. Given $u \in W^{2,2}(I_+)$ we have $\partial u \in W^{1,2}(I_+)$. By Corollary 8.2 the map $a \longmapsto \tau_{2a}\partial u \in W^{1,2}(I_-)$ is continuous, so $\int_0^1 \mathrm{d}t\, \tau_{2a_1+2(a_2-a_1)t}\, \partial u$ is a well-defined Bochner integral in $W^{1,2}(I_-)$.
With Sobolev embedding $W^{2,2}(I_+) \hookrightarrow C^1(I_+)$ we may assume that u is continuously differentiable. Thus, pointwise at a fixed $x \in I_-$ (corresponding to $x + 2a \in I_+$) the Fundamental Theorem of Calculus yields

$$u(x+2a_2) - u(x+2a_1) \;=\; \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} u\big(x + 2a_1 + 2(a_2-a_1)t\big) \;=\; 2(a_2-a_1) \int_0^1 \mathrm{d}t\, \tau_{2a_1+2(a_2-a_1)t}\, \partial u\,(x)$$

As a bounded linear map $ev_x : W^{1,2}(I_-) \hookrightarrow C^0(I_-) \longrightarrow \mathbb{B}$ commutes with the Bochner integral, so we get

$$ev_x\Big[\tau_{2a_2}u \;-\; \tau_{2a_1}u \;-\; 2(a_2-a_1)\int_0^1 \mathrm{d}t\, \tau_{2a_1+2(a_2-a_1)t}\, \partial u\Big] = 0 \qquad \text{for all } x \in I_-$$

□

Corollary 8.4 (Differentiation w.r.t. the shift parameter)
Given a fixed $u \in W^{2,2}(I_+)$, the map $a \in (a_0 \pm \Delta a) \longmapsto \tau_{2a}u \in W^{1,2}(I_-)$ is differentiable with derivative $a \longmapsto 2\,\tau_{2a}\,\partial u \in W^{1,2}(I_-)$.

Proof. This is a straightforward application of Lemma 8.3 and Corollary 8.2:
Using the triangle inequality for Bochner integrals in $W^{1,2}(I_-)$ as well as continuity of the map $a \longmapsto \tau_{2a}u \in W^{1,2}(I_-)$ we get

$$\left\|\frac{\tau_{2(a+\delta a)}u - \tau_{2a}u}{\delta a} - 2\,\tau_{2a}\partial u\right\|_{W^{1,2}(I_-)} \le 2\int_0^1 \mathrm{d}t\, \big\|\tau_{2(a+t\delta a)}\partial u - \tau_{2a}\partial u\big\|_{W^{1,2}(I_-)} \longrightarrow 0 \quad \text{as } \delta a \longrightarrow 0$$

□

8.1.2 Differentiation of the rescaling map

Essentially repeating our strategy from the previous section, we will differentiate the rescaled bump coefficient $R_{\lambda(a)}f$ by interpreting its difference quotient

$$\frac{R_{\lambda+\delta\lambda}f - R_\lambda f}{\delta\lambda}$$

as a Bochner integral in a suitably constructed Banach space $C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-)$.
By construction of this space, every element $\alpha \in C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-)$ acts through scalar multiplication on the module $W^{n,2}_{V_-}(\text{-}1, 0)$, with an inequality

$$\|\alpha v\|_{W^{n,2}_{V_-}(\text{-}1,0)} \leq \text{const.} \times \|\alpha\|_{C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-)} \|v\|_{W^{n,2}_{V_-}(\text{-}1,0)}$$

making it possible to differentiate $R_{\lambda(a)}f \cdot \tau_{2a}v$ by the product rule.

Let us begin by introducing the spaces $C^n_{\text{bounded}}(I, V)$ and discussing their relation to the ring $\mathcal{R}(I, V)$ from section 6.3.1.

Definition 8.5 (Generalized C^1-spaces)
Let $\mathbb{B}$ be a Banach space, $I \subset \mathbb{R}$ an open interval and $V(x)\partial_x$ a vector field on I with $V > 0$. As a generalization of the classical $C^1_{\text{bounded}}(I)$ we consider the space

$$C^1_{\text{bounded}}(I, V; \mathbb{B}) := \left\{ (u_0, u_1) \in C^0_{\text{bounded}}(I, \mathbb{B})^{\oplus 2} \,\middle|\, u_0 : I \longrightarrow \mathbb{B} \text{ is differentiable and } u_1 = V\partial u_0 \right\}$$

Whenever our choice of $\mathbb{B}$ is clear, we will simply write $C^1_{\text{bounded}}(I, V)$. In the following it will be sufficient to work with $\mathbb{B} = \mathbb{R}$.

Lemma 8.6 (Contravariance of the spaces $C^1_{bounded}(I, V)$)
Given $\Phi : I_x \longrightarrow I_y$ a diffeomorphism between open intervals in $\mathbb{R}$, the isometry

$$C^0_{bounded}(I_y)^{\oplus 2} \longrightarrow C^0_{bounded}(I_x)^{\oplus 2}, \quad (u_0, u_1) \longmapsto (u_0 \circ \Phi, u_1 \circ \Phi)$$

identifies $C^1_{bounded}(I_y, V)$ with $C^1_{bounded}(I_x, \Phi^*V)$.

Proof. Consider $(u_0, u_1) \in C^1_{\text{bounded}}(I_y, V)$. Then $\partial u_0 = \frac{u_1}{V} \in C^0(I_y)$ guarantees $u_0 \in C^1(I_y)$, so we can apply the chain rule to observe that $\partial[u_0 \circ \Phi] = \dfrac{\partial y}{\partial x}\,[\partial u_0] \circ \Phi$ and therefore

$$u_1 \circ \Phi \;=\; (V \circ \Phi) \cdot [\partial u_0] \circ \Phi \;=\; \frac{V \circ \Phi}{\partial y / \partial x}\, \partial[u_0 \circ \Phi] \;=\; \Phi^*V\, \partial[u_0 \circ \Phi]$$

□

Lemma 8.7 (Completeness of C^1)
$C^1_{bounded}(I, V)$ is a closed subspace of $C^0_{bounded}(I)^{\oplus 2}$ and therefore a Banach space itself.

Proof. Let $\Phi : I_x \longrightarrow I$ be the straightening diffeomorphism from Remark 6.3.
Then we have $\Phi^*V = \partial_x$, so by Lemma 8.6 the componentwise isometry

$$C^0_{\text{bounded}}(I)^{\oplus 2} \xrightarrow{\;\_ \circ \Phi\;} C^0_{\text{bounded}}(I_x)^{\oplus 2}$$

identifies our space $C^1_{bounded}(I, V)$ with the classical $C^1_{\text{bounded}}(I_x)$ which is known to be complete. □

Remark 8.8 (Completeness of C^{n+1})
Lemma 8.7 guarantees that for every $n \geq 0$

$$C^{n+1}_{\text{bounded}}(I, V) := \left\{ (u_0, u_1, ..., u_{n+1}) \in C^0_{\text{bounded}}(I)^{\oplus n+2} \,\middle|\, (u_j, u_{j+1}) \in C^1_{\text{bounded}}(I, V) \text{ for all } j = 0, ..., n \right\}$$

is a closed subspace of $C^0_{\text{bounded}}(I)^{\oplus n+2}$ and therefore a Banach space itself.

The next observation is a follow-up on Remark 6.9 and will later allow us to extract the pole divergence from $\left(\frac{\partial}{\partial a}\right)^n R_{1/a} f\, \tau_{2au}$:

Auxiliary Lemma 8.9 (Rescaling of derivatives)
Let $I \subset \mathbb{R}$ be an open subset and $\rho \in C^\infty(I)$ a smooth function with $\rho \neq 0$ everywhere. Assume that for $\mathbb{B}$ a normed vector space we are given maps $v_n : I \longrightarrow \mathbb{B}$, $n = 0, ..., N$ such that

- *v_N is continuous*
- *for $n = 0, ..., N-1$ the map v_n is differentiable with derivative $v_n' = \dfrac{v_{n+1}}{\rho}$*

Then we have $v_0 \in C^N(I, \mathbb{B})$ and

$$v_n = (\rho\partial)^n v_0 = \sum_{k=0}^{n} C_{n,k}[\rho]\, \rho^k \partial^k v_0 \tag{8.1}$$

$$\rho^n\, \partial^n v_0 = \sum_{k=0}^{n} C^{-1}_{n,k}[\rho]\, v_k \tag{8.2}$$

for all $n = 0, ..., N$.

Proof. Our claim that $v_0 \in C^N(I, \mathbb{B})$ follows from a chain of implications

$$\begin{array}{rcl} & & v_N \in C^0 \\ v_{N-1}' = \frac{v_N}{\rho} \in C^0 & \Longrightarrow & v_{N-1} \in C^1 \\ v_{N-2}' = \frac{v_{N-1}}{\rho} \in C^1 & \Longrightarrow & v_{N-2} \in C^2 \\ & \cdots\cdots\cdots\cdots & \\ v_0' = \frac{v_1}{\rho} \in C^{N-1} & \Longrightarrow & v_0 \in C^N \end{array}$$

Equation (8.1) is obtained by successive application of $\rho\,\partial$ using that $C_{n,k}\,\rho^k \in C^\infty(I)$ and $\partial^k v_0 \in C^{N-k}(I)$.

Equation (8.2) exploits the fact that $[C_{n,k}] \in \mathrm{SL}\big(N, C^\infty(I)\big)$ is invertible. □

Remark 8.10 (Relating $C^n_{\text{bounded}}(I, V)$ to $\mathcal{R}(I, V)$)

- The set $\mathcal{R}^n(I, V) := \left\{ f \in C^n(I) \,\middle|\, \sup_{x \in I} \|V^k[f]\| < \infty \ \text{ for all } k = 0, ..., n \right\}$ is a subring of $C^0(I)$.
- The ring from Remark 6.5 can be recovered as $\mathcal{R}(I, V) = \bigcap_{n \geq 0} \mathcal{R}^n(I, V)$.
- We have a linear isomorphism

$$\begin{array}{rcl} \mathcal{R}^n(I, V) & \xrightarrow{\ \sim\ } & C^n_{bounded}(I, V) \subset C^0_{bounded}\,(I)^{\oplus n+1} \\ u & \longmapsto & \big[u_j = V^j u\big]_{j=0,...,n} \end{array}$$

  Note that the surjectivity of this map is a result of Auxiliary Lemma 8.9.

Notation. When our choice of I and V is clear, we will sometimes write $\mathcal{R}^n$ instead of $\mathcal{R}^n(I, V)$.

Now we are ready to repeat our recipe from section 8.1.1. It turns out that the vector field $V_- = -x\partial_x$ shows up naturally as we differentiate $R_\lambda f$.

Lemma 8.11 (Domain rescaling acts continuously on C^0)
Given a fixed $f \in \mathcal{R}^1(\mathbb{R}_{<0}, V_-)$, the map $\lambda \in (0,\infty) \longmapsto R_\lambda f \in C^0_{bounded}(\mathbb{R}_{<0})$ is continuous.

Proof. For $\lambda, \lambda+\delta\lambda > 0$ and pointwise at $x \in \mathbb{R}_{>0}$ the function $f \in C^1(\mathbb{R}_{<0})$ obeys

$$R_{\lambda+\delta\lambda} f(x) - R_\lambda f(x) = \int_0^1 \mathrm{d}t\, \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} f\big((\lambda+t\delta\lambda)x\big)}_{\delta\lambda\cdot x\,\partial f|_{(\lambda+t\delta\lambda)x}} = -\,\delta\lambda \cdot \int_0^1 \mathrm{d}t\, \frac{1}{b} R_b V_-[f]\Big|_{b=\lambda+t\delta\lambda}(x) \tag{8.3}$$

Restricting to the case $2\,|\delta\lambda| < \lambda$ we have $\frac{1}{|\lambda+t\delta\lambda|} < \frac{2}{|\lambda|}$ and thus

$$\big\|R_{\lambda+\delta\lambda} f(x) - R_\lambda f(x)\big\| \le 2\left|\frac{\delta\lambda}{\lambda}\right| \cdot \sup_{y\in\mathbb{R}_{<0}} \big\|V_-[f]\big\|$$

where the right hand side does not depend on x anymore. ☐

Corollary 8.12 (Domain rescaling acts continuously on C^n)
Given a fixed $f \in \mathcal{R}^{n+1}(\mathbb{R}_{<0}, V_-)$, the map $\lambda \in (0,\infty) \longmapsto R_\lambda f \in C^n_{bounded}(\mathbb{R}_{<0}, V_-)$ is continuous.

Proof. Given a fixed $\lambda \in (0,\infty)$, the diffeomorphism $R_\lambda : \mathbb{R}_{<0} \longrightarrow \mathbb{R}_{<0},\ x \longmapsto \lambda \cdot x$ satisfies

$$R_\lambda^* V_- = -\frac{1}{\lambda} \cdot \lambda x = V_-$$

so $f \longmapsto R_\lambda f = f \circ R_\lambda \in C^0_{bounded}(\mathbb{R}_{<0})$ is covered by a **componentwise** map

$$\begin{aligned} C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-) &\longrightarrow C^n_{\text{bounded}}(\mathbb{R}_{<0}, R_\lambda^* V_-) = C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-) \\ \big(f, f_1, ..., f_n\big) &\longmapsto \big(R_\lambda f, R_\lambda f_1, ..., R_\lambda f_n\big) \end{aligned}$$

Note that we have $f_j = V^j f \in \mathcal{R}^{n-j+1} \subset \mathcal{R}^1$, so continuity follows from Lemma 8.11. ☐

Lemma 8.13 (Difference quotient of the rescaling map as a Bochner integral)
Given $\lambda, \lambda+\delta\lambda > 0$ and $f \in \mathcal{R}^{n+2}(\mathbb{R}_{<0}, V_-)$, the formula

$$R_{\lambda+\delta\lambda} f - R_\lambda f = -\,\delta\lambda \cdot \int_0^1 dt\, \frac{1}{b} R_b V_-[f]\Big|_{b=\lambda+t\delta\lambda} \quad \textit{holds in } C^n_{bounded}(\mathbb{R}_{<0}, V_-).$$

Proof. Having $f \in \mathcal{R}^{n+2}$ ensures $V_-[f] \in \mathcal{R}^{n+1}$ so the map $b \longmapsto \frac{1}{b} R_b V_-[f] \in C^n_{bounded}(\mathbb{R}_{<0}, V_-)$ is continuous and we get a well-defined Bochner integral

$$\int_0^1 \mathrm{d}t\, \frac{1}{b} R_b V_-[f]\Big|_{b=\lambda+t\delta\lambda} \in C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-)$$

The Bochner integral commutes with $ev_x : C^0_{\text{bounded}}(\mathbb{R}_{<0}, \mathbb{B}) \longrightarrow \mathbb{B}$ so equation (8.3) implies

$$ev_x\left[R_{\lambda+\delta\lambda} f - R_\lambda f \,+\, \delta\lambda \cdot \int_0^1 \mathrm{d}t\, \frac{1}{b} R_b V_-[f]\Big|_{b=\lambda+t\delta\lambda}\right] = 0 \qquad \text{for all } x \in \mathbb{R}_{<0}$$

☐

Corollary 8.14 (Differentiation w.r.t. the scale parameter)
Given a fixed $f \in \mathcal{R}^{n+2}(\mathbb{R}_{<0}, V_-)$, the map $\lambda \in (0,\infty) \longmapsto R_\lambda f \in C^n_{bounded}(\mathbb{R}_{<0}, V_-)$ is differentiable with derivative $\lambda \longmapsto -\frac{1}{\lambda} R_\lambda V_-[f] \in C^n_{bounded}(\mathbb{R}_{<0}, V_-)$.

Proof. Since $b \longmapsto \frac{1}{b} R_b V_-[f] \in C^n_{bounded}(\mathbb{R}_{<0}, V_-)$ is continuous, one has

$$\left\|\frac{R_{\lambda+\delta\lambda} f - R_\lambda f}{\delta\lambda} + \frac{1}{\lambda} R_\lambda V_-[f]\right\|_{C^n_{bounded}(\mathbb{R}_{<0}, V_-)} \le \int_0^1 \mathrm{d}t \left\|\frac{1}{b} R_b V_-[f]\Big|_\lambda^{\lambda+t\delta\lambda}\right\|_{C^n_{bounded}(\mathbb{R}_{<0}, V_-)} \longrightarrow 0 \ \text{ as } \delta\lambda \longrightarrow 0$$

☐

By considering $\lambda(a) = a^{-k}$, we observe that $\lambda \frac{\partial}{\partial \lambda} R_\lambda[f] = -R_\lambda\left[V_- f\right]$ and $a \frac{\partial}{\partial a} R_{\lambda(a)}[f] = k \cdot R_{\lambda(a)}\left[V_- f\right]$ agree up to a prefactor, so our discussion will not rely on any particular choice of $k \geq 1$ in "$R_{1/a^k} f \cdot \tau_{2a^k} u$":

Corollary 8.15 (Inverting the scale parameter)
Write $\lambda(a) = 1/a^k$. Then for $f \in \mathcal{R}^{n+2}(\mathbb{R}_{<0}, V_-)$, the derivative of the map $a \in (0, \infty) \longmapsto R_{\lambda(a)} f \in C^n_{bounded}(\mathbb{R}_{<0}, V_-)$ satisfies

$$a \frac{\partial}{\partial a} R_{\lambda(a)} f \;=\; k \cdot R_{\lambda(a)} V_-[f]$$

Proof. Since $[\lambda \longmapsto R_\lambda f]$ belongs to $C^1\left[(0,\infty), C^n_{bounded}(\mathbb{R}_{<0}, V_-)\right]$ the chain rule shows

$$\frac{\partial}{\partial a} R_{\lambda(a)} f \;=\; -\frac{\lambda'(a)}{\lambda(a)}\, R_{\lambda(a)} V_-[f] \;=\; \frac{k}{a}\, R_{\lambda(a)} V_-[f]$$

□

Let us conclude this section by an illustration of Auxiliary Lemma 8.9 and Remark 8.10:

Lemma 8.16 (The rescaling map has bounded logarithmic derivatives)
Given $f \in \mathcal{R}^{m+n+1}(\mathbb{R}_{<0}, V_-)$, the map $\left[a \longmapsto R_{\lambda(a)} f\right]$ belongs to $\mathcal{R}^m\left[(0,\infty), a\frac{\partial}{\partial a};\, C^n_{bounded}(\mathbb{R}_{<0}, V_-)\right]$.

Proof. We have $V_-^j[f] \in \mathcal{R}^{n+2}$ for $j = 0, ..., m-1$ and $V_-^m[f] \in \mathcal{R}^{n+1}$.
Thus, the sequence of maps $v_j : (0, \infty) \longrightarrow C^n_{bounded}(\mathbb{R}_{<0}, V_-)$, $j = 0, ..., m$ given by

$$v_j(a) := \; R_{\lambda(a)} V_-^j[f]$$

satisfies $a\frac{\partial}{\partial a} v_j = v_{j+1}$ for $j = 0, ..., m-1$ and v_m is continuous.
With Auxiliary Lemma 8.9 we conclude that $a \longmapsto v_j(a)$ is of class C^m. Moreover,

$$\left\| \left[a \frac{\partial}{\partial a}\right]^j v_0 \right\| = \left\| R_{\lambda(a)} V_-^j[f] \right\|_{C^n_{bounded}(\mathbb{R}_{<0}, V_-)} = \sum_{k=0}^{n} \sup_{x \,\in\, \mathbb{R}_{<0}} \left| R_{\lambda(a)} V_-^{j+k}[f] \right| = \sum_{k=0}^{n} \sup_{x \,\in\, \mathbb{R}_{<0}} \left| V_-^{j+k}[f] \right|$$

does not depend on a. □

8.1.3 Differentiating the off-diagonal part of the retraction

Let us now combine our findings from sections 8.1.1 and 8.1.2.

Notation.

- In the following we will continue to write $\lambda(a) = 1/a$.
- Moreover, we will consider all intervals $I \subset \mathbb{R}_x$ equipped with the metric $g = \mathrm{d}x^2$ and identify $W^{n,2}_V(I) := W^{n,2}_{V,g}(I)$ with a subspace of $L^2(I)$.

The following technical result builds on the observation that, thanks to the prefactor $R_{1/a}f \in C_0^\infty\big(\text{-}\frac{3}{2}a, \text{-}\frac{a}{2}\big)$, there is an unproblematic transition from $u \in W^{2,2}_{V_+}(0,1)$ to $R_{1/a}f \cdot \tau_{2a}u \in W^{1,2}_{V_-}(\text{-}1,0)$.

Lemma 8.17 (Transfer from $W^{1,2}_{V_+}$ to $W^{1,2}_{V_-}$ is differentiable w.r.t. the gluing parameter) *Given $f \in C_0^\infty(-\frac{3}{2}, -\frac{1}{2})$ and $u \in W^{2,2}_{V_+}(0,1)$ the map $a \longmapsto R_{\lambda(a)}f \cdot \tau_{2a}u \in W^{1,2}_{V_-}(-1,0)$ is differentiable and its derivative satisfies*

$$a\frac{\partial}{\partial a}\, R_{\lambda(a)}f \cdot \tau_{2a}u \;=\; R_{\lambda(a)}V_-[f] \cdot \tau_{2a}u + R_{\lambda(a)}\big(f \cdot h\big)\; \tau_{2a}V_+u \;\;\in W^{1,2}_{V_-}(-1,0)$$

with $h = \frac{2}{x+2} \in C^\infty(-2,0)$.

Proof. We work locally in a with $a \in (a_0 \pm \Delta a)$.
Given $u \in W^{2,2}_{V_+}(0,1)$ Lemma 6.13 applied to "f" $= \frac{1}{x} \in \mathcal{R}(I_+, V_+)$ shows that $u \in W^{2,2}_\partial(I_+)$ where $\partial u = \frac{1}{x}V_+u$ holds as an identity in the $\mathcal{R}(I_+,V_+)$-module $W^{1,2}_{V_+}(I_+)$.

There are ring homomorphisms

$$\mathcal{R}(I_+,V_+) \xrightarrow{\text{incl.}} \mathcal{R}(I_+, \partial = \tfrac{1}{x}V_+) \xrightarrow{\tau_{2a}} \mathcal{R}(I_-,\partial) \xrightarrow{\text{incl.}} \mathcal{R}(I_-, V_- = -x\partial)$$

accompanied by module homomorphisms

$$W^{n,2}_{V_+}(I_+) \xrightarrow{\quad\text{incl.}\quad} W^{n,2}_\partial(I_+) \xrightarrow{\quad\tau_{2a}\quad} W^{n,2}_\partial(I_-) \xrightarrow{\quad\text{incl.}\quad} W^{n,2}_{V_-}(I_-)$$

Thus, $\tau_{2a}\partial u = \dfrac{1}{x+2a}\,\tau_{2a}V_+u$ can be understood as an equation in the $\mathcal{R}(I_-,V_-)$-module $W^{1,2}_{V_-}(I_-)$.
Since we have $u \in W^{2,2}_\partial(I_+)$, Corollary 8.4 shows that the map $a \longmapsto \tau_{2a}u \in W^{1,2}_\partial(I_-)$ is differentiable with derivative $a \longmapsto 2\,\tau_{2a}\partial u \in W^{1,2}_\partial(I_-)$.
Using the bounded linear inclusion $W^{1,2}_\partial(I_-) \hookrightarrow W^{1,2}_{V_-}(I_-)$ the same statement holds in $W^{1,2}_{V_-}(I_-)$, i.e. $a \longmapsto \tau_{2a}u \in W^{1,2}_{V_-}(I_-)$ is differentiable and its derivative satisfies

$$a\frac{\partial}{\partial a}\,\tau_{2a}u \;=\; 2a\,\tau_{2a}\partial u \;=\; \frac{2a}{x+2a}\,\tau_{2a}V_+u \;\;\in W^{1,2}_{V_-}(I_-)$$

Now consider the general situation where $a \longmapsto s(a) \in C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-)$ and $a \longmapsto v(a) \in W^{n,2}_{V_-}(I_-)$ are differentiable with derivatives $s'(a) \in C^n_{\text{bounded}}(\mathbb{R}_{<0}, V_-)$ and $v'(a) \in W^{n,2}_{V_-}(I_-)$ respectively.
Then by the identity

$$\|s \cdot v\|_{W^{n,2}_{V_-}(I_-)} \;\le\; \text{const.} \times \|s\|_{C^n_{\text{bounded}}(\mathbb{R}_{<0},V_-)}\;\|v\|_{W^{n,2}_{V_-}(I_-)}$$

the map $a \longmapsto s(a)v(a)$ is differentiable with derivative $s'(a)v(a) + s(a)v'(a)$.
Consequently, in our case $a \longmapsto R_{\lambda(a)}f \cdot \tau_{2a}u \in W^{1,2}_{V_-}(I_-)$ is differentiable with derivative

$$a\frac{\partial}{\partial a}\, R_{\lambda(a)}f\;\tau_{2a}u \;=\; R_{\lambda(a)}V_-[f]\tau_{2a}u + R_{\lambda(a)}\big(f \cdot h\big)\; \tau_{2a}V_+u \;\;\in W^{1,2}_{V_-}(I_-) \tag{8.4}$$

where $h := \frac{2}{x+2} \in C^\infty(-2,0)$ and by assumption $f \in C_0^\infty(-3/2,-1/2)$.
Notice that for all $a \in (a_0 {\pm} \Delta a)$ both $R_{\lambda(a)}f\;\tau_{2a}u$ and $\frac{\partial}{\partial a}R_{\lambda(a)}f\;\tau_{2a}u$ belong to $C_0^\infty(I_-) \cdot W^{1,2}_{V_-}(I_-)$.
For $v \in C_0^\infty(I_-) \cdot W^{1,2}_{V_-}(I_-)$ one has $v \in W^{1,2}_{V_-}(-1,0)$ with $\|v\|_{W^{1,2}_{V_-}(-1,0)} = \|v\|_{W^{1,2}_{V_-}(I_-)}$.

Therefore the statement (8.4) continues to hold when $W^{1,2}_{V_-}\big(I_-\big)$ is replaced by $W^{1,2}_{V_-}(-1,0)$ and our claim can be patched together by varying the interval $(a_0 \pm \Delta a)$. □

Lemma 8.17 shows that the logarithmic derivative $a\frac{\partial}{\partial a}\big[R_{1/a}f\cdot\tau_{2a}u\big]$ is again a sum of terms

$$\text{"}\ R_{1/a}\big[\text{bump}\big]\ \tau_{2a}\big[\text{Sobolev function}\big]\ \text{"}$$

Coming to our main result of this section, we will iteratively calculate the bump coefficients appearing in $\left(a\frac{\partial}{\partial a}\right)^n\big[R_{1/a}f\tau_{2a}u\big]$, before applying Auxiliary Lemma 8.9 to conclude that $\left(\frac{\partial}{\partial a}\right)^n\big[R_{1/a}f\tau_{2a}u\big]$ involves a pole of order exactly n.

Definition 8.18 (Bump coefficients $N_{k,l}$ and $\chi_{m,l}$)
As in Lemma 8.17 let us write $h=\frac{2}{x+2}\in C^\infty(-2,0)$.
Given $f\in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ we define the coefficients $N_{k,l}[f]\in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ iteratively by

$$\begin{aligned} N_{0,l} &= f\cdot\delta_{0,l}\\ N_{k+1,l} &= V_-[N_{k,l}]+N_{k,l-1}\cdot h\end{aligned}$$

In the following, by $C_{m,k}$ we will always mean the coefficients $C^V_{m,k}[\rho]$ with $V=\partial$ and $\rho(a)=a$. For instance, $\big[C_{m',k}\big]\in SL(m,\mathbb{Z})$ and $N_{k,l}[f]\in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ combine into coefficients

$$\chi_{m,l}[f]=\sum_k C^{-1}_{m,k}N_{k,l}[f]\in C_0^\infty\left(-\tfrac{3}{2},-\tfrac{1}{2}\right)$$

Proposition 8.19 (Pole order arising from derivatives in a)
Given $f\in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ and $u\in W^{N+1,2}_{V_+}(0,1)$, the map $a\longmapsto R_{\lambda(a)}f\cdot\tau_{2a}u\in W^{1,2}_{V_-}(-1,0)$ belongs to $C^N\Big((0,\infty),W^{1,2}_{V_-}(-1,0)\Big)$, with derivatives given by

$$\left(\frac{\partial}{\partial a}\right)^n R_{\lambda(a)}f\cdot\tau_{2a}u \;=\; \frac{1}{a^n}\sum_{l=0}^{n}R_{\lambda(a)}\,\chi_{n,l}[f]\cdot\tau_{2a}V_+^l u\quad\in W^{1,2}_{V_-}(-1,0)$$

Proof. We apply Auxiliary Lemma 8.9 in the case $I=(0,\infty)$, $\rho(a)=a$, $\mathbb{B}=W^{1,2}_{V_-}(-1,0)$:
Given $u\in W^{N+1,2}_{V_+}(0,1)$ and $f\in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ let us define

$$v_n(a)=\sum_{l=0}^{n}R_{\lambda(a)}N_{n,l}[f]\ \tau_{2a}V_+^l u\ \in W^{1,2}_{V_-}(-1,0)\qquad\text{for } n=0,...,N$$

These maps are continuous in a, as can be seen by combining Corollaries 8.12 and 8.2. Moreover, restricting to $n=0,....,N-1$ we have

$$l\le n\le N-1\implies V_+^l u\in W^{2,2}_{V_+}(0,1)$$

so by Lemma 8.17 the map $a\longmapsto v_n(a)$ is differentiable with derivative

$$\begin{aligned} a\frac{\partial}{\partial a}v_n &= \sum_l R_{\lambda(a)}V_-[N_{n,l}]\ \tau_{2a}V_+^l u+R_{\lambda(a)}(N_{n,l}\cdot h)\ \tau_{2a}V_+^{l+1}u\\ &= \sum_l R_{\lambda(a)}\big[\underbrace{V_-[N_{n,l}]+N_{n,l-1}\cdot h}_{N_{n+1,l}}\big]\ \tau_{2a}V_+^l u \qquad = v_{n+1}\end{aligned}$$

Taking into account that $C_{m,k}\big[\rho(a)=a\big]\in\mathbb{Z}$, our claim follows from equation (8.2) of Auxiliary Lemma 8.9. $\square$

Having settled the 'longitudinal' derivatives in a-direction, we still have to discuss the 'transversal' derivatives in direction of the level sets. For the term $R_{1/a}f\cdot\tau_{2a}u\in W^{n,2}_{V_-}(\text{-}1,0)$ these arise as powers of $V_- = \text{-}x\partial_x$. Just as for the 'longitudinal' derivatives $\left(a\frac{\partial}{\partial a}\right)^n$, we start by a technical argument for the case $n=1$, before introducing yet another variant of bump coefficients $M_{k,l}\in C_0^\infty\left(\text{-}\frac{3}{2},\text{-}\frac{1}{2}\right)$ to formulate our result in the case of general n.

Lemma 8.20 (Translating V_- into V_+)
Let us consider $f\in C_0^\infty\left(-\frac{3}{2},-\frac{1}{2}\right)$ *and* $w\in W^{1,2}_{V_+}(0,1)$.
Then $R_{\lambda(a)}f\cdot\tau_{2a}w$ *belongs to* $C_0^\infty(I_-)\cdot W^{1,2}_{V_-}(I_-)\subset W^{1,2}_{V_-}(-1,0)$ *with*

$$V_-\left[R_{\lambda(a)}f\cdot\tau_{2a}w\right] \;=\; R_{\lambda(a)}V_-[f]\cdot\tau_{2a}w \,+\, R_{\lambda(a)}\left[f\cdot(h-1)\right]\cdot\tau_{2a}V_+w$$

Proof. At fixed $a\in(a_0\pm\Delta a)$ we consider the diffeomorphism $\Phi: I_-\longrightarrow I_-+2a\subset I_+,\; x\longmapsto x+2a$. With $\sqrt{\Phi^*g}=\sqrt{g}=1$ we have an implication

$$\big(w,V_+w\big)\in W^{1,2}_{V_+}(I_+)\;\Longrightarrow\;\big(w\circ\Phi,\,[V_+w]\circ\Phi\big)\in W^{1,2}_{\Phi^*V_+}(I_-)$$

and in $C^\infty(I_-)$ we calculate

$$V_- \;=\; -x \;=\; \left[\frac{2a}{x+2a}-1\right](x+2a)\;=\;R_{\lambda(a)}(h-1)\;\Phi^*V_+$$

Note that the prefactor $R_{\lambda(a)}(h-1)$ is bounded on $\overline{I_-}\subset(-2a,0)$, so by part 1) of Auxiliary Lemma 6.12 one has $\tau_{2a}w=w\circ\Phi\in W^{1,2}_{V_-}(I_-)$ with

$$\begin{aligned} V_-[\tau_{2a}w] \;=\; V_-[w\circ\Phi] \;&=\; R_{\lambda(a)}(h-1)\cdot\Phi^*V_+[w\circ\Phi]\\ &=\; R_{\lambda(a)}(h-1)\cdot[V_+w]\circ\Phi \quad=\; R_{\lambda(a)}(h-1)\cdot\tau_{2a}V_+w \end{aligned}$$

Regarding the multiplication by $R_{\lambda(a)}f\in C_0^\infty(I_-)$, part 2) of Auxiliary Lemma 6.12 ensures that $R_{\lambda(a)}f\cdot\tau_{2a}w\,\in W^{1,2}_{V_-}(I_-)$ with

$$\begin{aligned} V_-\left[R_{\lambda(a)}f\cdot\tau_{2a}w\right] \;&=\; V_-\left[R_{\lambda(a)}f\right]\cdot\tau_{2a}w \,+\, R_{\lambda(a)}f\cdot V_-[\tau_{2a}w]\\ &=\; R_{\lambda(a)}V_-[f]\cdot\tau_{2a}w \,+\, R_{\lambda(a)}\left[f\cdot(h-1)\right]\cdot\tau_{2a}V_+w \quad\in L^2(I_-) \end{aligned}$$

□

Definition 8.21 (Bump coefficients $M_{k,l}$)
As before let us write $h = \frac{2}{x+2} \in C^\infty(-2,0)$.
Given $f \in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ we define the coefficients $M_{k,l}[f] \in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ iteratively by

$$\begin{aligned} M_{0,l} &= f \cdot \delta_{0,l} \\ M_{k+1,l} &= V_-[M_{k,l}] + M_{k,l-1} \cdot (h-1) \end{aligned}$$

Proposition 8.22 (No poles arising from V_-)
Given $f \in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$ and $w \in W^{n,2}_{V_+}(0,1)$ one has $R_{\lambda(a)} f \cdot \tau_{2a} w \in W^{n,2}_{V_-}(-1,0)$ with

$$V_-^k\big[R_{\lambda(a)} f \cdot \tau_{2a} w\big] = \sum_{l=0}^{k} R_{\lambda(a)} M_{k,l}[f] \cdot \tau_{2a} V_+^l w \qquad \textit{for} \quad k = 0,...,n$$

Proof. Every element $w \in W^{n,2}_{V_+}(0,1)$ can be represented by a tuple $(w, V_+w, ..., V_+^n w) \in L^2(0,1)^{\oplus n+1}$.
Let us define

$$v_k = \sum_{l=0}^{k} R_{\lambda(a)} M_{k,l}[f] \cdot \tau_{2a} V_+^l w \ \in L^2(-1,0)$$

For $k = 0, ..., n-1$ we claim that $(v_k, v_{k+1}) \in W^{1,2}_{V_-}(-1,0)$:
Indeed, for $l \le k \le n-1$ we have $V_+^l w \in W^{1,2}_{V_+}(0,1)$ and $M_{k,l}[f] \in C_0^\infty(-\frac{3}{2},-\frac{1}{2})$,
so Lemma 8.20 implies $v_k \in W^{1,2}_{V_-}(-1,0)$ with

$$\begin{aligned} V_-[v_k] &= \sum_l R_{\lambda(a)} V_-[M_{k,l}] \cdot \tau_{2a} V_+^l w + R_{\lambda(a)} \big[M_{k,l} \cdot (h-1)\big]\ \tau_{2a} V_+^{l+1} w \\ &= \sum_l R_{\lambda(a)} \underbrace{\big[V_-[M_{k,l}] + M_{k,l-1} \cdot (h-1)\big]}_{M_{k+1,l}}\ \tau_{2a} V_+^l w \qquad = v_{k+1} \end{aligned}$$

□

Recall from the introduction that our initial motivation to consider the vector field $V_- = \text{-}x\partial_x$ was that V_- commutes with the rescaling map. According to Proposition 8.22, this now manifests in the absence of poles in $V_-^k\big[R_{1/a} f \cdot \tau_{2a} u\big]$. Poles of this type, if there were some, would destroy sc^0-continuity of the retraction at $a = 0$. The pole from Proposition 8.19, on the other hand, can be compensated by a 'weight difference' as we shall see in Proposition 8.27.

8.2 A simple criterion for sc-smoothness

It seems intuitively clear that to prove sc-smoothness of a fibre-linear map, we only need to care about derivatives in the base direction, while keeping all arguments coming from the fibre fixed. We will rigorously justify this idea in Proposition 8.25, thereby providing a clean and efficient way to verify sc-smoothness in our case at hand.

In this section, let $E = (E_n)_{n\geq 0}$, $F = (F_n)_{n\geq 0}$, $G = (G_n)_{n\geq 0}$ be sc-Banach spaces and take $B \subset \mathbb{R}$ to be an open subset.

As a preparation, let us talk about sc^1-differentiability:

Lemma 8.23 (Sc^1 fibre-linear maps)
Assume we are given sc^0-maps $\alpha : B \oplus G \longrightarrow F$ (linear in G-direction) and $\beta : B \oplus G^1 \longrightarrow F$ such that for every fixed $e \in G_1$

$$b \longmapsto \alpha_b(e) \in F_0 \text{ is differentiable with derivative } b \longmapsto \beta_b(e) \in F_0\,.$$

Then α is sc^1.

Proof. At fixed $(b, e) \in B \oplus G_1$ the assignment

$$(\delta b, \delta e) \longmapsto D\alpha(b, e, \delta b, \delta e) = \delta b \cdot \beta_b(e) + \alpha_b(\delta e) \quad \text{defines a map in } \mathcal{L}(B \oplus G_0, F_0)\,.$$

The map $\quad T\alpha : B \oplus G^1 \oplus B \oplus G \xrightarrow{(\alpha^1, D\alpha)} F^1 \oplus F \quad$ is sc^0.

It remains to interpret $D\alpha$ as the differential of $\alpha|_1 : B \oplus G_1 \longrightarrow F_0$.
Indeed, restricting to $e, \delta e \in G_1$ we find

$$\frac{1}{|\delta b| + \|\delta e\|_1} \|\alpha_{b+\delta b}(e + \delta e) - \alpha_b(e) - \delta b \cdot \beta_b(e) - \alpha_b(\delta e)\|_{F_0}$$
$$\leq \underbrace{\frac{1}{|\delta b|} \|\alpha_{b+\delta b}(e) - \alpha_b(e) - \delta b \cdot \beta_b(e)\|_{F_0}}_{\longrightarrow 0 \text{ as } \delta b \to 0 \quad \text{by assumption}} + \underbrace{\left\| \alpha_{b+\delta b}\left[\frac{\delta e}{\|\delta e\|_1}\right] - \alpha_b\left[\frac{\delta e}{\|\delta e\|_1}\right] \right\|_{F_0}}_{\leq \sup_{v\in K} \|\alpha_{b+\delta b} v - \alpha_b v\|_{F_0}}$$

The embedding $G_1 \hookrightarrow G_0$ being compact means that $K := \overline{B_1^{G_1}(0)}^{G_0}$ is a compact subset of G_0, so by Auxiliary Lemma 8.24 we obtain

$$\sup_{v\in K} \|\alpha_{b+\delta b} v - \alpha_b v\|_{F_0} \longrightarrow 0 \quad \text{as } \delta b \longrightarrow 0$$

□

Auxiliary Lemma 8.24 (Continuity w.r.t. compact-open topology turns uniform)
Let $\alpha : B \oplus G_0 \longrightarrow F_0$ be a continuous map between normed vector spaces and assume that $K \subset G_0$ is a compact subset. Then $\alpha : B \longrightarrow C^0_{bounded}(K, F_0)$ is continuous.

Proof. Given $b \in B$, $\epsilon > 0$ cover $\{b\} \times K$ by open subsets $B_{\delta_i}(b) \times U_i$ such that

$$\|\alpha(x) - \alpha(y)\|_{F_0} < \epsilon \quad \text{for all } x, y \in B_{\delta_i}(b) \times U_i$$

Since K is compact, a finite cover will be enough and we can set $\delta := \min \delta_i > 0$
Then however we observe that

$$|\delta b| < \delta \implies \|\alpha_{b+\delta b} v - \alpha_b v\|_{F_0} < \epsilon \text{ for all } v \in K$$

□

The following criterion shows that sc-smoothness of a fibre-linear map boils down to pointwise differentiability in the base direction together with sc^0-continuity of all derivatives:

Proposition 8.25 (Bootstrapped criterion for the sc-smoothness of fibre-linear maps) *Assume we are given a sequence of sc^0-maps $\partial^n\pi : B \oplus E^n \longrightarrow F\,,\; n \geq 0$ such that*

1) $\pi_b(e) = \big(\partial^0\pi\big)_b(e)$ is linear in $e \in E_0$

2) For fixed $e \in E_{n+1}$ the map $b \longmapsto \big(\partial^{n+1}\pi\big)_b(e) \in F_0$ is the derivative of $b \longmapsto \big(\partial^n\pi\big)_b(e) \in F_0$

Then $\pi = \partial^0\pi : B \oplus E \longrightarrow F$ is sc^∞

Proof. Since $b \longmapsto \big(\partial^{n+1}\pi\big)_b(e) \in F_0$ is the derivative of $b \longmapsto \big(\partial^n\pi\big)_b(e) \in F_0\,$, we see that by induction $\big(\partial^n\pi\big)_b(e)$ is linear in $e \in E_n$ for all $n \geq 0\,$.
Thus, Lemma 8.23 applied to "α" $= \partial^n\pi$, "β" $= \partial^{n+1}\pi$ shows

$$\mathcal{P}(1) \qquad \partial^n\pi : B \oplus E^n \longrightarrow F \text{ is } sc^1 \text{ for all } n \geq 0$$

Note that the sc^0-map

$$\begin{aligned} D\big(\partial^n\pi\big) : B \oplus E^{n+1} \oplus B \oplus E^n &\longrightarrow F \\ (b, e, \delta b, \delta e) &\longmapsto \delta b \cdot \big(\partial^{n+1}\pi\big)_b e + \big(\partial^n\pi\big)_b \delta e \end{aligned}$$

is the sum of

$$B \oplus E^{n+1} \oplus B \oplus E^n \xrightarrow[sc^\infty]{\text{proj.}} B \oplus E^{n+1} \oplus B \xrightarrow{\partial^{n+1}\pi \oplus \mathrm{id}} F \oplus B \xrightarrow[sc^\infty]{\text{mult.}} F$$

and

$$B \oplus E^{n+1} \oplus B \oplus E^n \xrightarrow[sc^\infty]{\text{proj.}} B \oplus E^n \xrightarrow{\quad\partial^n\pi\quad} F$$

Let us assume by induction that at some $k \geq 1$ it holds that

$$\mathcal{P}(k) \qquad \partial^n\pi : B \oplus E^n \longrightarrow F \text{ is } sc^k \;\text{ for all } n \geq 0$$

Then by the sc^k-chain rule (iterate [HWZ21] Thm 1.3.1) the combination of $\partial^n\pi$ and $\partial^{n+1}\pi$ being sc^k implies that

$$D\big(\partial^n\pi\big) : B \oplus E^{n+1} \oplus B \oplus E^n \longrightarrow F \quad \text{is } sc^k.$$

On the other hand, [HWZ21] Prop 1.2.2 guarantees that

$$(\partial^n\pi)^1 : B \oplus E^{n+1} \longrightarrow F^1 \quad \text{is } sc^k.$$

Taking the last two results together, we conclude that

$$T(\partial^n\pi) : B \oplus E^{n+1} \oplus B \oplus E^n \xrightarrow{\big[(\partial^n\pi)^1,\, D(\partial^n\pi)\big]} F^1 \oplus F \quad \text{is } sc^k$$

which is nothing else but the definition of $\partial^n\pi$ being sc^{k+1}.
Therefore we have shown the implication $\mathcal{P}(k) \implies \mathcal{P}(k+1)$ and our claim follows by induction. $\square$

As is apparent in our formulation of Propositions 7.5 and 7.6, the topology transition at vanishing gluing parameter manifests in a singularity of the retraction. This singularity will be removed by the following extension to our sc^∞-criterion from Proposition 8.25.

Theorem 8.26 (Sc-smoothness of fibre-linear maps with removable singularity)
Assume that for $B = (-\epsilon, \epsilon) \setminus \{0\}$ we are given a family $\partial^n \pi : B \oplus E^n \longrightarrow F$, $n \geq 0$ satisfying the requirements of Proposition 8.25
Assume in addition that

3) At every fixed $e \in E_{n+k}$ the limits $\lim_{b \searrow 0} \left(\partial^n \pi\right)_b e$, $\lim_{b \nearrow 0} \left(\partial^n \pi\right)_b e \in F_k$ exist and agree

4) For all $n, k \geq 0$ there exist constants $\mathcal{K}_{n,k}, \epsilon_{n,k} > 0$ such that

$$\left\| \left(\partial^n \pi\right)_b (\cdot) \right\|_{\mathcal{L}(E_{n+k}, F_k)} \leq \mathcal{K}_{n,k} \qquad \forall\, b \in B_{\epsilon_{n,k}}(0) \setminus \{0\}$$

Then the requirements of Proposition 8.25 are fulfilled for $B = (-\epsilon, \epsilon)$ so the extended $\pi : (-\epsilon, \epsilon) \oplus E \longrightarrow F$ is sc^∞.

Proof. On $e \in E_{n+k}$ we define $\left(\partial^n \pi\right)_0 (e) := \lim_{b \searrow 0} \left(\partial^n \pi\right)_0 (e) = \lim_{b \nearrow 0} \left(\partial^n \pi\right)_0 (e) \in F_k$.
This definition is compatible with all lower levels $E_{n+(k-1)}, \ldots, E_{n+0}$ containing the given e and immediately shows that

- $\left(\partial^n \pi\right)_0 (E_{n+k}) \subset F_k$
- $\left(\partial^n \pi\right)_0 (e)$ is linear in $e \in E_0$

To see that $\partial^n \pi : (-\epsilon, \epsilon) \oplus E^n \longrightarrow F$ is sc^0 it remains to work locally around the point $(0, e)$ with $e \in E_{n+k}$ and variations $\delta e \in E_{n+k}$:

$$\|\partial^n \pi_b(e + \delta e) - \partial^n \pi_0(e)\|_{F_k} \leq \underbrace{\|\partial^n \pi_b(e) - \partial^n \pi_0(e)\|_{F_k}}_{\longrightarrow 0 \text{ as } b \longrightarrow 0} + \underbrace{\|\partial^n \pi_b\|_{\mathcal{L}(E_{n+k}, F_k)}}_{\leq \mathcal{K}_{n,k}} \|\delta e\|_{E_{n+k}}$$

As a last step, we have to show that condition 2) of Proposition 8.25 also holds at $b = 0$. Since condition 2) is already assumed for $b \neq 0$, we see that at fixed $e \in E_{n+1}$ one has

$$\begin{aligned} &\left[b \longmapsto \partial^n \pi_b(e)\right] \in C^1\big((-\epsilon, \epsilon) \setminus \{0\}, F_0\big) \\ \text{with derivative} \quad &\left[b \longmapsto \partial^{n+1} \pi_b(e)\right] \in C^0\big((-\epsilon, \epsilon), F_0\big) \end{aligned}$$

Hence, the Fundamental Theorem of Calculus yields

$$\partial^n \pi_b(e) - \partial^n \pi_{b_0}(e) = \int_{b_0}^{b} \mathrm{d}b' \; \partial^{n+1} \pi_{b'}(e)$$

For $b > b_0 > 0$ we take the limit $b_0 \longrightarrow 0$ whereas for $0 > b > b_0$ we take the limit $b \longrightarrow 0$. This is possible because by assumption 3) we have continuously extended $\partial^{n+1} \pi_b(e)$ to $b = 0$. In the situation $b > 0$ for instance we obtain

$$\left\| \frac{\partial^n \pi_b(e) - \partial^n \pi_0(e)}{b} - \partial^{n+1} \pi_0(e) \right\|_{F_0} \leq \frac{1}{b} \int_0^b \mathrm{d}b' \;\; \left\| \partial^{n+1} \pi_{b'}(e) - \partial^{n+1} \pi_0(e) \right\|_{F_0} \longrightarrow 0$$

$\square$

8.3 Application to our case

Finally, let us combine the formulae obtained in section 8.1 to show that our retraction has a removable singularity in the sense of Theorem 8.26. As advertised in the introduction, this involves compensating the pole divergence from Proposition 8.19 by a 'weight difference' between different levels of regularity.

Setup

As already indicated in Section 7.2 we will use the following sc-Banach spaces:

$$F_n = W^{n+1,2}_{V_-\,;\,\rho^n}\big(-1,0\big) \qquad E_n = W^{n+1,2}_{V_+\,;\,\rho^n}\big(0,1\big) \qquad \text{where } \rho = \tfrac{1}{|x|} > 1$$

With $n \geq 0$, we consider the sequence of maps

$$\partial^n \pi : (0,\epsilon) \oplus E_n \longrightarrow F_0$$
$$\partial^n \pi_a(u) := \frac{1}{a^n} \sum_{l=0}^{n} R_{\lambda(a)} \chi_{n,l} [\tau_{+1} \alpha \gamma] \cdot \tau_{2a} V_+^l u \ \in F_0 = W^{1,2}_{V_-}(-1,0)$$

Proposition 8.27 (Verifying the conditions of Theorem 8.26)
At fixed $u \in E_{n+1}$ the map $a \longmapsto \partial^{n+1}\pi_a u \in F_0$ is the derivative of $a \longmapsto \partial^n \pi_a u \in F_0$. Moreover, for all pairs $n,k \geq 0$ we observe the following:

- *$u \in E_{n+k}$ implies $\partial^n \pi_a(u) \in F_k$*
- *There exists $\mathcal{K}_{n,k} > 0$ independent of a such that for all $a \in (0,\infty)$:*
$$\big\|\partial^n \pi_a(\,\cdot\,)\big\|_{\mathcal{L}(E_{n+k},\,F_k)} \leq \mathcal{K}_{n,k}$$
- *For every fixed $u \in E_{n+k}$ we have $\lim_{a\to 0} \partial^n \pi_a u = 0$ in F_k*
- *The map $\partial^n \pi : (0,\epsilon) \oplus E_{n+k} \longrightarrow F_k$ is continuous*

Proof. Given a fixed $u \in E_{n+1} \subset W^{n+2,2}_{V_+}(0,1)$, Proposition 8.19 shows that the map $a \mapsto R_{\lambda(a)} \tau_{+1} \alpha \gamma \cdot \tau_{2a} u$ belongs to $C^{n+1}\Big((0,\epsilon), W^{1,2}_{V_-}(-1,0)\Big)$ with derivatives

$$\left(\frac{\partial}{\partial a}\right)^n R_{\lambda(a)} \tau_{+1} \alpha \gamma \cdot \tau_{2a} u \;=\; \partial^n \pi_a u$$
$$\frac{\partial}{\partial a} \partial^n \pi_a u \;=\; \left(\frac{\partial}{\partial a}\right)^{n+1} R_{\lambda(a)} \tau_{+1} \alpha \gamma \cdot \tau_{2a} u \;=\; \partial^{n+1} \pi_a u$$

Now let us work at fixed $a > 0$. Assuming $u \in E_{n+k}$ for some $k \geq 0$ ensures that one has

$$V_+^l u \in W^{k+1,2}_{V_+}(0,1) \qquad \text{for all } l \leq n$$

so Proposition 8.22 implies $\partial^n \pi_a u = \frac{1}{a^n} \sum_{l=0}^{n} R_{\lambda(a)} \chi_{n,l} \cdot \tau_{2a} V_+^l u \ \in W^{k+1,2}_{V_-}(-1,0)$. More precisely, the V_--derivatives of order $m = 0, ..., k+1$ are given by

$$V_-^m \big[\partial^n \pi_a u\big] = \frac{1}{a^n} \sum_{s=0}^{n+m} R_{\lambda(a)} L_s^{m,n} \cdot \tau_{2a} V_+^s u \ \in L^2(-1,0)$$
$$\text{with} \quad L_s^{m,n} = \sum_{\substack{j+l=s \\ 0 \leq j \leq m \\ 0 \leq l \leq n}} M_{m,j}\big[\chi_{n,l}\big] \ \in C_0^\infty\left(-\tfrac{3}{2}, -\tfrac{1}{2}\right)$$

We claim that these belong not only to $L^2(-1,0)$ but in fact to $L^2(-1,0)_{\rho^k}$, thus showing that $\partial^n \pi_a u \in F_k = W^{k+1,2}_{V_-\,;\,\rho^k}(-1,0)$.

Indeed, with $L \in C_0^\infty\left(-\frac{3}{2}, -\frac{1}{2}\right)$ and $w \in L^2(0,1)_{\rho^{n+k}}$ we calculate

$$\begin{aligned}
\int_{(-1,0)} \mathrm{d}x \, \left\|\rho^k \cdot R_{\lambda(a)} L \cdot \tau_{2a} w\right\|^2 &= \int_{I_-} \mathrm{d}x \, \left\|\tau_{2a}\rho^k \cdot R_{\lambda(a)} \left[\frac{L}{(h-1)^k}\right] \cdot \tau_{2a} w\right\|^2 \\
&= \int_{I_-+2a} \mathrm{d}x \, \left\|\rho^k \cdot R_{\lambda(a)} \underbrace{\left[\tau_{-2}\frac{L}{(h-1)^k}\right]}_{C_0^\infty\left(\frac{1}{2}, \frac{3}{2}\right)} \cdot w\right\|^2 \\
&= \int_{(0,1)} \mathrm{d}x \, \left\|\rho^{n+k} \, |x|^n \cdot R_{\lambda(a)} \left[\tau_{-2}\frac{L}{(h-1)^k}\right] \cdot w\right\|^2
\end{aligned}$$

where for the first equality we have used that with

$$h = \frac{2}{x+2} \in C^\infty\big((-2,0), \mathbb{R}_{>1}\big) \implies \frac{1}{h-1} = -\frac{x+2}{x} \in C^\infty\big((-2,0), \mathbb{R}_{>0}\big)$$

one has $$\rho = -\frac{1}{x} = -\frac{1}{x+2a}\frac{x+2a}{x} = \tau_{2a}\rho \cdot R_{\lambda(a)}\left[\frac{1}{h-1}\right] \in C^\infty\big(-2a, 0\big).$$

Our calculation shows that

$$\left\|\frac{1}{a^n} R_{\lambda(a)} L \, \tau_{2a} w\right\|_{L^2(-1,0)_{\rho^k}} \leq 2^n \cdot \sup \left|\frac{L}{(h-1)^k}\right| \cdot \left\|\chi_{(0,2a)} \cdot w\right\|_{L^2(0,1)_{\rho^{n+k}}} \tag{8.5}$$

By invoking dominated convergence, this inequality implies $\lim\limits_{a\to 0} \left\|\frac{1}{a^n} R_{\lambda(a)} L \, \tau_{2a} w\right\|_{L^2(-1,0)_{\rho^k}} = 0.$

Since as remarked above all V_--derivatives of $\partial^n \pi_a u$ are built from terms of the form "$R_{\lambda(a)} L \cdot \tau_{2a} w$", we conclude that at any fixed $u \in E_{n+k}$ one has $\lim\limits_{a\to 0} \partial^n \pi_a u = 0$ in F_k.

On the other hand, (8.5) also implies

$$\left\|\frac{1}{a^n} R_{\lambda(a)} L \, \tau_{2a} w\right\|_{L^2(-1,0)_{\rho^k}} \leq K_{n,k} \cdot \|w\|_{L^2(0,1)_{\rho^{n+k}}}$$

where the constant $K_{n,k} > 0$ depends on n, k and L but not on a. Thus, after collecting all summands, we can find uniform bounds

$$\left\|\partial^n \pi_a(\,\cdot\,)\right\|_{\mathcal{L}(E_{n+k}, F_k)} \leq \mathcal{K}_{n,k}$$

Given a fixed $u \in E_{n+k}$, Lemma 8.28 ensures that the map $a \in (0,\epsilon) \longmapsto \partial^n \pi_a u \in F_k$ is continuous. Hence, by

$$\left\|\partial^n \pi_{a+\delta a}(u+\delta u) - \partial^n \pi_a u\right\|_{F_k} \leq \left\|\partial^n \pi_{a+\delta a} u - \partial^n \pi_a u\right\|_{F_k} + \underbrace{\left\|\partial^n \pi_{a+\delta a}\right\|_{\mathcal{L}(E_{n+k}, F_k)}}_{\leq \mathcal{K}_{n,k}} \left\|\delta u\right\|_{E_{n+k}}$$

we conclude that $\partial^n \pi : (0,\epsilon) \oplus E_{n+k} \longrightarrow F_k$ is continuous as well. □

In the last step, we have used the following result:

Lemma 8.28 (The weight does not destroy continuity of our map)
Given $f \in C_0^\infty(-\frac{3}{2}, -\frac{1}{2})$ and $v \in L^2(0,1)$ the map $a \in (0,\epsilon) \longmapsto R_{\lambda(a)} f \, \tau_{2a} v \in L^2_{\rho^n}(-1,0)$ is continuous.

Proof. Working with $a \in (a_0 \pm \Delta a)$ ensures that $R_{\lambda(a)} f$ is supported in the "comoving interval" $I_- = I_-(a_0)$.
On I_- one has $\|w\|_{L^2_{\rho^n}(I_-)} \leq C_{I_-}^n \, \|w\|_{L^2(I_-)}$ where $C_{I_-} := \sup\limits_{x\in I_-} \rho(x) < \infty$.
More explicitly, with $\rho = \frac{1}{|x|}$ and $I_- = [-r \cdot a_0, -s \cdot a_0]$ one observes $C_{I_-}^n \sim \frac{1}{a_0^n}$.
Using $\|\alpha \cdot w\|_{L^2(I_-)} \leq \|\alpha\|_{C^0_{bounded}(\mathbb{R}_{<0})} \|w\|_{L^2(I_-)}$ the claim is a combination of Lemmas 8.11 and 8.1.
□

Appendix A

Transition between real and complex sc-Hilbert spaces

A.1 Complexification of real Hilbert scales

Note that there are two a priori unrelated approaches to complexification, depending on whether we are dealing with a Hilbert space or only a Banach space:

- Hilbert space complexification:
  Given a real Hilbert space H, we equip its complexification $H^{\mathbb{C}} := H \otimes_{\mathbb{R}} \mathbb{C}$ with the hermitean product $\langle x \otimes \lambda, y \otimes \mu\rangle_H^{\mathbb{C}} := \bar{\lambda}\mu\langle x, y\rangle_H \in \mathbb{C}$.
  Setting $v = w$ makes the imaginary part of

  $$\langle v_0 + iv_1, w_0 + iw_1\rangle_H^{\mathbb{C}} = \big[\langle v_0, w_0\rangle_H + \langle v_1, w_1\rangle_H\big] + i\big[\langle v_0, w_1\rangle_H - \langle v_1, w_0\rangle_H\big]$$

  vanish, whereas from the real part we read off that the norm induced by $\langle \cdot, \cdot\rangle_H^{\mathbb{C}}$ is equivalent to $\|v_0 + iv_1\|_{H \oplus H} = \|v_0\|_H + \|v_1\|_H$ and therefore complete itself.
  Note that $\|\cdot\|_{H \oplus H}$ is a real but not a complex norm, by failure of $\|\lambda v\| = |\lambda| \cdot \|v\|$ for general $\lambda \in \mathbb{C}$. This does not prevent it from being equivalent to the complex norm induced by $\langle \cdot, \cdot\rangle_H^{\mathbb{C}}$.

- Banach space complexification:
  Given a real Banach space W, its complexification $W^{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$ can be understood as a complex Banach space by defining

  $$\|v\|_W^{\mathbb{C}} := \max_{t\in\mathbb{R}} \|\cos(t)v_0 - \sin(t)v_1\|_W = \max_{t\in\mathbb{R}} \big\| \operatorname{Re}\big[e^{it}v\big] \big\|_W \tag{A.1}$$

  By inserting $t = 0$ and $t = \frac{\pi}{2}$ we obtain

  $$\frac{1}{2}\big(\|v_0\|_W + \|v_1\|_W\big) \le \|v\|_W^{\mathbb{C}}$$

  whereas $|\cos(t)|, |\sin(t)| \le 1$ guarantees that

  $$\|v\|_W^{\mathbb{C}} \le \|v_0\|_W + \|v_1\|_W$$

  Thus, $\|\cdot\|_W^{\mathbb{C}}$ is equivalent to $\|\cdot\|_{W \oplus W}$ and therefore complete itself. However, unlike $\|\cdot\|_{W \oplus W}$, it succeeds in being a complex norm since by writing $\lambda = |\lambda| e^{i\theta}$ we observe that

  $$\|\lambda v\|_W^{\mathbb{C}} = |\lambda| \cdot \max_{t\in\mathbb{R}} \big\| \operatorname{Re}\big[e^{i(t+\theta)}v\big] \big\|_W = |\lambda| \cdot \|v\|_W^{\mathbb{C}}$$

  As mentioned in [MST], the choice (A.1) is known as the "Taylor complexification" of W.

Let $H \supset W_1 \supset \ldots$ be a filtration of real vector spaces. Complexification amounts to considering the filtration of real vector spaces given by $H \oplus H \supset W_1 \oplus W_1 \supset \ldots$ and regarding $H^{\mathbb{C}} = H \oplus H$ as a complex vector space with $i \in \mathbb{C}$ acting by

$$I = \begin{bmatrix} & -\mathrm{id}_H \\ \mathrm{id}_H & \end{bmatrix}$$

Since all higher levels are preserved by I in the sense that $I(W_k \oplus W_k) \subset W_k \oplus W_k$, we can regard the $W^{\mathbb{C}} = W_k \oplus W_k$ as complex subspaces of $H^{\mathbb{C}}$, leaving us with a filtration of complex vector spaces

$$H^{\mathbb{C}} \supset W_1^{\mathbb{C}} \supset \ldots \supset W_k^{\mathbb{C}} \supset \ldots$$

Now assume that our original $H \supset W_1 \supset \ldots$ was a filtration of real Banach spaces with bounded inclusions, the norm $\|\cdot\|_H$ arising from an inner product $\langle \cdot, \cdot \rangle_H$.
Then H gets complexified as a Hilbert space, whereas the higher levels $W_k^{\mathbb{C}}, k \geq 1$ are equipped with their Taylor norms (A.1). In any case, $\|\cdot\|_{W_k}^{\mathbb{C}}$ is equivalent to the real norm $\|(v_0, v_1)\|_{W_k \oplus W_k} = \|v_0\|_{W_k} + \|v_1\|_{W_k}$, so the filtration $H^{\mathbb{C}} \supset W_1^{\mathbb{C}} \supset \ldots$ has bounded inclusions as well.
The following simple observation will be used in section 5.1 Proposition 5.8:
Lemma A.1 (Complexification preserves Density and Compactness)
Given $H \supset W_1 \supset \ldots$ a filtration of Banach spaces with bounded inclusions such that $\|\cdot\|_H$ arises from an inner product, let us consider the filtration $H^{\mathbb{C}} \supset W_1^{\mathbb{C}} \supset \ldots$ described above. Then $\left(W_k^{\mathbb{C}}\right)_{k \geq 0}$ is an almost/honest sc-Banach space if and only if $(W_k)_{k \geq 0}$ was an almost/honest sc-Banach space.

Proof. Density: Note that

$$(W^{\mathbb{C}})_\infty = \bigcap_{k \geq 0} (W_k \oplus W_k) = \bigcap_{k \geq 0} W_k \oplus \bigcap_{k \geq 0} W_k = (W_\infty)^{\mathbb{C}}$$

are two descriptions of the same set. Since the norm on $W_k^{\mathbb{C}}$ is equivalent to $\|\cdot\|_{W_k \oplus W_k}$, we have an equivalence between W_∞ being dense in W_k and $(W_\infty)^{\mathbb{C}} = W_\infty \oplus W_\infty$ being dense in $W_k^{\mathbb{C}} = W_k \oplus W_k$.
Compactness: A subsequence argument shows that if $W_{k+1} \hookrightarrow W_k$ is compact, then so is $W_{k+1} \oplus W_{k+1} \hookrightarrow W_k \oplus W_k$. The reverse implication can be seen by considering sequences $(v_n, 0)$ with constant second entry. $\square$

A.2 Symmetrisation of complex Hilbert scales

As a reverse operation to the complexification process of the previous section, some situations will make it necessary to forget the complex structure and regard a given complex sc-Hilbert space $H \supset W_1 \supset \ldots$ as a real sc-Hilbert space $H^{\mathrm{Re}} \supset W_1^{\mathrm{Re}} \supset \ldots$
In this case, all sets and norms stay the same, the only material change being that instead of a hermitean form $\langle \cdot, \cdot \rangle_H$ the space $H^{\mathrm{Re}} = H$ now carries the symmetric inner product

$$\langle v, w \rangle_H^{\mathrm{Re}} := \mathrm{Re}\langle v, w \rangle_H = \frac{\langle v, w \rangle_H + \langle w, v \rangle_H}{2}$$

Still, $\langle \cdot, \cdot \rangle_H^{\mathrm{Re}}$ and $\langle \cdot, \cdot \rangle_H$ induce the same norm on H, so the transition from $\left(H, \langle \cdot, \cdot \rangle_H\right)$ to $\left(H, \langle \cdot, \cdot \rangle_H^{\mathrm{Re}}\right)$ will not affect questions of density or compactness.

Bibliography

[HWZ21] H. Hofer, K. Wysocki, and E. Zehnder. *Polyfold and Fredholm Theory.* Springer Verlag, 2021.

[HWZ07] H. Hofer, K. Wysocki, and E. Zehnder. "A general Fredholm theory. I. A splicing-based differential geometry". In: *J. Eur. Math. Soc. (JEMS)* 9.4 (2007), pp. 841–876.

[HWZ10] H. Hofer, K. Wysocki, and E. Zehnder. "Sc-smoothness, retractions and new models for smooth spaces". In: *Discrete Contin. Dyn. Syst.* 28.2 (2010), pp. 665–788.

[RS] J. Robbin and D. Salamon. "The spectral flow and the Maslov index". In: *Bulletin of the London Mathematical Society* 27.1 (1995), pp. 1–33.

[FW] U. Frauenfelder and J. Weber. "The shift map on Floer trajectory spaces". In: *Journal of symplectic geometry* 19.2 (2021), pp. 351–397.

[We] K. Wehrheim. *Fredholm notions in scale calculus and Hamiltonian Floer theory.* arXiv:1209.4040, (2012).

[FFGW] O. Fabert, J. Fish, R. Golovko, and K. Wehrheim. "Polyfolds: A First and Second Look". In: *EMS Surv. Math. Sci.* 3 (2016), pp. 131–208.

[Sa] D. Salamon. "Lectures on Floer homology". In: *Symplectic geometry and topology (IAS/Park City Math. Ser.)* Vol. 7. Providence, USA: Amer. Math. Soc., 1999, pp. 143–229.

[Wa] N. Waterstraat. "Fredholm Operators and Spectral Flow". In: *Rend. Sem. Mat. Univ. Politec. Torino* 75 (2017), pp. 7–51.

[Te] G. Teschl. *Mathematical Methods in Quantum Mechanics.* American Mathematical Society, 2009.

[Ka] T. Kato. *Perturbation Theory for Linear Operators.* Springer Verlag, 1976.

[ReSi] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I, Functional Analysis.* Academic Press, 1980.

[Ad] R. A. Adams. *Sobolev Spaces.* 2nd ed. Academic Press, 2003.

[MST] G. Munoz, Y. Sarantopoulos, and A. Tonge. "Complexifications of real Banach spaces, polynomials and multilinear maps". In: *Studia Mathematica* 134.1 (1999), pp. 1–33.

Danksagung

Herzlich danken möchte ich zunächst meinem Betreuer, Prof. Dr. Peter Albers, für seine Ermutigung und Geduld sowie für wertvolle Anregungen und Hinweise. Darüber hinaus gilt ein herzlicher Dank meinen Eltern, Großeltern und Freunden, insbesondere aber meiner Mutter Regine und Großmutter Ilse für ihre liebevolle Fürsorge und Unterstützung, ohne die diese Arbeit nicht möglich gewesen wäre.

Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den