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On Lerman's Cut Construction

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1 Summary

On the way through literature the author stumbled across Lerman's paper *Contact cuts* [Ler01], originally just to understand the broader context in some other work. However, besides the motivation of merely comprehending the general concept, he realized that the proofs presented by Lerman are quite tricky and leave out many necessary information. Thus, the idea arose to 'round off' Lerman's arguments in [Ler01] and check his proofs in detail. As a consequence, many but not all of his theorems can be found in this thesis, sometimes slightly modified or even corrected. Unfortunately, this also means that there is a motley assortment of propositions, not all of them are interconnected in a way the authour wished for in the first place, especially since there will be no final application of the cut construction: This was initially not intended and, additionally, the author simply lacks the knowledge in contact geometry to understand the last section in *Contact cuts*. Instead, the style below will be considerably technical with little to no examples or motivation and it is strongly recommended to at least take a look at Lerman's paper while reading the thesis. Nonetheless, the author hopes to still make a small contribution by reviewing Lerman's cut construction and correcting details if necessary, and at best help the reader to better understand the subject matter of *Contact cuts*.

Here is a brief outline of what we will consider in the separate chapters:

Chapter 2 has two main purposes: To introduce the reader to smooth group actions on manifolds, supplying him or her with the fundamental theorems in this field that the advanced reader may already know, and establishing the basic results on *slices*. This concept and the derived propositions will be essential for exactly one theorem, 4.3.3, which tells us that under certain conditions we always can choose an invariant contact form (under the group action) for a given contact distribution. This is significant because, when we consider contact reduction in Section 4.3, the invariance of the contact form becomes the essential condition. But, since Theorem 4.3.3 holds, we can always choose an appropriate contact form.

In Chapter 3 we turn to our main source [Ler01] and present the construction of topological cuts, note that the concrete definition of the smooth structure is only given in the proof of Proposition 3.2.1, so it is inevitable to read that one too. Afterwards, we generalize by cutting along the boundary of a manifold with boundary.

We will turn away briefly from the cut construction in Chapter 4 to study symplectic and contact reduction. In particular, we are going to give complete proofs of the famous Marsden-Weinstein-Meyer Theorem and an analog contact version by Geiges.

When we have studied reduction, we can finally define a canonical symplectic respectively contact form on the cut in Chapter 5 and obtain Lerman's symplectic respectively contact cut. The goal of the last section 5.3 is, to develop two helpful propositions which combine symplectic and contact cuts. They can also be found in [Ler01] but, as we will point out later, one of them most probably is not entirely correct in Lerman's paper.

Some propositions of Chapters 2 and 3 have been outsourced into the appendix if they have appeared to be too off-topic or are interesting results by themselves but cannot be included coherently into the main content.

2 Fundamental Results on Group Actions on Manifolds

In this chapter we will recapitulate basic definitions and theorems regarding group actions on topological spaces in general but also prove further, crucial results that we will use later in Chapter 4.

Therefore, most of the first section in this chapter, containing the more general definitions, should be well-known and, hence, the familiar reader may skim over it. However, since the introduced terminology is highly dependent on the author, with the aim of avoiding unnecessary misunderstandings, perhaps it is still useful to look at the definitions at least. In addition, in Section 2.1 we will introduce notation recurring later on. It is declared intention to keep the naming conventions very close to *Introduction to Smooth Manifolds* [Lee13], the standard work by Lee, which also happens to be the origin of nearly all important results in the first section.

In Section 2.3 we introduce the concept of slices for group actions. Our main source will be Palais [Pal61]. It is in this section that we will prove our main, relative to the rest of the chapter lesser-known, propositions and lemmata from topological group theory, which will (only) be important for one preliminary theorem in Chapter 4 and of course for the Quotient Manifold Theorem in this chapter.

As already indicated, the fourth section contains a rigid proof of the Quotient Manifold Theorem, which the reader probably already knows. Nonetheless, for completeness, the outstanding significance of the theorem in this work and in general and, because major parts of the proof have been elaborated in the preceding section anyway, we will still present it.

2.1 Group Actions on Manifolds

2.1.1 Topological Groups

A ***topological group*** G is a group endowed with a topology such that both $m : G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ and $i : G \to G$, $g \mapsto g^{-1}$ are continuous. If in addition G is a topological manifold and carries a smooth structure such that above maps are even smooth, it is called a ***Lie group***. We will often denote the neutral element in a topological group by e.

Let us first prove some basic results on topological groups.

Lemma 2.1.1 (see [Dik11])**.** *Let G be a topological group and $V \subseteq G$ be a neighborhood of $e \in G$. Then there exists an open neighborhood U of e in G with:*

(i) $U \subseteq U \cdot U \subseteq V$ *and*
(ii) $U^{-1} \subseteq V$

Proof. We first show that for any given open neighborhood V of e in G there exist U_1 and U_2, open neighborhoods of e respectively, with $U_1 \subseteq U_1 \cdot U_1 \subseteq V$ and $U_2^{-1} \subseteq V$.
This is indeed the case: The preimage $m^{-1}(V)$ is open in $G \times G$ and since the products of open sets form a base of the product topology of $G \times G$, we can choose open sets $\tilde{U}_1, \tilde{U}_2$ satisfying $(e,e) \in \tilde{U}_1 \times \tilde{U}_2 \subseteq m^{-1}(V)$. Now define $U_1 := \tilde{U}_1 \cap \tilde{U}_2$. Clearly U_1 is open and, since it contains e, $U_1 \subseteq U_1 \cdot U_1$ also holds. Furthermore, $U_1 \cdot U_1 = m(U_1 \times U_1) \subseteq m(\tilde{U}_1 \times \tilde{U}_2) \subseteq V$. Let U_2 be $i^{-1}(V)$. This proves our first statement.
Now let V be an arbitrary neighborhood of e. By considering the interior of V, me may assume that V is open. Choose an open set U_1 fulfilling $e \in U_1 \subseteq U_1 \cdot U_1 \subseteq V$. Now let U_2 be an open neighborhood of e with $U_2^{-1} \subseteq U_1$. Define U to be $U_1 \cap U_2$. Then U is an open neighborhood of e and satisfies (i) and (ii). □

Corollary 2.1.2. *For every neighborhood $V \subset G$ of $e \in G$ in a topological group G there exists an open neighborhood U of e with $UU \cup UU^{-1} \subseteq V$. In particular, $U \subseteq V$ and $U^{-1} \subseteq V$.*

Proof. By the preceding lemma we can choose an open $e \in V' \subseteq G$ with $V'V' \subseteq V$ and $V'^{-1} \subseteq V$. Again, using the same lemma, there exists an open neighborhood U of e with $UU \subseteq V' \cap V'^{-1}$, $U^{-1} \subseteq V' \cap V'^{-1}$. Then clearly $UU \cup UU^{-1} \subseteq V$ holds. □

Lemma 2.1.3. *For an open subset U in a topological group G and any subset $V \subset G$ the sets VU and UV are open.*

Proof. For each $v \in V$ the set vU (respectively Uv) is open because it is the image of the open set U under the homeomorphism $G \to G$, $g \mapsto vg$ (respectively $g \mapsto gv$). Thus,

$$VU = \bigcup_{v \in V} vU \quad \text{and} \quad \bigcup_{v \in V} Uv$$

are open in G. □

Lemma 2.1.4 (see [Dik11])**.** *Let G be a topological group and $K \subseteq G$ a compact subset.*

(a) If $C \subseteq G$ is a closed subset, then CK and KC are both closed.

(b) If $C \subseteq G$ is compact in G, then CK and KC are both compact.

(c) If V is an open neighborhood of K (i.e. V open and $K \subseteq V$), then there exists an open neighborhood U of e satisfying $UK \subseteq V$.

Proof.
(a) We only show that CK is closed, the other case works analogously. For that, it suffices to prove that, if $(x_\alpha)_{\alpha \in A}$ is a net contained in CK and $x \in CK$ is a limit point of $(x_\alpha)_{\alpha \in A}$, then $x \in CK$. For every $\alpha \in A$ choose $c_\alpha \in C$ and $k_\alpha \in K$ such that $x_\alpha = c_\alpha k_\alpha$. Since K is compact, there exists a convergent subnet $(k_{\alpha_\beta})_{\beta \in B}$ with a limit point $k \in K$. Hence, $(x_{\alpha_\beta}, k_{\alpha_\beta}) \to (x,k)$ in $G \times G$ (It is trivial that, if X is an arbitrary topological space, $Y \subseteq X$ is equipped with the subspace topology, $(y_i)_{i \in I}$ is a net in X, contained in Y, and $y \in Y$, then $y_i \to y$ in X iff $y_i \to y$ in Y. Thus, $k_{\alpha_\beta} \to k$ in K implies $k_{\alpha_\beta} \to k$ in G. Then use the fact that convergence in a product space is equivalent to pointwise convergence.) and, since m and i are continuous, also $c_{\alpha_\beta} = x_{\alpha_\beta} k_{\alpha_\beta}^{-1} \to xk^{-1}$ in G. Since C is closed and $(c_{\alpha_\beta})_{\beta \in B} = (x_{\alpha_\beta} k_{\alpha_\beta}^{-1})_{\beta \in B}$ is a net in C, it follows that $xk^{-1} \in C$ and therefore $x = (xk^{-1})\, k \in CK$.
(b) Products of compact sets are compact, so $C \times K$ is compact. Since m is continuous, $CK = m(C \times K)$ is compact. Observe, that this argument holds because the product topology and the subspace topology on $C \times K$ coincide, which is a general topological fact.
(c) $V^C = G \setminus V$ is disjoint with K, therefore $e \notin V^C K^{-1}$. Since $i(K) = K^{-1}$ is compact and V^C is closed, $V^C K^{-1}$ is closed by (a). Set $U := (V^C K^{-1})^C$. Then U is an open neighborhood of e. Now assume $UK \not\subseteq V$. Then there exists $x = uk \in UK \cap V^C$, $u \in U$, $k \in K$. Thus, $u = xk^{-1} \in V^C K^{-1} \cap U$, which is a contradiction.
□

2.1.2 Group Actions

Let G be a topological group, operating on a topological space X via the (algebraic) group action $\theta : G \times X \longrightarrow X$. If θ is continuous, then we call θ a ***continuous action*** and X a ***G-space***. If additionally G is a Lie group, X a smooth manifold (from now on the term 'manifold' without further qualification means a smooth manifold) and θ is a smooth map, then θ is a ***smooth action*** and X is a ***smooth G-space***. To keep the notation brief, we use the convention that, whenever we speak of an action and the corresponding group G is a topological one, respectively a Lie group, and the space on which the group acts is a topological space, respectively a manifold, it is a continuous action, respectively a smooth action. Furthermore, if the action is apparent from the context, then we will often abbreviate: $g \cdot x := \theta(g,x)$, $g \in G$, $x \in X$.

Let $\theta : G \times X \longrightarrow X$ be a continuous action of G on X. For any $g \in G$ we denote by

$$\theta_g : X \longrightarrow X\,, \quad x \mapsto \theta(g,x)$$

the action of g on X. Then we have the natural embedding

$$\left\{\begin{matrix}\text{continuous}\\ G\text{-actions on } X\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{group homomorphisms}\\ G \to \mathrm{Homeo}(X)\end{matrix}\right\}$$

$$\theta \longmapsto \begin{cases} G \to \mathrm{Homeo}(X) \\ g \mapsto \theta_g \end{cases}$$

(respectively even 'Diffeo(X)' in the smooth case).
For $x \in X$ let

$$\theta^{(x)} : G \longrightarrow X\,, \quad g \mapsto \theta(g,x)$$

and the ***orbit*** of x will be denoted by

$$G \cdot x := \big\{\theta(g,x) \bigm| g \in G\big\} = \theta^{(x)}(G)\ .$$

Finally, the notation for the ***stabilizer*** , respectively the ***isotropy group*** , of x is

$$G_x := \big\{g \in G \bigm| \theta(g,x) = x\big\} = \big(\theta^{(x)}\big)^{-1}(x). \tag{2.1}$$

Clearly, θ_g and $\theta^{(x)}$ are both continuous (respectively smooth if θ is a smooth action). Therefore, if X is a T_1-space, then $\{x\}$ is closed in X and, thus, by equation (2.1) the stabilizer of x is closed in G. It follows that, if X is a smooth G-space, then G_x is a closed subgroup of G and, by the closed subgroup theorem ([Lee13, Corollary 20.13]), G_x is a Lie supgroup of G ('submanifold' and 'Lie subgroup' mean that these structures are endowed with a smooth structure such that the inclusion map is an embedding, i.e. a smooth immersion that is also a topological embedding; note, that these definitions are not compatible with [Lee13, Corollary 20.13] where 'submanifold' respectively 'Lie subgroup' without any further qualification denote what we will refer to as 'immersed submanifold' respectively 'immersed Lie group', i.e. a subspace endowed with a topology and smooth structure such that the inclusion map is a smooth immersion).

Let

$$X/G := \big\{G \cdot x \bigm| x \in X\big\}$$

be the ***orbit space*** of the action. It is the quotient set of X by the equivalence relation $\sim$, where $x \sim y :\Leftrightarrow x \in G \cdot y$. We equip X/G with the quotient topology, given by the canonical projection $\Pi_X : X \to X/G$.

Lemma 2.1.5. *Let θ be a continuous action of G on X. Then the quotient map $\Pi_X : X \to X/G$ is an open map.*

Proof. Let U be an arbitrary open subset of X. Then

$$\Pi_X^{-1}(\Pi_X(U)) = G \cdot U = \{g \cdot x \mid g \in G,\, x \in U\} = \bigcup_{g \in G} g \cdot U$$

and $g \cdot U = \theta_g(U)$. Since θ_g is a homeomorphism for any $g \in G$, every set $g \cdot U$ is open and therefore so is its union $\Pi_X^{-1}(\Pi_X(U))$. Thus, by definition of the quotient topology, $\Pi_X(U)$ is open in X/G. □

If X is a smooth G-space, the natural question arises, whether the orbit space carries a canonical smooth structure such that the projection Π_X is smooth. However, in the general case X/G does not even have to be Hausdorff and is therefore definitely not a topological manifold in these cases.

Example 2.1.6 (see [Lee13, Example 21.2 (d)]).
Let $G := \mathrm{GL}(n, \mathbb{R})$ act on $X := \mathbb{R}^n$ via $(A, x) \mapsto Ax$. The orbit through $0 \in \mathbb{R}^n$ is $\{0\}$. Let $x, y \neq 0$ be arbitrary. Then there are ordered bases $\mathcal{B} = (x, b_2, \ldots, b_n)$ and $\mathcal{C} = (y, c_2, \ldots, c_n)$, containing x respectively y and a linear isomorphism mapping $\mathcal{B}$ onto $\mathcal{C}$. The transformation matrix A of this isomorphism is regular and $Ax = y$. Hence, $y \in G \cdot x$. In summary, we have shown that $X/G = \{\,\{0\}\,,\, \mathbb{R}^n \backslash \{0\}\,\}$ and thus the quotient topology is given by $\tau_{X/G} = \{\,\emptyset\,,\, X/G\,,\, \{\mathbb{R}^n \backslash \{0\}\}\,\}$. The only open neighborhood of $\{0\}$ in X/G is X/G and therefore X/G is not Hausdorff.

Due to this complication we would like to introduce a concept, such that we get at least a sufficient condition for the Hausdorff property of the orbit space.
Recall that a map $F : X \to Y$ between two topological spaces is said to be proper if the preimage of any compact subset $K \subseteq Y$ is compact in X.

Definition 2.1.7. Let G be a Hausdorff topological group acting on the Hausdorff space X via the continuous action θ. Then θ is called a ***proper action*** if the map $G \times X \to X \times X\,,\, (g, x) \mapsto (g \cdot x, x) = (\theta(g, x), x)$ is proper.

Proposition 2.1.8. *Let $\theta : G \times X \to X$ be a continuous action and G and X be Hausdorff. If θ is a proper map, then it is a proper action.*

Proof. Since products of Hausdorff spaces are itself Hausdorff, $G \times X$ and $X \times X$ are Hausdorff. Let $K \subseteq X \times X$ be an arbitrary compact subset. Then it is closed in $X \times X$ (because compact subsets in Hausdorff spaces are closed). By the universal property of product spaces, the map $\Theta : G \times X \to X \times X\,,\, (g, x) \mapsto (g \cdot x, x)$ is continuous and therefore $\Theta^{-1}(K)$ is closed in $G \times X$. Clearly, $\Theta^{-1}(K) \subseteq \theta^{-1}(\mathrm{pr}_1(K))$, where $\mathrm{pr}_1 : X \times X \to X$ is the projection onto the first factor. Since continuous images of compact sets are compact and by assumption θ is proper as map, $\theta^{-1}(\mathrm{pr}_1(K))$ is compact. Closed subsets of compact spaces are compact, so $\Theta^{-1}(K)$ is compact. □

Proposition 2.1.9. *Suppose the Lie group G acts continuously on the manifold M via $\theta : G \times M \to M$. If θ is a proper action, then the orbit space M/G is Hausdorff.*

Proof. Again, let Θ denote the proper map $(g, p) \mapsto (g \cdot p, p)\,,\, g \in G\,,\, p \in M$, and Π_M the orbit map. We define the relation $\mathcal{R} \subseteq M \times M$ by

$$\mathcal{R} = \Theta(G \times M)\ .$$

Then $(p, q) \in \mathcal{R} \Leftrightarrow \Pi_M(p) = \Pi_M(q)$, for all $(p, q) \in M \times M$. Θ is closed as a proper map between $G \times M$ and the topological manifold $M \times M$ (cf. Lemma A.1.2). Hence, $\mathcal{R}$ is closed in $M \times M$. The proposition follows from A.1.1 (note that Π_M is an open quotient map by Lemma 2.1.5). □

Proposition 2.1.10. *Let θ be a continuous action of a Lie group G on the manifold M. Then the following are equivalent:*

(i) *θ is a proper action.*
(ii) *For all nets $(g_i)_{i\in I}$ in G and $(p_i)_{i\in I}$ in M the following holds:*
$(p_i)_{i\in\mathbb{I}}$ and $(g_i\cdot p_i)_{i\in I}$ both converge
$\Longrightarrow$ a subnet of $(g_i)_{i\in I}$ converges
(iii) *For all sequences $(g_n)_{n\in\mathbb{N}}$ in G and $(p_n)_{n\in\mathbb{N}}$ in M the following holds:*
$(p_n)_{n\in\mathbb{N}}$ and $(g_n\cdot p_n)_{n\in\mathbb{N}}$ both converge
$\Longrightarrow$ a subsequence of $(g_n)_{n\in\mathbb{N}}$ converges
(iv) *For every compact subset $K\subseteq M$ the set*
$G_K := ((K,K)) := \{g\in G \mid (g\cdot K)\cap K \neq \varnothing\}$ is compact.

Proof. As before, put $\Theta : G\times M \to M\times M$, $(g,p)\mapsto(g\cdot p,p)$, so that (i) is equivalent to Θ being a proper map. We will show (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i).

(i) $\Rightarrow$ *(ii)*: Suppose $p_i\to p$ and $g_i\cdot p_i\to q$ in M. Choose compact neighborhoods K_p , K_q of p respectively q. Then there is some $i_0\in I$ such that $p_i\in K_p$, $g_i\cdot p_i\in K_q\quad\forall\, i\succeq i_0$ (where $\succeq$ denotes the relation belonging to the directed set I). Define $J:=\{i\in I\mid i\succeq i_0\}$, then $(p_i)_{i\in J}$ and $(g_i\cdot p_i)_{i\in J}$ are subnets that lie in K_p , K_q respectively. Hence, $(g_i,p_i)\in\Theta^{-1}(K_q\times K_p)$ for $i\in J$. By assumption $\Theta^{-1}(K_q\times K_p)$ is compact, thus there exists a subnet $((g_{i_\alpha},p_{i_\alpha}))_{\alpha\in A}$ of $((g_i,p_i))_{i\in J}$, which converges in $G\times M$. In particular $(g_{i_\alpha})_{\alpha\in A}$ is a convergent subnet of $(g_i)_{i\in I}$.

(ii) $\Rightarrow$ *(iii)*: If a sequence, contained in a manifold, has a convergent subnet, then it also has a convergent subsequence (cf. Lemma A.1.3).

(iii) $\Rightarrow$ *(iv)*: Any subset of a manifold is second countable Hausdorff, so compactness and sequential compactness are equivalent for such a subset. Therefore, it suffices to show that each sequence $(g_n)_{n\in\mathbb{N}}$ in G_K has a convergent subsequence with limit in G_K: By definition of G_K, we can choose a sequence $(p_n)_{n\in\mathbb{N}}$ with $p_n\in(g_n\cdot K)\cap K$, thus $p_n\in K$ and $g_n^{-1}\cdot p_n\in K$. Since K is compact, we can assume that (p_n) and $(g_n^{-1}\cdot p_n)$ converge in K (first take a convergent subsequence of (p_n), say (p_{n_k}), then take a converging subsequence of $(g_{n_k}^{-1}\cdot p_{n_k})$, say $(g_{n_{k_m}}^{-1}\cdot p_{n_{k_m}})$; then we find a convergent subsequence of $(g_{n_{k_m}})$, which is also a convergent subsequence of (g_n)). By (iii) we can choose a convergent subsequence $(g_{n_k}^{-1})$. Thus, (g_{n_k}) is the requested subsequence. Let $g_{n_k}\to g$, $p_{n_k}\to p\in K$. Then $g_{n_k}^{-1}\cdot p_{n_k}\to g^{-1}\cdot p\in K$ and hence $g\in G_K$.

(iv) $\Rightarrow$ *(i)*: Let $L\subseteq M\times M$ be an arbitrary compact subset. Let pr_1 , $\mathrm{pr}_2 : M\times M\to M$ denote the natural projections on the first respectively second factor. Then $K:=\mathrm{pr}_1(L)\cup\mathrm{pr}_2(L)$ is compact as a finite union of compact sets. Clearly, $L\subseteq K\times K$ holds. If $(g,p)\in\Theta^{-1}(K\times K)$, then $g\cdot p\in(g\cdot K)\cap K\neq\varnothing$, so $(g,p)\in G_K\times K$. Hence,

$$\Theta^{-1}(L)\subseteq\Theta^{-1}(K\times K)\subseteq G_K\times K.$$

By (iv), $G_K\times K$ is compact and therefore $\Theta^{-1}(L)$ is compact as closed set (it is the continuous preimage of the compact, and thus closed, set L) in a compact one. □

Corollary 2.1.11. *Suppose θ is a continuous action of a compact Lie group G on the manifold M. Then θ is a proper action.*

Proof. Since G is compact, every sequence in G has a convergent subsequence, so condition (iii) in Proposition 2.1.10 holds.
(Alternatively, this statement follows directly from Lemma A.1.4 with Proposition 2.1.10 (iv).) □

Proposition 2.1.12. *Let $\theta : G\times M\to M$ be a smooth and proper action of a Lie group G on the manifold M and $p\in M$ be an arbitrary point. Then the following hold:*

(a) The orbit map $\theta^{(p)} : G\to M$ is a proper map.

(b) The orbit $G\cdot p$ is closed in M.

(c) The isotropy group G_p is compact.

(d) If $G_p = \{e\}$, then $\theta^{(p)}$ is a smooth embedding and $G \cdot p$ is a submanifold with proper inclusion map $\iota : G \cdot p \hookrightarrow M$ (i.e. the orbit is a properly/closed embedded submanifold).

Proof.
(a) Let $K \subseteq M$ be an arbitrary compact subset. K is closed in M, thus $(\theta^{(p)})^{-1}(K)$ is closed in G. If $g \cdot p \in K$, then $g \cdot p \in (g \cdot (K \cup \{p\})) \cap (K \cup \{p\}) \neq \varnothing$, so $(\theta^{(p)})^{-1}(K) \subseteq G_{K \cup \{p\}}$. By Proposition 2.1.10 (iv), $G_{K \cup \{p\}}$ is compact, hence $(\theta^{(p)})^{-1}(K)$ is compact too.

(b) Follows directly from (a) and Lemma A.1.2.

(c) Set $K := \{p\}$. Then the isotropy group G_p coincides with $G_K = ((K, K))$, which is compact by Proposition 2.1.10 (iv).

(d) If $\theta^{(p)}(g_1) = \theta^{(p)}(g_2)$, then $g_2^{-1} g_1 \cdot p = p$ and thus $g_2^{-1} g_1 \in G_p = \{e\}$. Hence, $g_1 = g_2$ and so $\theta^{(p)}$ is injective. Consider the transitive G-action on G through left multiplication. Clearly, $\theta^{(p)}$ is G-equivariant with this action. By the Equivariant Rank Theorem (cf. Lemma A.1.5 for the theorem and the definition of 'equivariance'), $\theta^{(p)}$ has constant rank. Since it is also injective, it is an immersion by the Global Rank Theorem (cf. [Lee13, Theorem 4.14]). By Lemma A.1.2, it is closed, so in total it is indeed a smooth embedding. Therefore, $G \cdot p = \theta^{(p)}(G)$ is an (embedded) submanifold of M. For any compact subset $K \subseteq M$ we have $\iota^{-1}(K) = K \cap (G \cdot p) = \theta^{(p)}((\theta^{(p)})^{-1}(K))$ and, because $\theta^{(p)}$ is proper, it follows that $\iota^{-1}(K)$ is compact.

□

Even if the action is neither proper nor free, there is a natural smooth structure on the orbit through p. This case will be covered by the next proposition. However, the argument strongly depends on the Quotient Manifold Theorem 2.4.4, although the required version is slightly weaker than the one we will prove later in this chapter. For a more detailed explanation we refer to the preface of Section 2.4 and the appendix A.1.
In addition, we need the well-known fact that a subgroup H of a Lie group G is a Lie subgroup (i.e. a subgroup, endowed with a Lie group structure, such that the inclusion map is an embedding) if and only if it is closed in G. Therefore, using the Quotient Manifold Theorem for the right action of H on G, if H is closed in G, there is a unique smooth structure on the quotient group G/H for which the canonical projection $\Pi : G \to G/H$ is a smooth submersion.
With the natural transitive G-action on G/H, given by

$$g_1 \cdot (g_2 H) := (g_1 g_2) H\,, \;\; g_1, g_2 \in G\,, \tag{2.2}$$

G/H becomes a smooth G-space.
In particular, since the stabilizer G_p of $p \in M$ is a closed subgroup in G, the quotient group G/G_p is a homogeneous G-space (i.e. the action, operating on the manifold, is smooth and transitive) in a natural way.

Proposition 2.1.13. *Given a smooth action θ of the Lie group G on the manifold M and a point p in M. Then there is a unique topological respectively smooth structure on the orbit $G \cdot p$ through p, such that $G \cdot p$ is homeomorphic respectively diffeomorphic to G/G_p via the canonical bijection (see equation (2.3)). With these structures $G \cdot p$ is an immersed submanifold of M. If in addition θ is a proper action, then $G \cdot p$ is a (properly embedded) submanifold of M.*

Proof. We have the canonical bijection

$$\begin{aligned} F : G/G_p &\xrightarrow{\sim} G \cdot p \\ g\, G_p &\longmapsto g \cdot p \end{aligned} \tag{2.3}$$

Clearly, there is a unique topology respectively unique smooth structure on $G \cdot p$, such that F becomes a homeomorphism respectively a diffeomorphism (simply the quotient topology induced by F and, if $\mathcal{A} := \{(U_i, \varphi_i)\}_{i \in I}$ is a smooth atlas of G/G_p, the smooth structure on $G \cdot p$ is induced by the atlas $\{(F(U_i), \varphi \circ F^{-1})\}_{i \in I}$). To show that $G \cdot p$ is an immersed

submanifold of M, it suffices to prove that the map $G/G_p \hookrightarrow M$, $g\,G_p \mapsto g \cdot p$ is a smooth immersion. By the universal property of submersions this map is smooth if and only if the map $G \to M$, $g \mapsto g \cdot p$ is smooth, which is obviously the case. By the Equivariant Rank Theorem A.1.5 the map $G/G_p \hookrightarrow M$ has constant rank and, since it is injective, is an immersion by the Global Rank Theorem.
Now suppose that θ is a proper action. The map $\theta^{(p)}|^{G\cdot p} : G \to G\cdot p$, $g \mapsto g\cdot p$ is continuous (with respect to the topology of the manifold $G \cdot p$) if and only if the map $G \to G/G_p$, $g \mapsto g\,G_p$ is continuous, which indeed is true. We have the following commutative diagram of continuous functions:

$$\begin{tikzcd}
G \arrow[rr, "\theta^{(p)}"] \arrow[dr, "\theta^{(p)}|^{G\cdot p}"'] & & M \\
& G \cdot p \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

where $\iota : G \cdot p \hookrightarrow M$ is the inclusion map. For any compact set $K \subseteq M$ we have $\iota^{-1}(K) = \theta^{(p)}|^{G\cdot p}((\theta^{(p)})^{-1}(K))$, which is compact because $\theta^{(p)}$ is a proper map by Proposition 2.1.12. Thus, the inclusion map is proper and by A.1.2 also closed, hence a proper embedding. □

Remark 2.1.14. Consider the situation of Proposition 2.1.13 with θ a proper action and $G_p = \{e\}$. Then the smooth structures on $G \cdot p$ of Propositions 2.1.12 and 2.1.13 coincide since for every submanifold $S \subseteq M$ there is exactly one topological and smooth structure on S such that S endowed with these structures is a submanifold of M.

2.2 Thin and Small Sets

A topological space X is called ***locally compact*** if every point $x \in X$ has a compact neighborhood. Recall that if X is Hausdorff, X is locally compact if and only if for every point $x \in X$ and every neighborhood U of x there exists a compact neighborhood $K \subseteq U$ of x.
We call a topological group G a ***locally compact group*** if it is a locally compact Hausdorff space. Note that every Lie group is a locally compact group.

For the rest of the section let G denote a locally compact group, acting continuously on the completely regular Hausdorff space (this is also called a Tychonoff space) X. Admittedly, this might seem like a needless generalization, since we only tend to use the results of this section in the restricted case of a Lie group acting on a smooth manifold. However, by premising weaker conditions, the applied techniques of the following proofs will be straightforward and, more importantly, compatible with the proofs in [Pal61].

Definition 2.2.1. Let the locally compact group G act continuously on the completely regular Hausdorff space X.

(a) For every pair of subsets $Y, Z \subseteq X$ we define:

$$((Y, Z)) := \{g \in G \mid (g \cdot Y) \cap Z \neq \varnothing\}$$

(b) $Y \subseteq X$ ***is thin relative to*** $Z \subseteq X$ if $((Y, Z))$ is precompact in G (i.e. $\overline{((Y, Z))}$ is compact in G).

(c) $Y \subseteq X$ is ***thin*** if Y is thin relative to Y.

(d) X is a ***Cartan G-space*** if every point of X has a thin neighborhood.

Lemma 2.2.2. *Suppose the locally compact group G acts continuously on the completely regular Hausdorff space X.*

(a) For any two subsets $Y, Z \subseteq X$:

$$Y \text{ thin relative to } Z \iff Z \text{ thin relative to } Y$$

Therefore, we will often just say that Y and Z are relatively thin.

(b) If Y is thin relative to Z, then $g_1 \cdot Y$ is thin relative to $g_2 \cdot Z$ for all $g_1, g_2 \in G$.

(c) All subsets $Y' \subseteq Y$ and $Z' \subseteq Z$ of relatively thin sets Y, Z are relatively thin. Hence, a subset of a thin set is thin itself.

(d) If Y_i is thin relative to Z_i, $i = 1, \dots n$, then $\bigcup_i Y_i$ is thin relative to $\bigcap_i Z_i$.

(e) For compact, relatively thin sets K_1, K_2, the set $((K_1, K_2))$ is compact.

(f) If G is compact, then any two subsets of X are thin relative to each other. In particular, every subset is thin.

Proof.

(a) Follows directly from $((Y,Z)) = ((Z,Y))^{-1}$. Due to symmetry, it suffices to show '$\subseteq$'. Suppose $(g \cdot Y) \cap Z \neq \varnothing$, say $g \cdot y \in Z$, $y \in Y$. Then $y \in (g^{-1} \cdot Z) \cap Y \neq \varnothing$ and thus $g^{-1} \in ((Z,Y))$.

(b) $(gg_1 \cdot Y) \cap (g_2 \cdot Z) = g_2 \cdot [\,(g_2^{-1} g g_1) \cdot Y \cap Z\,]$. Hence,

$$g \in ((g_1 \cdot Y\,,\, g_2 \cdot Z)) \iff g_2^{-1} g g_1 \in ((Y,Z)) \iff g \in g_2 \cdot ((Y,Z)) \cdot g_1^{-1}\ .$$

We conclude $((g_1 \cdot Y, g_2 \cdot Z)) = g_2 \cdot ((Y,Z)) \cdot g_1^{-1}$, which has compact closure because $g \mapsto g_2 g g_1^{-1}$ is a homeomorphism.

(c)
$$((Y', Z')) \subseteq ((Y,Z)) \subseteq \overline{((Y,Z))} \implies \overline{((Y',Z'))} \subseteq \overline{((Y,Z))}\ ,$$

and closed subsets of compact sets are compact, so $\overline{((Y',Z'))}$ is compact.

(d) If $(g \cdot \bigcup_i Y_i) \cap \bigcap_i Z_i \neq \varnothing$, then clearly $(g \cdot Y_i) \cap Z_i \neq \varnothing$ for some $i = 1, \dots, n$. Therefore $((\bigcup_i Y_i\,, \bigcap_i Z_i)) \subseteq \bigcup_i ((Y_i, Z_i))$. Because $\bigcup_i \overline{((Y_i, Z_i))}$ is compact as finite union of compact sets, the closure of $((\bigcup_i Y_i\,, \bigcap_i Z_i))$ is compact.

(e) Follows directly from Lemma A.1.4.

(f) Trivial.

□

Lemma 2.2.3. *In a Cartan G-space X every orbit is closed. Thus, X/G is a T_1-space. In addition, the isotropy group of $x \in X$ is compact.*

Proof. Let $x \in X$ be arbitrary. Choose a thin neighborhood V of x. Then we have $G_x \subseteq ((V,V))$ and, since G_x is closed and $((V,V))$ is precompact, G_x is compact. Now suppose an arbitrarily given net $g_i \cdot x$ in $G \cdot x$ converges to $y \in X$. Choose a thin neighborhood U of y. There exists some i_0 with $g_i \cdot x \in U$ for $i \succeq i_0$. Since $U \ni g_i \cdot x = (g_i g_{i_0}^{-1}) \cdot (g_{i_0} \cdot x) \in (g_i g_{i_0}^{-1}) \cdot U$, we can conclude $g_i g_{i_0}^{-1} \in ((U,U))$. Because U is thin, we can choose a converging subnet $(g_{i_j} g_{i_0}^{-1})_j$ with limit, say, $g g_{i_0}^{-1}$. Thus $g_{i_j} \to g$ and $g_{i_j} \cdot x \to g \cdot x$ and, since limits in Hausdorff spaces are unique, it follows that $y = g \cdot x \in G \cdot x$. So, every orbit is closed in X.
In particular, every singleton $\{G \cdot x\} \subseteq X/G$ is closed in X/G because $\Pi_X^{-1}(\{G \cdot x\}) = G \cdot x$ is closed in X. This however is equivalent to X/G being a T_1-space. □

Lemma 2.2.4. *Let X be a Cartan G-space, $x \in X$ and the orbit through x endowed with the subspace topology. Then the map $G \to G \cdot x$, $g \mapsto g \cdot x$ is an open continuous map.*

Proof. Clearly, above map is continuous as a restriction. We will show that, given any open subset W containing e, the set $W \cdot x$ is a neighborhood of x in $G \cdot x$.
Now let U be an arbitrary open subset in G. If for every $g \in U$ the set $U \cdot x$ is a neighborhood of $g \cdot x$ with respect to the subspace topology on $G \cdot x$, then $U \cdot x$ is open in $G \cdot x$. For $g \in U$ let $W := g^{-1}U$. Clearly, W is an open neighborhood of e and thus $W \cdot x$ is a neighborhood of x. As restrictions, the maps $\theta_g|_{G\cdot x}^{G\cdot x}$ and $\theta_{g^{-1}}|_{G\cdot x}^{G\cdot x}$ are continuous and therefore $\theta_g|_{G\cdot x}^{G\cdot x} : G\cdot x \to G\cdot x$ is a homeomorphism. So, $\theta_g|_{G\cdot x}^{G\cdot x}(W \cdot x) = g \cdot (W \cdot x) = (gW) \cdot x = U \cdot x$ is a neighborhood of $g \cdot x$ in $G \cdot x$.
It remains to show that for any open neighborhood W of e the set $W \cdot x$ is a neighborhood of x. Assume, $W \cdot x$ is no neighborhood of x. Then there exists a net $(g_i \cdot x)_{i \in I}$ with $g_i \cdot x \notin W \cdot x$ but $g_i \cdot x \to x$ (e.g. let I be the collection of all open neighborhoods of x in $G \cdot x$; this is a directed set with the inclusion relation).

Thus $g_i \notin WG_x \ \forall\, i \in I.$ $(*)$

Choose a thin neighborhood V of x. Then $g_i \in ((V,V)) \subseteq \overline{((V,V))}$ for i large enough, so there exists some subnet (g_{i_j}) that converges in G, say $g_{i_j} \to g \in G$. Thus $g \cdot x = \lim_j (g_{i_j} \cdot x) = x$ and therefore $g \in G_x$. Since W is open and contains e, the set WG_x is an open neighborhood of $g \in G_x \subseteq WG_x$. Hence, $g_{i_{j_0}} \in WG_x$ for some j_0, which contradicts $(*)$. □

Proposition 2.2.5. *Let X be a Cartan G-space and $x \in X$. Then the natural bijection*

$$F : G/G_x \to G \cdot x\,,\ gG_x \mapsto g \cdot x\,,$$

(cf. equation (2.3)*) is a homeomorphism if we endow $G \cdot x$ with the subspace topology.*

Proof. The map $\theta^{(x)}|^{G\cdot x} : G \to G \cdot x\,,\ g \mapsto g \cdot x$ is continuous. By the universal property of the quotient topology, the natural bijection F is continuous. Clearly,

$$\theta^{(x)}|^{G\cdot x} = F \circ \Pi_{G/G_x},$$

where $\Pi_{G/G_x} : G \twoheadrightarrow G/G_x$ is the natural projection. If $U \subseteq G/G_x$ is open, the preimage $\Pi_{G/G_x}^{-1}(U)$ is open and therefore $F(U) = \theta^{(x)}|^{G\cdot x}(\Pi_{G/G_x}^{-1}(U))$ is open by Lemma 2.2.4. Hence, F is a bijective, open, continuous map, so F is a homeomorphism. □

Lemma 2.2.6. *Let X be a Cartan G-space and $x \in X$. Then for every neighborhood U of G_x in G there is an open neighborhood V of x in X satisfying $G_x \subseteq ((V,V)) \subseteq U$.*

Proof. By Lemma 2.2.3 the isotropy group G_x is compact and clearly $\{e\} \cdot G_x = G_x \subseteq \mathrm{Int}(U)$. By Lemma A.1.10 we can choose an open neighborhood $\tilde{U}$ of e in G with $U' := \tilde{U}G_x \subseteq \mathrm{Int}(U) \subseteq U$. The set U' is an open neighborhood of G_x by Lemma 2.1.3 and by definition a union of left G_x cosets. $\Pi_{G/G_x}^{-1}(\Pi_{G/G_x}(G\backslash U')) = G\backslash U'$ because $G\backslash U'$ is a union of left G_x cosets. Hence $\Pi_{G/G_x}(G\backslash U')$ is closed in G/G_x and by Proposition 2.2.5 $F \circ \Pi_{G/G_x}(G\backslash U') = (G\backslash U') \cdot x$ is closed in $G \cdot x$. Since $G \cdot x$ is closed in X by Lemma 2.2.3, $(G\backslash U') \cdot x$ is closed in X. Since $G_x \subseteq U'$, we have $x \notin (G\backslash U') \cdot x$.

We claim that there exists a closed, thin neighborhood W of x, disjoint with $(G\backslash U') \cdot x$:
By assumption, X is a completely regular space, in particular, it is a regular space. So, for every closed subset A in X and every $y \notin A$, there exists a closed neighborhood B of y, disjoint with A.
Since $(G\backslash U') \cdot x$ is a closed set in X, not containing x, there exists a closed neighborhood W_1 of x such that $W_1 \cap (G\backslash U') \cdot x = \varnothing$.
Now choose an open neighborhood $\widetilde{W}$ of x, which is thin. Then there exists a closed neighborhood $W_2 \subseteq \widetilde{W}$ of x because X is regular.
Define $W := W_1 \cap W_2$. Clearly, W is a closed neighborhood of x, disjoint with $(G\backslash U') \cdot x$.

Since W is contained in $\widetilde{W}$, it follows that W is thin (by Lemma 2.2.2 (c)).

We put

$$K := \overline{G\backslash U' \cap ((W, W))} \subseteq G\,.$$

The set K is closed and contained in $\overline{((W, W))}$, so it is compact and, since U' is open, K is contained in $G\backslash U'$. If $k \in K$ then $k \cdot x \in (G\backslash U') \cdot x \subseteq X\backslash W$. Since W is closed, for every $k \in K$ we can choose open neighborhoods $L_k \subseteq G$ of k and $V_k \subseteq X$ of x such that $k \cdot x \in L_k \cdot V_k \subseteq X\backslash W$. Since K is compact, we can choose $L_{k_1}, \ldots, L_{k_n}$, covering K. Set $V := (\bigcap_{i=1}^{n} V_{k_i}) \cap \mathrm{Int}(W) \subseteq W$. Clearly, V is an open neighborhood of x, thus $G_x \subseteq ((V, V))$.

Let us prove $((V, V)) \subseteq U'$ by contradiction:
Assume, there is some $g \in ((V, V)) \cap (G\backslash U')$. Then $g \cdot V \cap V \neq \varnothing$, so a fortiori $g \cdot W \cap W \neq \varnothing$ and thus $g \in ((W, W)) \cap (G\backslash U') \subseteq K$. So, there is some $i \in \{1, \ldots, n\}$ with $g \in L_{k_i}$, hence

$$g \cdot V \subseteq L_{k_i} \cdot V \subseteq L_{k_i} \cdot V_{k_i} \subseteq X\backslash W \subseteq X\backslash V \quad ,$$

which contradicts $g \cdot V \cap V \neq \varnothing$.

The lemma follows because $U' \subseteq U$. □

Corollary 2.2.7. *Let θ be a proper, continuous action of the Lie group G on the manifold M. Then M is a Cartan G-space.*

Proof. Let $p \in M$ be arbitrary. Since M is a manifold, we can choose a compact neighborhood K of p. By Proposition 2.1.10 (iv) the set $((K, K))$ is compact and closed (since G is Hausdorff) in G. So, K is a thin neighborhood of p. □

2.3 Slices

In this chapter G will be a Lie group, acting smoothly on the manifold M via the group action $\theta : G \times M \to M$. When nothing else stated, let H be a closed subgroup of G. Thus, H is a Lie subgroup of G and G/H carries a canonical smooth structure such that it is a homogeneous G-space.

Definition 2.3.1. A subset $\varnothing \neq S \subseteq M$ is an ***H-kernel*** if there is a continuous G-equivariant map $f : G \cdot S \to G/H$ with $f^{-1}(\{H\}) = S$. If additionally $G \cdot S$ is open in M, S is called an ***H-slice***. For $p \in M$, we call S a ***slice at*** p if p lies in S and S is a G_p-slice.

We also write $f^{-1}(H)$ instead of $f^{-1}(\{H\})$.
First, let us show that, if such a function f exists, then it is unique.

Lemma 2.3.2. *For an H-kernel S there exists a unique equivariant map $f : G \cdot S \to G/H$ with $f^{-1}(H) = S$.*

Proof. Let $f_1, f_2 : G \cdot S \to G/H$ be G-equivariant with $f_1^{-1}(H) = S = f_2^{-1}(H)$. For an arbirary $p \in G \cdot S$ we can choose $g \in G$, $s \in S$ with $p = g \cdot s$. Then:

$$\begin{aligned}
f_1(p) &= f_1(g \cdot s) && \\
&= g \cdot f_1(s) && (f_1 \text{ is } G\text{-equivariant}) \\
&= g \cdot H && (s \in f_1^{-1}(H)) \\
&= gH && (\text{equation } (2.2)) \\
&= f_2(p) && (\text{analogously for } f_2)
\end{aligned}$$

□

For an H-kernel S we will denote the corresponding unique map by $f^S : G\cdot S \to G/H$. The calculation in the proof above shows

$$f^S(g\cdot s) = gH \quad \forall\, g\in G\,,\, s\in S \tag{2.4}$$

Remark 2.3.3. Let S be an H-kernel in M. Then, by equation (2.4), $f^S(h\cdot s) = hH = H$ for any $h\in H\,,\, s\in S$, so S is H-invariant and therefore an H-space. If in addition S is a submanifold of M, then we have the following commutative diagram

$$\begin{tikzcd} H\times S \arrow[r, "\iota_H\times\iota_S"] \arrow[dr, "\theta|^S_{H\times S}"'] & G\times M \arrow[r, "\theta"] & M \\ & S \arrow[ur, "\iota_S"'] & \end{tikzcd}$$

where $\iota_H : H\hookrightarrow G$ and $\iota_S : S\hookrightarrow M$ denote the inclusion maps of H respectively S. Clearly, $\theta\circ(\iota_H\times\iota_S)$ is smooth and, since ι_S is an embedding, $\theta|^S_{H\times S}$ is smooth too. Hence, S is a smooth H-space.

Remark 2.3.4. Suppose S is an H-kernel in M and consider S as an H-space. For $s\in S$, obviously, $H_s = G_s\cap H \subseteq G_s$ holds. If $g\in G_s$, then

$$gH \overset{\text{eqn. (2.4)}}{=\!=\!=} f^S(g\cdot s) = f^S(s) = H.$$

Thus $g\in H$ and therefore $g\in G_s\cap H = H_s$.
In total, we have shown that $G_s = H_s$ for every $s\in S$.

Before we formulate the next important result, let us prove the following lemma, which will be relevant in the proof of (d) in Theorem 2.3.6.

Lemma 2.3.5. *Let G be a locally compact group with compact subgroup H. Then there exist open sets $U, V\subseteq G$ with $H\subseteq U\subseteq UU\cup UU^{-1}\subseteq V$ and V precompact.*

Proof. Since G is locally compact and H is compact, we can cover H with finitely many open, precompact sets. Then their union V is an open, precompact neighborhood of H. Let $m : G\times G\to G$ denote the multiplication map. Since $H\times H\subseteq m^{-1}(H)\subseteq m^{-1}(V)$, Lemma A.1.7 gives us open sets W_1, W_2 with $H\times H\subseteq W_1\times W_2\subseteq m^{-1}(V)$. Let $W := W_1\cap W_2$, then $H\times H\subseteq W\times W\subseteq m^{-1}(V)$. Now put $U := W\cap W^{-1}$. ☐

Theorem 2.3.6. *Let M be a smooth G-space and H a closed subgroup of G.*
If S is an H-kernel in M, then:

(a) S is closed in $G\cdot S$.

(b) S is H-invariant and an H-space by restricting the action on M.

(c) $((S,S)) = H$, in particular:
$\forall g\in G : (g\cdot S)\cap S\neq\varnothing \Longrightarrow g\in H$

If in addition H is compact, then:

(d) S has a thin, open neighborhood in $G\cdot S$.

Conversely, suppose (a) – (d) hold: Then S is an H-kernel and H is compact.

Proof. Suppose first, S is an H-kernel:

(a) $S = (f^S)^{-1}(H)$, f^S is continuous and $\{H\}$ is closed in G/H.
(b) see Remark 2.3.3.
(c) "$\supseteq$": For $h\in H$ we have $(h\cdot S)\cap S = S\cap S\neq\varnothing$.
"$\subseteq$": Suppose $(g\cdot S)\cap S\neq\varnothing$. Then there is $s\in S$ with $g\cdot s\in S$ and therefore $gH = f^S(g\cdot s) = H$ and so $g\in H$.

(d) As Lie group, G is a locally compact group. By Lemma 2.3.5 we can choose V open, precompact and V' open in G such that $H \subseteq V' \subseteq V'V' \cup V'V'^{-1} \subseteq V$. By assumption H is compact, so by Lemma 2.1.4 (c) there exists an open neighborhood U' of e with $U := U'H \subseteq V'$. By Lemma 2.1.3 and, because we have $e \in U' \cap H$, U is an open neighborhood of e in G. Thus, $\widetilde{U} := \Pi_{G/H}(U)$ is an open neighborhood of H in G/H (notice that G/H is the orbit space of the smooth H-action $H \times G \to G$, $(h,g) \mapsto gh^{-1}$ on G, so $\Pi_{G/H}$ is the projection map onto the H-orbit and thus open by Lemma 2.1.5). By definition, U is a union of left H-cosets, so $\Pi_{G/H}^{-1}(\widetilde{U}) = U$. Then $W := (f^S)^{-1}(\widetilde{U})$ is an open neighborhood of S in $G \cdot S$ because f^S is continuous. Furthermore, if $(g \cdot W) \cap W \neq \varnothing$, then there is some $w \in W$ such that $g \cdot w \in W$. Choose a $\hat{g} \in G$ with $f^S(w) = \hat{g}H$. Then

$$\Pi_{G/H}(\hat{g}) = \hat{g}H = f^S(w) \in \widetilde{U}$$

and similarly

$$\Pi_{G/H}(g\hat{g}) = g\hat{g}H = g \cdot \hat{g}H = g \cdot f^S(w) = f^S(g \cdot w) \in \widetilde{U}.$$

Therefore, we get $\hat{g}, g\hat{g} \in \Pi_{G/H}^{-1}(\widetilde{U}) = U \subseteq V'$ and thus $g = (g\hat{g})\hat{g}^{-1} \in V'V'^{-1} \subseteq V$. In conclusion, we have shown that $((W,W)) \subseteq V$ and, since V is precompact, that W is thin. This shows (d).

Now suppose, (a) – (d) hold:
By (c) and (d) we get $H = ((S,S)) \subseteq ((W,W))$ for some thin neighborhood W of S. Since H is closed, we conclude that H is indeed compact.
Now we want to prove that S is an H-kernel. We define the following function:

$$\begin{aligned} f : G \cdot S &\longrightarrow G/H \\ g \cdot s &\longmapsto gH \end{aligned}$$

This is indeed a well-defined map: Suppose $g_1 \cdot s_1 = g_2 \cdot s_2$. Then $g_2^{-1} g_1 \cdot s_1 = s_2 \in (g_2^{-1} g_1 \cdot S) \cap S$, so $g_2^{-1} g_1 \in H$ by (c) and hence $g_1 H = g_2 H$.
Clearly, f is G-equivariant and $S \subseteq f^{-1}(H)$. On the other hand, if $gH = H$ then $g \in H$ and by (b) $g \cdot s \in S$, so $S = f^{-1}(H)$.
It remains to show the continuity of f:
Suppose the net $(g_i \cdot s_i)_{i \in I} \in G \cdot S$ converges to $g \cdot s \in G \cdot S$. We have to show that $g_i H = f(g_i \cdot s_i) \to f(g \cdot s) = gH$ in G/H. Without loss of generality we may assume that $g = e$ (because we have both $g^{-1} g_i \cdot s_i \to s$ and $g^{-1} g_i H \to H \implies g_i H = g \cdot (g^{-1} g_i H) \to g \cdot H = gH$). We will show $g_i H \to H$ by contradiction: Assume $g_i H \nrightarrow H$. Thus, there exists an open neighborhood U of $H \in G/H$ such that for every $i_0 \in I$ there exists an $i \succeq i_0$ with $g_i H \notin U$. With $I' := \{i \in I \mid g_i H \notin U\}$ we have constructed the subnet $(g_i)_{i \in I'}$ with $g_i H \notin U \ \forall i \in I'$. The preimage U' of U under $\Pi_{G/H}$ is an open neighborhood of H such that $g_i \notin U'$ for all $i \in I'$.

A fortiori, no subnet of $(g_i)_{i \in I'}$ converges to a point in H. $(*)$

Because of condition (d) we can choose a thin neighborhood V of S in GS. Since $g_i \cdot s_i \to s \in S \subseteq V$, we have $g_i \cdot s_i \in V$ for i large enough and hence $g_i \in ((V,V))$. Since V is thin, we can choose a converging subnet (g_{i_j}) of $(g_i)_{i \in I'}$ with limit $g \in G$. Then

$$s_{i_j} = g_{i_j}^{-1} \cdot (g_{i_j} \cdot s_{i_j}) \to g^{-1} \cdot s.$$

Because of assumption (a) we also know $g^{-1} \cdot s \in S$, and by (c) therefore $g^{-1} \in H$. Since H is a group, it follows that $g \in H$ and therefore there exists a subnet of $(g_i)_{i \in I'}$ with limit point in H. This contradicts $(*)$. □

Definition 2.3.7. A ***local cross section in*** G/H is a smooth map $\chi : U \to G$, where $U \subseteq G/H$ is an open neighborhood of $H \in G/H$, such that $\chi(H) = e$ and $\chi(\gamma) \in \gamma \ \forall \gamma \in U$.

Observe that local cross sections are smooth local sections of the natural projection. Since every submersion admits a smooth local section through a fixed point, in particular the projection $G \twoheadrightarrow G/H$ does so too. Thus, a local cross sections exists for an arbitrary Lie group G with respect to any closed subgroup $H \leq G$.

Definition 2.3.8. We shall say that a subset $S^* \subseteq M$ of the smooth G-space M is a ***near slice at*** $p \in M$ if $p \in S^*$, S^* is G_p-invariant and there is a local cross section $\chi : U \to G$ in G/G_p with the property that $(u, s) \mapsto \chi(u) \cdot s$ is a homeomorphism of $U \times S^*$ onto an open neighborhood of p in M.

Lemma 2.3.9. *If M is a Cartan G-space and S^* is a near slice at p, then there exists $S \subseteq S^* \subseteq M$ which is a slice at p and open in S^*.*

Proof. Choose a local cross section $\chi : U \to G$ in G/G_p such that $(u, s) \mapsto \chi(u) \cdot s$ is a homeomorphism of $U \times S^*$ onto an open neighborhood of p. Let U' be the preimage of U under Π_{G/G_p}. By Lemma 2.2.6 there is some open neighborhood V of p with $((V, V)) \subseteq U'$ (because $\Pi_{G/G_p}(G_p) = \{G_p\} \subseteq U$ holds, U' is indeed an open neighborhood of G_p). G_p is compact by Lemma 2.2.3, thus there exists an open neighborhood V' of p with $V'' := G_p \cdot V' \subseteq V$ (see Lemma A.1.10). Since we have

$$V'' = \bigcup_{g \in G_p} g \cdot V',$$

V'' is an open neighborhood of p, that is G_p-invariant. Additionally $((V'', V'')) \subseteq ((V, V)) \subseteq U'$, so without loss of generality, instead of considering V'', we may just assume that V is G_p -invariant.
Now, let $S := S^* \cap V$. Then S is open in S^*, contains p and is G_p-invariant (because S^* and V are G_p-invariant). We also have the identity:

$$U' \cdot S = \{\chi(u) \cdot s \mid u \in U,\, s \in S\} \tag{2.5}$$

"$\supseteq$": $\chi(u) \in u \subseteq U' = \Pi^{-1}_{G/G_p}(U)$ because χ is a local cross section in G/G_p .

"$\subseteq$": Suppose $u' \in U'$, $s' \in S$. Let $u := \Pi_{G/G_p}(u') \in U$. Since χ is a local cross section in G/G_p, we have $u'\, G_p = u = \chi(u)\, G_p$, so $\chi(u)^{-1}u' \in G_p$ and, because S is G_p-invariant, it follows that $s := \chi(u)^{-1}u' \cdot s'$ is in S and hence $u' \cdot s' = \chi(u) \cdot s \in \{\chi(\tilde{u}) \cdot \tilde{s} \mid \tilde{u} \in U,\, \tilde{s} \in S\}$.

Since S is open in S^* and $(u, s) \mapsto \chi(u) \cdot s$ is a homeomorphism of $U \times S^*$ onto an open neighborhood of p, the set $U' \cdot S$ is open in M by equation (2.5). Therefore, also $G \cdot S = G \cdot (U' \cdot S)$ is open in M.

So, to finish the proof we just have to show that S is a G_p-kernel:
Suppose that $(g \cdot S) \cap S \neq \varnothing$. A fortiori, $(g \cdot V) \cap V \neq \varnothing$ and therefore $g \in ((V, V)) \subseteq U'$ and $gG_p \in U$. Choose $s_1 \in S$, $h \in G_p$ with $s_2 := g \cdot s_1 \in S$ and $\chi(gG_p) = gh$. By rearranging the equations we get

$$\chi(gG_p) \cdot (h^{-1} \cdot s_1) = \chi(gG_p)h^{-1} \cdot s_1 = g \cdot s_1 = s_2 = \chi(G_p) \cdot s_2$$

and, because $(u, s) \mapsto \chi(u) \cdot s$ is a homeomorphism of $U \times S^*$ onto its image and $h^{-1} \cdot s_1 \in S \subseteq S^*$ (since S is G_p-invariant), we obtain $gG_p = G_p$ and thus $g \in G_p$.
We conclude that $((S, S)) = G_p$ (the other inclusion is trivial). Therefore the map $f : G \cdot S \to G/G_p$, $g \cdot s \mapsto gG_p$ is well-defined. As in Theorem 2.3.6 we easily see that f is G-equivariant and $f^{-1}(G_p) = S$ and it remains to prove the continuity of f. Given any converging net $g_i \cdot s_i \to g \cdot s$ we have to show $g_iG_p \to gG_p$. With the same argument as in the preceding Theorem 2.3.6 we may assume $g = e$. Since V is a neighborhood of S, we may additionally assume that $g_i \cdot s_i \in V \quad \forall i$ and therefore $g_i \in ((V, V)) \subseteq U'$ and $g_iG_p \in U$. For

each i choose $h_i \in G_p$ with $g_i = \chi(g_i G_p) h_i$ and, since S is G_p-invariant, we have $h_i \cdot s_i \in S$. Since $\chi(g_i G_p) \cdot (h_i \cdot s_i) = g_i \cdot s_i \to s = \chi(G_p) \cdot s$, using again that $(\tilde{u}, \tilde{s}) \mapsto \chi(\tilde{u}) \cdot \tilde{s}$ is a homeomorphism on $U \times S^*$ onto its open image, we conclude that $(g_i G_p, h_i \cdot s_i) \to (G_p, s)$. Therefore $g_i G_p \to G_p$ holds, which we wanted to prove. $\square$

Recall from Proposition 2.1.13 that for every smooth group action on a manifold M the orbit $G \cdot p$ through p is an immersed submanifold in a natural way. Dependent on the context, when we write $T_q(G \cdot p)$ with $q \in G \cdot p$, we often mean the image of this tangent space under the inclusion map $\iota : G \cdot p \hookrightarrow M$, so $d_q \iota(T_q(G \cdot p))$.

Lemma 2.3.10. *Let M be a smooth G-space, $p \in M$ a point with compact isotropy group G_p and N a G_p-invariant submanifold of M, containing p. Assume that N and $G \cdot p$ are transverse at p, i.e. $T_pN \oplus T_p(G \cdot p) = T_pM$. Then there exists a local cross section $\chi : U \to G$ in G/G_p and a G_p-invariant open submanifold S^* of N containing p, such that $(u,s) \mapsto \chi(u) \cdot s$ is a diffeomorphism of $U \times S^*$ onto an open neighborhood of p. In particular, S^* is a near slice at p.*

Proof. Let $\chi^* : U^* \to G$ be a local cross section in G/G_p. Define the function

$$F : U^* \times N \to M\,, \quad (u,s) \mapsto \chi^*(u) \cdot s.$$

By definition of a local cross section χ^* is smooth, so F is smooth. We claim that the differential of F at $(G_p, p) \in U^* \times N$ is invertible. If we accept this for the moment, by the inverse function theorem there exist open neighborhoods around (G_p, p) respectively $F(G_p, p)$ such that the restriction of F to these neighborhoods is a diffeomorphism. Therefore, by taking even smaller open neighborhoos if necessary, there exist open subsets $G_p \in U \subseteq U^*$, $p \in N' \subseteq N$, such that $F|_{U \times N'} : U \times N' \to F(U \times N')$ is a diffeomorphism. Since the isotropy group G_p is compact, by Lemma A.1.10 we can choose an open neighborhood N'' of p with $S^* := G_p \cdot N'' \subseteq N'$. Then S^* is an open submanifold of N containing p, which is invariant under G_p and the map $(u,s) \mapsto \chi^*(u) \cdot s$ is a diffeomorphism of $U \times S^*$ onto the open neighborhood $F(U \times S^*)$ of p. Then the Lemma follows by putting $\chi := \chi^*|_U$.

It remains to show that $d_{(G_p,p)}F : T_{(G_p,p)}(U^* \times N) \to T_pM$ is invertible:
The dimension of the vector space $T_{(G_p,p)}(U^* \times N) \cong T_{G_p}U^* \oplus T_pN$ is

$$\begin{aligned}
\dim(T_{(G_p,p)}(U^* \times N)) &= \dim(U^*) + \dim(N) \\
&= \dim(G/G_p) + (\dim(M) - \dim(G \cdot p)) \qquad (2.6)\\
&= \dim(G/G_p) + (\dim(M) - \dim(G/G_p)) \\
&= \dim(M) \\
&= \dim(T_pM)
\end{aligned}$$

where we used the transversality of N and $G \cdot p$ in equation (2.6). Thus, it suffices to show that the differential of F at (G_p, p) is surjective. Since $F(G_p, s) = s \quad \forall s \in N$, we clearly have $T_pN \subseteq d_{(G_p,p)}F(\,T_{(G_p,p)}(\{G_p\} \times N)\,)$. The surjectivity of $d_{(G_p,p)}F$ follows from the assumption $T_pN \oplus T_p(G \cdot p) = T_pM$ togeter with the following statement:

$$T_p(G \cdot p) \subseteq d_{(G \cdot p,p)}F(\,T_{(G_p,p)}(U^* \times \{p\})\,) \qquad (2.7)$$

To prove equation (2.7), let us assume that $v = \dot{\gamma}(0) \in T_p(G \cdot p)$ is arbitrary, where $\gamma : I_\gamma \to G \cdot p$ is a smooth path with $\gamma(0) = p$. Let $\Phi : G/G_p \to G \cdot p$ denote the canonical diffeomorphism, given by equation (2.3). By continuity, we can assume that $(\Phi^{-1} \circ \gamma)(I_\gamma) \subseteq U^*$. Then the path

$$\delta : I_\gamma \to U^* \times N\,, \quad \delta(t) := (\Phi^{-1} \circ \gamma(t), p)$$

is a smooth path in $U^* \times \{p\}$. For $t \in I_\gamma$ choose $g_t \in G$, such that $\gamma(t) = g_t \cdot p$. Since χ^* is a local cross section, for each t we can choose $\hat{g}_t \in G_p$ with $\chi^*(g_t G_p) = g_t \hat{g}_t$. Thus,

$$\begin{aligned} F \circ \delta(t) &= \chi^*(\Phi^{-1} \circ \gamma(t)) \cdot p \\ &= \chi^*(g_t G_p) \cdot p \qquad = \qquad g_t \hat{g}_t \cdot p \\ &= g_t \cdot (\hat{g}_t \cdot p) \qquad = \qquad g_t \cdot p \\ &= \gamma(t) \end{aligned}$$

and therefore we have $F \circ \delta = \gamma$. By differentiating $v = d_{(G_p,p)}F\,(\,\dot{\delta}(0)\,)$, which proves equation (2.7). □

Proposition 2.3.11. *Let $\theta : G \times M \to M$ be a smooth G-action on M and $H \leq G$ be a compact subgroup. Then there exists an H-invariant Riemannian metric γ for M, i.e. $\theta_h^* \gamma = \gamma$ for all $h \in H$.*

Proof. Recall that for any manifold there is a Riemannian metric and that any compact topological group admits, up to a positive multiplicative constant, an unique nontrivial, regular, bi-invariant Borel measure, the so called Haar measure. Choose a Riemannian metric $\tilde{\gamma}$ on M. Let μ denote the unique Haar measure on H with $\int_H \mathrm{d}\mu = 1$ Define the metric γ on M by

$$\gamma_p(v,w) := \int_H ((\theta_g)^*\tilde{\gamma})_p(v,w)\ \mathrm{d}\mu(g) \qquad p \in M\,,\ v,w \in T_pM, \tag{2.8}$$

where '$(\theta_g)^*\tilde{\gamma}$' denotes the pullback of $\tilde{\gamma}$ with respect to θ_g. Since H is compact, the integral above is indeed finite, so γ is well-defined. Clearly, γ_p is bilinear and symmetric. Since θ_g is bijective for all g, we see that γ_p is also positive-definite. Thus, γ_p is an inner product for all $p \in M$.

Additionally, $\gamma : M \to T^*M \otimes T^*M$ is smooth:
Given arbitrary vector fields $X, Y \in \mathfrak{X}(M) := \Gamma(TM)$, we have to prove that $M \to \mathbb{R}\,,\ p \mapsto \gamma_p(X_p, Y_p)$ is smooth. Let $p_0 \in M$ be arbitrary. Choose a chart (W, φ) around p_0. Let $U \subseteq W$ be a precompact, open neighborhood of p_0 with $\overline{U} \subseteq W$. We prove the smoothness of $F : \varphi(U) \to \mathbb{R}\,,\ q \mapsto \gamma_{\varphi^{-1}(q)}(X_{\varphi^{-1}(q)}, Y_{\varphi^{-1}(q)})$ via induction.
In a first step we will prove that the function

$$\tilde{F} : H \times W \to \mathbb{R}\,, \quad (g,p) \mapsto ((\theta_g)^*\tilde{\gamma})_p(X_p, Y_p)$$

is smooth. When we make use of the identification $T_{(g,p)}(H \times W) \cong T_gH \oplus T_pW$, we have

$$d_p\theta_g \cdot X_p = d_{(g,p)}\theta \cdot ((0_{TG})_g, X_p),$$

where $0_{TG} \in \Gamma(TG)$ denotes the zero section of TG (i.e. the trivial vector field). Thus, we get

$$\begin{aligned} ((\theta_g)^*\tilde{\gamma})_p(X_p, Y_p) &= \tilde{\gamma}_{g \cdot p}(d_p\theta_g \cdot X_p\ ,\ d_p\theta_g \cdot Y_p) \\ &= \tilde{\gamma}_{\theta(g,p)}(\ d_{(g,p)}\theta \cdot ((0_{TG})_g, X_p)\ ,\ d_{(g,p)}\theta \cdot ((0_{TG})_g, Y_p)\ ) \\ &= (\theta^*\tilde{\gamma})(\ (0_{TG}, X)\ ,\ (0_{TG}, Y)\ )_{(g,p)} \end{aligned}$$

The pullback $\theta^*\tilde{\gamma}$ is a covariant tensor field and $(0_{TG}, X)$ respectively $(0_{TG}, Y)$ are (smooth) vector fields, so $\theta^*\tilde{\gamma}(\ (0_{TG}, X)\ ,\ (0_{TG}, Y)\ )$ is a smooth function. Together with the previous computation, we have proven that $\tilde{F}$ is indeed a smooth map.
Since $\overline{U}$ is compact and contained in W, the restriction of $\tilde{F}$ to $H \times U$ is smooth and bounded and, therefore, so is the map $\tilde{F} \circ (\mathrm{id}_H \times \varphi^{-1})|_{H \times \varphi(U)} : H \times \varphi(U) \to \mathbb{R}$. Because we have

$$F(q) = \int_H (\tilde{F} \circ (\mathrm{id}_H \times \varphi^{-1}))(g,q)\ \mathrm{d}\mu(g)\ \ ,$$

it follows that F is continuous. This was the base case $n = 0$.

Now, let $n \in \mathbb{N}$ be arbitrary and suppose $F \in C^n(\varphi(U))$ and that we have

$$\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} F(q) = \int_H \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} \Big(\tilde{F} \circ (\mathrm{id}_H \times \varphi^{-1})(g, \cdot\,) \Big)(q) \ \mathrm{d}\mu(g)$$

for all $q \in \varphi(U)$ and all $i_1, \dots i_n \in \{1, \dots, \dim(M)\}$.

Let $j, i_1, \dots, i_n \in \{1, \dots, \dim(M)\}$ be arbitrary. Since H and $\overline{U} \subseteq W$ are compact, their product $H \times \overline{U}$ is so too and thus the map

$$\begin{aligned} H \times \varphi(U) &\longrightarrow \mathbb{R} \\ (g, q) &\longmapsto \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} \Big(\tilde{F} \circ (\mathrm{id}_H \times \varphi^{-1})(g, \cdot\,) \Big)(q) \end{aligned}$$

is smooth and bounded. By the Leibniz integral rule (for measure spaces) it follows $\partial_{i_1} \dots \partial_{i_n} F \in C^1(\varphi(U))$ and for every $q \in \varphi(U)$ we have

$$\begin{aligned} &\frac{\partial}{\partial x^j} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} F(q) \\ &= \frac{\partial}{\partial x^j} \bigg|_{\tilde{q}=q} \left(\int_H \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} \Big(\tilde{F} \circ (\mathrm{id}_H \times \varphi^{-1})(g, \cdot\,) \Big)(\tilde{q}) \ \mathrm{d}\mu(g) \right) \\ &= \int_H \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} \Big(\tilde{F} \circ (\mathrm{id}_H \times \varphi^{-1})(g, \cdot\,) \Big)(q) \ \mathrm{d}\mu(g) \end{aligned}$$

This was the induction step.
In conclusion, $F \in C^n(\varphi(U))$ for all $n \in \mathbb{N}$, so F is smooth. Because p_0 was arbitrary, we have shown that $\gamma(X, Y)$ is smooth, which in turn proves that γ is smooth.

In total, we now know that γ is in fact a Riemannian metric on M. It remains to show the H-invariance of γ.
Let $h \in H$, $p \in M$, $v, w \in T_pM$ be arbitrary. Then

$$\begin{aligned} (\theta_h^* \gamma)_p(v, w) &= \gamma_{h \cdot p}(d_p\theta_h \cdot v, d_p\theta_h \cdot w) \\ &= \int_H (\theta_g^* \tilde{\gamma})_{h \cdot p}(d_p\theta_h \cdot v,\ d_p\theta_h \cdot w) \ \mathrm{d}\mu(g) \\ &= \int_H (\theta_h^*(\theta_g^* \tilde{\gamma}))_p(v, w) \ \mathrm{d}\mu(g) \\ &= \int_H ((\theta_g \circ \theta_h)^* \tilde{\gamma})_p(v, w) \ \mathrm{d}\mu(g) \\ &\overset{(*)}{=} \int_H (\theta_{gh}^* \tilde{\gamma})_p(v, w) \ \mathrm{d}\mu(g) \\ &\overset{(**)}{=} \int_H (\theta_g^* \tilde{\gamma})_p(v, w) \ \mathrm{d}\mu(g) \\ &= \gamma_p(v, w) \end{aligned}$$

where we used the identity $\theta_g \circ \theta_h = \theta_{gh}$ (since θ is a group action) for $(*)$ and the bi-invariance of μ for $(**)$. This shows the H-invariance of γ, as desired. □

Remark 2.3.12. Instead of the circuitous equation (2.8) we will often just write

$$\gamma = \int_H g^* \tilde{\gamma} \ \mathrm{d}\mu(g). \tag{2.9}$$

Remark 2.3.13. In the above prove we have shown that for any Riemannian metric $\tilde{\gamma}$ the map γ, defined as in equation (2.9), is smooth. Completely analogously one can show the following, more general, version:
Let M smooth G-space, $H \leq G$ a compact subgroup, $k \in \mathbb{N}_0$ and

$$\tilde{\sigma} \in \Gamma(\underbrace{T^*M \otimes \ldots \otimes T^*M}_{k\text{-times}}) = \Gamma((T^*M)^{\otimes k})$$

be a smooth section of the vector bundle $(T^*M)^{\otimes k} \twoheadrightarrow M$. Then the section

$$\sigma = \int_H h^* \tilde{\sigma} \ \mathrm{d}\mu(h)$$

of $(T^*M)^{\otimes k} \twoheadrightarrow M$ is smooth and H-invariant.

Lemma 2.3.14. *Let $\theta : G \times M \to M$ be a smooth G-action on M and $p \in M$ a point with compact isotropy group G_p. Then there exists some G_p-invariant submanifold N in M, which contains p and is transverse to $G \cdot p$ at p.*

Proof. By Proposition 2.3.11 we can choose a G_p-invariant Riemannian metric γ for M. Choose Riemannian normal coordinates $(\tilde{V}, \tilde{\varphi})$ at p with $\tilde{\varphi}(\tilde{V}) = B_{\tilde{r}}(0)$ the open ball of radius $\tilde{r} > 0$ at the origin and $\tilde{\varphi}(p) = 0$. Since $G_p \cdot \{p\} = \{p\} \subseteq \tilde{V}$ and G_p is compact, we can choose an open neighborhood $\tilde{\mathcal{O}}$ of p with $\tilde{U} := G_p \cdot \tilde{\mathcal{O}} \subseteq \tilde{V}$ due to Lemma A.1.10. Clearly, $\tilde{U}$ is an open neighborhood of p, which is G_p-invariant. Let $\tilde{W} = \tilde{\varphi}(\tilde{U})$.
Choose an open neighborhood $V \subseteq \tilde{U}$ of p with $\tilde{\varphi}(V) = B_r(0)$ for some $0 < r < \tilde{r}$. Again, by using A.1.10, there exists an open neighborhood $\mathcal{O}$ of p with $U := G_p \cdot \mathcal{O} \subseteq V$ and U is a G_p-invariant open neighborhood of p. Define $\varphi := \tilde{\varphi}|_U : U \to \tilde{\varphi}(U) =: W \subseteq \mathbb{R}^m$, where $m := \dim(M)$. Thus, we have the inclusions $0 \in W \subseteq B_r(0) \subseteq \tilde{W} \subseteq B_{\tilde{r}}(0) \subseteq \mathbb{R}^m$ of open sets.

For the rest of the proof we interpret tangent spaces of submanifolds of $\mathbb{R}^m$ as subsets of $\mathbb{R}^m$, i.e. if we write $T_q^{\mathbb{R}^m} S$ for some (immersed) submanifold S of $\mathbb{R}^m$ and a point $q \in S$, we mean the image of the tangent space T_qS under the canonical isomorphism $\alpha_{\mathrm{id}_{\mathbb{R}^m},q} : T_q\mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m$ (the coordinates for $T_q\mathbb{R}^m$ induced by the trivial chart $(\mathbb{R}^m, \mathrm{id}_{\mathbb{R}^m})$).
We put

$$\begin{aligned} L &:= \varphi(U \cap G \cdot p) \subseteq W \subseteq \mathbb{R}^m \\ N &:= \varphi^{-1}(\ W \cap (T_0^{\mathbb{R}^m} L)^{\perp}\ ) \subseteq U \subseteq M\ , \end{aligned}$$

where '$\perp$' indicates the orthogonal complement in $\mathbb{R}^m$ with respect to the Euclidean inner product. Notice that this definition does indeed make sense since $U \cap G \cdot p$ is an immersed submanifold of M by Proposition 2.1.13, so the tangent space of the immersed submanifold L is well-defined.

The intersection of the open set W with the linear subspace $(T_0^{\mathbb{R}^m} L)^{\perp}$ is a submanifold of $\mathbb{R}^m$ for obvious reasons. Thus, N is a submanifold of M, containing p.

Our next goal is to prove that N and $G \cdot p$ are transverse at p:
Denote by $\iota : U \cap G \cdot p \hookrightarrow U$, $\iota_L : L \hookrightarrow W$ the inclusion maps. They are injective immersions. By definition of the smooth structure on L, the map

$$\varphi|_{U \cap G \cdot p} : U \cap G \cdot p \xrightarrow{\sim} \varphi(U \cap G \cdot p) = L$$

is a diffeomorphism. Clearly, we have:

$$
\begin{tikzcd}
T_p\left(U\cap G\cdot p\right) \arrow[r, "\simeq", "d_p(\varphi|_{U\cap G\cdot p})"'] \arrow[d, hook, "d_p\iota"'] & T_0\, L \arrow[d, hook, "d_0\iota_L"] \\
T_pU \arrow[r, "\simeq", "d_p\varphi"'] & T_0W \cong T_0\mathbb{R}^m
\end{tikzcd}
\tag{2.10}
$$

It follows

$$
\begin{aligned}
d_p\varphi\left(T_p\left(G\cdot p\right)\right) &= d_p\varphi\left(\, d_p\iota\left(T_p\left(G\cdot p\right)\right)\,\right) && \text{(notation convention)}\\
&= d_0\iota_L\circ d_p(\varphi|_{U\cap G\cdot p})\left(T_p\left(G\cdot p\right)\right) && \text{(follows from 2.10)}\\
&= d_0\iota_L\left(T_0L\right) &&\\
&= T_0L && \text{(notation convention)}
\end{aligned}
$$

So, we have

$$
\alpha_{\mathrm{id}_{\mathbb{R}^m},0}\circ d_p\varphi\ \left(T_p\left(G\cdot p\right)\right) = T_0^{\mathbb{R}^m} L\ . \tag{2.11}
$$

The tangent space of an open subset of a linear subspace of $\mathbb{R}^m$ is the linear subspace itself, so we get:

$$
\begin{aligned}
\alpha_{\mathrm{id}_{\mathbb{R}^m},0}\circ d_p\varphi\ \left(T_pN\right) &= \alpha_{\mathrm{id}_{\mathbb{R}^m},0}\ \left(\ T_0\left(\ W\ \cap\ (T_0^{\mathbb{R}^m}L)^\perp\ \right)\ \right)\\
&= (T_0^{\mathbb{R}^m}L)^\perp
\end{aligned}
\tag{2.12}
$$

As orthogonal complements, clearly the following statement holds:

$$
T_0^{\mathbb{R}^m}L\oplus(T_0^{\mathbb{R}^m}L)^\perp = \mathbb{R}^m.
$$

Since the composition $\alpha_{\mathrm{id}_{\mathbb{R}^m},0}\circ d_p\varphi$ is a linear isomorphism, we can derive from equations (2.11) and (2.12) that

$$
T_pN\oplus T_p\left(G\cdot p\right) = T_pM
$$

holds. Hence, N and $G\cdot p$ are transverse at p.

It remains to show the G_p-invariance of N:
Since $\tilde{U}$ is G_p-invariant by construction, we can define the (natural) smooth G_p-action

$$
\begin{aligned}
\theta^{\mathbb{R}^m}: G_p\times\tilde{W} &\longrightarrow \tilde{W}\\
(g,q) &\longmapsto \theta^{\mathbb{R}^m}(g,q) := \tilde{\varphi}(\ \theta(g,\tilde{\varphi}^{-1}(q))\ )
\end{aligned}
$$

on $\tilde{W} = \tilde{\varphi}(\tilde{U})$. Obviously, with this new definition, $\tilde{\varphi}^{-1}$, and therefore also $\tilde{\varphi}$, become G_p-equivariant diffeomorphisms between the G_p-spaces $\tilde{U}$ and $\tilde{W}$. Hence, to prove the G_p-invariance of N, it suffices to show that $\varphi(N) = \tilde{\varphi}(N)$ is G_p-invariant (with respect to the action $\theta^{\mathbb{R}^m}$). We will abbreviate the operation of g on q by $g\cdot q$ too, the meaning of this expression will be apparent from the context.

Denote by γ_{normal} the pullback of the metric γ by $\tilde{\varphi}^{-1}$ onto $\tilde{\varphi}(\tilde{V}) = B_{\tilde{r}}(0)\subseteq\mathbb{R}^m$, so

$$
\gamma_{\mathrm{normal}} := (\tilde{\varphi}^{-1})^*(\gamma|_{\tilde{V}}).
$$

Because $\tilde{\varphi}$ are Riemannian normal coordinates, we have $(\gamma_{\mathrm{normal}})_0 = \langle\,\cdot,\,\cdot\rangle_{\mathrm{euc}}$, the Euclidean inner product, and that the geodesics through the origin 0 are exactly the straight lines through it, lying in the open ball $B_{\tilde{r}}(0) = \tilde{\varphi}(\tilde{V})$. The following computation shows that

$\gamma_{\text{normal}}|_{\tilde{W}}$ is G_p-invariant with respect to the action $\theta^{\mathbb{R}^m}$. Let $g \in G_p$ be arbitrary:

$$\begin{aligned}
(\theta_g^{\mathbb{R}^m})^*(\gamma_{\text{normal}}|_{\tilde{W}}) &= (\,\tilde{\varphi}^{-1}|_{\tilde{W}}^{\tilde{U}} \circ \theta_g^{\mathbb{R}^m}\,)^*\,\gamma|_{\tilde{U}} \\
&\overset{(*)}{=} (\,\theta_g|_{\tilde{U}}^{\tilde{U}} \circ \tilde{\varphi}^{-1}|_{\tilde{W}}^{\tilde{U}}\,)^*\gamma|_{\tilde{U}} \\
&= (\tilde{\varphi}^{-1}|_{\tilde{W}}^{\tilde{U}})^*(\,(\theta_g|_{\tilde{U}}^{\tilde{U}})^*\gamma|_{\tilde{U}}\,) \\
&\overset{(**)}{=} (\tilde{\varphi}^{-1}|_{\tilde{W}}^{\tilde{U}})^*\,\gamma|_{\tilde{U}} \\
&= \gamma_{\text{normal}}|_{\tilde{W}}
\end{aligned}$$

We have used the G_p-equivariance of $\tilde{\varphi}^{-1}$ for $(*)$ and the G_p-invariance of γ for $(**)$.

We now want to show the G_p-invariance of $\varphi(N) = W \cap (T_0^{\mathbb{R}^m} L)^\perp$:

Let $g \in G_p$, $q \in W \cap (T_0^{\mathbb{R}^m} L)^\perp$ be arbitrary. Since U is G_p-invariant, $W = \varphi(U)$ is so too, and thus $g \cdot q \in W$. We have to show that $g \cdot q \perp T_0^{\mathbb{R}^m} L$. The case $q = 0$ is trivial because $g \cdot q = g \cdot 0 = 0$ holds (since $g \in G_p$ and $\tilde{\varphi}(p) = 0$), so let us assume $q \neq 0$. Let $\bar{v} \in T_0^{\mathbb{R}^m} L$ be arbitrary and define $v \in T_0 L$ by $\bar{v} = \alpha_{\mathrm{id}_{\mathbb{R}^m},0}(v)$. Choose $v' \in T_0 L$ with $v = d_0\iota_L \cdot v'$. We claim the following:

$$d_0\theta_h^{\mathbb{R}^m} \cdot v \in T_0 L \qquad \forall\, h \in G_p. \tag{2.13}$$

To prove (2.13) it suffices to show that for $h \in G_p$ the map $\theta_h^{\mathbb{R}^m}|_L^L : L \to L$ is smooth with respect to the smooth structures of the immersed submanifolds, then we have

$$d_0\theta_h^{\mathbb{R}^m} \cdot v = d_0(\theta_h^{\mathbb{R}^m} \circ \iota_L) \cdot v' = d_0(\iota_L \circ \theta_h^{\mathbb{R}^m}|_L^L) \cdot v' \in T_0 L.$$

By the universal property of injective immersions, if we show that $\theta_h^{\mathbb{R}^m}|_L^L$ is continuous, then its smoothness is a direct consequence. We have

$$\begin{array}{llr}
 & \begin{cases} G/G_p \to G/G_p \\ \tilde{g}G_p \mapsto h\cdot(\tilde{g}G_p) \end{cases} & \text{is continuous} \\
\iff & \begin{cases} G\cdot p \to G\cdot p \\ \tilde{g}\cdot p \mapsto h\cdot(\tilde{g}\cdot p) \end{cases} & \text{is continuous} \\
\implies & \begin{cases} U\cap G\cdot p \to U\cap G\cdot p \\ \tilde{g}\cdot p \mapsto h\cdot(\tilde{g}\cdot p) \end{cases} & \text{is continuous} \\
\iff & \theta_h^{\mathbb{R}^m}|_L^L & \text{is continuous}
\end{array}$$

Clearly, the above map $G/G_p \to G/G_p$ is indeed continuous, so $\theta_h^{\mathbb{R}^m}|_L^L$ is smooth, and therefore we have proven (2.13).

Let

$$\begin{aligned} \delta : I = (-\varepsilon, \varepsilon) &\to B_r(0) \subseteq \tilde{W} \subseteq \mathbb{R}^m \\ t &\mapsto tq \end{aligned}$$

be the geodesic with velocity $\alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}(q)$ at $t = 0$, where ε is maximal, i.e. $\varepsilon > 0$ with $\varepsilon|q|_{euc} = r$. Notice that $\varepsilon > 1$ because $q \in W \subseteq B_r(0)$. Since $\gamma_{\text{normal}}|_{\tilde{W}}$ is G_p-invariant, the map $\theta_g^{\mathbb{R}^m} : \tilde{W} \xrightarrow{\sim} \tilde{W}$ is a Riemannian isometry. Thus, the composition $\theta_g^{\mathbb{R}^m} \circ \delta : I \to \tilde{W}$ is a geodesic through the origin 0 with starting velocity $d_0\theta_g^{\mathbb{R}^m} \cdot \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}(q)$.
Since we use normal coordinates, the unique maximal geodesic (with respect to $(B_{\tilde{r}}(0), \gamma_{\text{normal}})$) through the origin with starting velocity $d_0\theta_g^{\mathbb{R}^m} \cdot \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}(q)$ is the straight line

$$(-\tilde{\varepsilon}, \tilde{\varepsilon}) \to B_{\tilde{r}}(0)\,,\ t \mapsto t \,\cdot\, \alpha_{\mathrm{id}_{\mathbb{R}^m},0}\,(\,d_0\theta_g^{\mathbb{R}^m} \cdot \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}(q)\,)$$

(again, let $\tilde{\varepsilon}$ be maximal, see above).
Because $\theta_g^{\mathbb{R}^m} \circ \delta$ is a geodesic (with respect to $(B_{\tilde{r}}(0), \gamma_{\text{normal}})$) with the same starting velocity, it follows that

$$\theta_g^{\mathbb{R}^m} \circ \delta(t) = t \cdot \alpha_{\mathrm{id}_{\mathbb{R}^m},0}\,(\, d_0\theta_g^{\mathbb{R}^m} \cdot \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}(q)\,) \qquad \forall\, t \in I \subseteq (-\tilde{\varepsilon}, \tilde{\varepsilon})$$

In particular, we have

$$g \cdot q = \theta_g^{\mathbb{R}^m} \circ \delta(1) = \alpha_{\mathrm{id}_{\mathbb{R}^m},0}\,(\, d_0\theta_g^{\mathbb{R}^m} \cdot \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}(q)\,). \tag{2.14}$$

Thus, we can finally compute the value of the inner product of $g \cdot q$ and $\bar{v}$:

$$\begin{aligned}
\langle g \cdot q, \bar{v}\rangle_{\text{euc}} &= (\gamma_{\text{normal}})_0\ (\,\alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}\,(g \cdot q)\ ,\ \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}\,(\bar{v})\,) \\
&\overset{(*)}{=} (\gamma_{\text{normal}})_0\ (\,d_0\theta_g^{\mathbb{R}^m} \cdot \alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}\,(q)\ ,\ v\,) \\
&= ((\theta_g^{\mathbb{R}^m})^*\,\gamma_{\text{normal}}\,|_{\tilde{W}})_0\ (\alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}\,(q)\ ,\ d_0\theta_{g^{-1}}^{\mathbb{R}^m} \cdot v\,) \\
&\overset{(**)}{=} (\gamma_{\text{normal}})_0\ (\alpha^{-1}_{\mathrm{id}_{\mathbb{R}^m},0}\,(q)\ ,\ d_0\theta_{g^{-1}}^{\mathbb{R}^m} \cdot v\,) \\
&= \langle\, q\ ,\ \alpha_{\mathrm{id}_{\mathbb{R}^m},0}\,(\,d_0\theta_{g^{-1}}^{\mathbb{R}^m} \cdot v\,)\,\rangle_{\text{euc}} \\
&\overset{(***)}{=} 0
\end{aligned}$$

We have used equation (2.14) for $(*)$ and the G_p-invariance of $\gamma_{\text{normal}}\,|_{\tilde{W}}$ for $(**)$. The last equation $(***)$ holds because $q \in (T_0^{\mathbb{R}^m} L)^\perp$ and because by equation (2.13) we have $d_0\theta_{g^{-1}}^{\mathbb{R}^m} \cdot v \in T_0 L$ and thus $\alpha_{\mathrm{id}_{\mathbb{R}^m},0}\,(\,d_0\theta_{g^{-1}}^{\mathbb{R}^m} \cdot v\,) \in T_0^{\mathbb{R}^m} L$.

In conclusion $g \cdot q \perp \bar{v}$ and, because $\bar{v}$ was arbitrary, we have indeed shown $g \cdot q \perp T_0^{\mathbb{R}^m} L$. Thus $\varphi(N)$, and therefore N, is G_p-invariant. ☐

Theorem 2.3.15 (Existence of slices)**.** *Let M be a smooth Cartan G-space and $p \in M$. Then there exists a slice S at p, which is a submanifold of M, and a local cross section $\chi : U \to G$ in G/G_p, such that the following statements hold:*

(i) For all $g_0 \in G$ the map

$$F_{g_0} : (g_0 \cdot U) \times S \to G \cdot S\,,\ (u, s) \mapsto g_0\chi(g_0^{-1} \cdot u) \cdot s$$

is a diffemorphism onto an open neighborhood of $g_0 \cdot S$ in M.

(ii) The map $f^S : G \cdot S \to G/G_p$ is smooth.

Proof. The isotropy group G_p is compact by Lemma 2.2.3. By the preceding Lemma 2.3.14 there exists some G_p-invariant submanifold $p \in N \subseteq M$ such that N is transverse to $G \cdot p$ at p. By Lemma 2.3.10 there is an open subset S^* of N, which is a near slice at p and a local cross section $\chi : U \to G$, such that $U \times S^* \to (u, s) \mapsto \chi(u) \cdot s$ is a diffeomorphism onto an open neighborhood of p. Applying Lemma 2.3.9, there is an open subset S in S^* that is a slice at p. Since N is a submanifold and S is open in N, clearly S is a submanifold of M. Also, it follows immediately that $F_e : U \times S \to G \cdot S$ is a diffeomorphism onto an open neighborhood of S. As a consequence, statement (i) already follows from homogeneity.
Denoty by $\Phi : G/G_p \xrightarrow{\sim} G \cdot p$ the canonical diffeomorphism (cf. equation (2.3)). Since S is a G_p-slice, the map f^S is continuous and hence so is $\Phi \circ f^S$. By Proposition 2.1.13 the map $\Phi \circ f^S$ is smooth if the composition $\iota \circ \Phi \circ f^S$ with the inclusion map $\iota : G \cdot p \hookrightarrow M$ is smooth. By equation (2.4) we have $\Phi \circ f^S(g \cdot s) = g \cdot p$. Therefore, to prove the smoothness of f^S, it suffices to show that the map $\iota \circ \Phi \circ f^S : G \cdot S \to M\,,\ g \cdot s \mapsto g \cdot p$ is smooth.
Let $q_0 = g_0 \cdot s_0$ be arbitrary. Using statement (i), $\iota \circ \Phi \circ f^S$ is smooth on $F_{g_0}((g_0 \cdot U) \times S)$ if and only if the map $\iota \circ \Phi \circ f^S \circ F_{g_0}\ :\ (g_0 \cdot U) \times S \to M$ is smooth.
The map $\theta^{(p)} : G \to M\,,\ g \mapsto g \cdot p$ is smooth and, by the universal property of surjective submersions, $\eta : G/G_p \to M\,,\ g\,G_p \mapsto g \cdot p$ is so too. Since χ is a local cross section, we have

$\iota \circ \Phi \circ f^S \circ F_{g_0}(u,s) = \eta\,(\,g_0\chi(g_0^{-1}\cdot u)\;G_p\,) = \eta\,(u)$, and therefore the map $\iota \circ \Phi \circ f^S \circ F_{g_0}$ is indeed smooth.
Because q_0 was arbitrary in $G \cdot S$, we have proven (ii). □

Corollary 2.3.16. *If S is a slice at p as in the above Theorem 2.3.15, then for any open subset W in S the set $G \cdot W$ is open in M.*

Proof. By statement (i) the set $F_e(U \times W)$ is open in $G \cdot S$ and, since $G \cdot S$ is open in M, it is open in M. Thus, $G \cdot F_e(U \times W) = G \cdot W$ is open in M. □

Remark 2.3.17. If S is a slice at p as in the above theorem, then by statement (i) the dimension of S (as manifold) is

$$\dim(S) = \dim(G \cdot S) - \dim(U) = \dim(M) - \dim(G) + \dim(G_p)\ . \tag{2.15}$$

Proposition 2.3.18 (G-invariant partition of unity)**.** *Let $\theta : G \times M \to M$ be a smooth and proper action of G on M. Then there exists a partition of unity for M of G-invariant functions, i.e. there is a countable family $(f_n)_{n\in N}$, $N \subseteq \mathbb{N}$, of smooth functions $f_n : M \to \mathbb{R}$ and a collection $(S_n)_{n\in N}$ of submanifolds of M such that:*

(i) $0 \leq f_n \leq 1$ *for all $n \in N$. For every $n \in N$ there is some $p \in M$ with $f_n(p) > 0$.*

(ii) For every $n \in N$ there is a $p(n) \in M$ such that S_n is a slice at $p(n)$.

(iii) The support $\operatorname{supp}(f_n) := \overline{\{p \in M \mid f_n(p) \neq 0\}}$ *of f_n is contained in $G \cdot S_n$ and the collection* $(\operatorname{supp}(f_n))_{n\in N}$ *is a locally finite covering of M.*

(iv) $\sum_{n\in N} f_n(p) = 1$ *for all $p \in M$*

(v) $\theta_g^* f_n = f_n \circ \theta_g = f_n$ *for all $n \in N$ and $g \in G$*

Proof. We begin by showing that M/G is a paracompact space:
The following statement is a general topological fact, for a proof see [aut21b] :
If X is a second-countable Hausdorff topological space, which is locally compact, then it is paracompact.
As manifold M is locally compact and by Lemma 2.1.5 the projection $\Pi_M : M \twoheadrightarrow M/G$ is an open map. So, M/G is locally compact too. By Proposition 2.1.9 M/G is Hausdorff. Since Π_M is open and M is second-countable, M/G is second-countable too. Thus M/G is paracompact.

For every open cover of a paracompact Hausdorff space X there is a subordinate partition of unity. In particular, for every open cover $(U_i)_{i\in I}$ of X there is an open cover $(V_i)_{i\in I}$ of X such that $(\overline{V_i})_{i\in I}$ is locally finite and $\overline{V_i} \subseteq U_i$ for all $i \in I$. As shown above, these two properties hold particularly for M/G.

In the next step we will prove the following statement:

($*$) Given a collection $(U_n)_{n\in N}$, $N \subseteq \mathbb{N}$, such that $U_n \subseteq S_n$ is open in S_n, where S_n is a slice at $p(n) \in M$ as in Theorem 2.3.15. Suppose, U_n is $G_{p(n)}$-invariant and $(G \cdot U_n)_{n\in N}$ covers M. Then there exists an open cover $(\Omega_n)_{n\in N}$ of M/G such that $\overline{\Omega_n} \subseteq \Pi_M(U_n)$, $(\overline{\Omega_n})_{n\in N}$ is locally finite and, if we put $V_n := \Pi_M^{-1}(\Omega_n) \cap S_n$, then V_n is open in S_n and $G_{p(n)}$-invariant, $(G \cdot V_n)_{n\in N}$ is an open cover of M, $\overline{V_n}^{S_n} \subseteq U_n$ for $n \in N$ (where $\overline{\,\cdot\,}^{S_n}$ denotes the closure in S_n) and $(G \cdot \overline{V_n}^{S_n})_{n\in N}$ is locally finite.

Proof of ($*$). Since M/G is a paracompact Hausdorff space and $(\Pi_M(U_n))_{n\in N} = (\Pi_M(G \cdot U_n))_{n\in N}$ is an open cover of M/G (using Corollary 2.3.16 and the openness of Π_M), we can choose an open covering $(\Omega_n)_{n\in N}$ of M/G such that $(\overline{\Omega_n})_{n\in N}$ is locally finite and $\overline{\Omega_n} \subseteq \Pi_M(U_n)$. Put $V_n := \Pi_M^{-1}(\Omega_n) \cap S_n$. Then V_n is open in S_n and $G_{p(n)}$-invariant (since

$\Pi_M^{-1}(\Omega_n)$ is G-invariant and S_n is a slice at $p(n)$, so S_n is $G_{p(n)}$-invariant).
Given an arbitrary $p \in M$. Then there is some $n \in N$ with $\Pi_M(p) \in \Omega_n \subseteq \Pi_M(U_n)$. So there exist $g \in G$, $u \in U_n \subseteq S_n$ with $p = g \cdot u$. We have $\Pi_M(u) = G \cdot (g^{-1} \cdot p) = \Pi_M(p) \in \Omega_n$ and therefore $u \in V_n$. This shows that $\bigcup_n (G \cdot V_n) = M$. By Corollary 2.3.16 the set $G \cdot V_n$ is open in M, so $(G \cdot V_n)_{n \in N}$ is an open cover of M.
Now, let $p \in \overline{V_n}^{S_n}$ be arbitrary. We have

$$\overline{V_n}^{S_n} \subseteq \Pi_M^{-1}(\overline{\Omega_n}) \cap S_n \subseteq \Pi_M^{-1}(\Pi_M(U_n)) \cap S_n = G \cdot U_n \cap S_n.$$

Choose $g \in G$, $u \in U_n$ with $p = g \cdot u$. Applying Theorem 2.3.6 (c), we get $g \in G_{p(n)}$. Since U_n is $G_{p(n)}$-invariant, we conclude $p \in U_n$.
Since $(\overline{\Omega_n})_{n \in N}$ is locally finite, the collection $(\Pi_M^{-1}(\Omega_n))_{n \in N}$ is so too. Because we have $G \cdot \overline{V_n}^{S_n} \subseteq G \cdot \Pi_M^{-1}(\Omega_n) = \Pi_M^{-1}(\Omega_n)$, the cover $(G \cdot \overline{V_n}^{S_n})_{n \in N}$ is locally finite.

(Proof of $(*)$)

By Corollary 2.2.7, M is a smooth Cartan G-space. For every p in M choose a slice S_p at p as in Theorem 2.3.15. Now, choose an open neighborhood $p \in U_p \subseteq S_p$ of p in S_p such that U_p is precompact in S_p (this is possible because S_p is a submanifold). G_p is compact by Lemma 2.2.3, hence we can assume that U_p is G_p-invariant (use Corollary A.1.10). By Corollary 2.3.16 the set $G \cdot U_p$ is open in M. Since M is second countable, there is a countable subcover $(G \cdot U_n)_{n \in N}$, $N \subseteq \mathbb{N}$, of $(G \cdot U_p)_{p \in M}$.
By $(*)$ we can find $\Omega_n \subseteq M/G$, $n \in n$, such that $(\overline{\Omega_n})_{n \in N}$ is locally finite, $V_n := \Pi_M^{-1}(\Omega_n) \cap S_n$ is open in S_n and $G_{p(n)}$-invariant, $(G \cdot V_n)_{n \in N}$ is an open cover of M, $\overline{V_n}^{S_n} \subseteq U_n$ and $(G \cdot \overline{V_n}^{S_n})_{n \in N}$ is locally finite. Using $(*)$ again, there are W_n, $n \in N$, such that $W_n \subseteq S_n$ is open in S_n and $G_{p(n)}$-invariant, $\overline{W_n}^{S_n} \subseteq V_n$ and $(G \cdot W_n)_{n \in N}$ covers M. We have

$$W_n \subseteq \overline{W_n}^{S_n} \subseteq V_n \subseteq \overline{V_n}^{S_n} \subseteq U_n \subseteq \overline{U_n}^{Sn} \subseteq S_n,$$

and, since U_n is precompact in S_n, the subsets W_n , V_n , are precompat in S_n too. Assume $W_n \neq \varnothing$ for all $n \in N$ (otherwise consider $\tilde{N} := \{n \in N \mid W_n \neq \varnothing\}$).
Now put

$$K_n := \overline{W_n}^{S_n} \neq \varnothing\,,\ L_n := \overline{V_n}^{S_n}$$

Then K_n and L_n are compact subsets of M, $(G \cdot K_n)_n$ covers M and $(G \cdot L_n)_n$ is a locally finite covering of M. By [Lee13, Proposition 2.25], for each $n \in N$ there is a smooth bump function $\rho_n : S_n \to \mathbb{R}$ such that $0 \leq \rho_n \leq 1$, $\rho_n \equiv 1$ on K_n and $\operatorname{supp}(\rho_n) \subseteq V_n \subseteq L_n$.
Let $\hat{\rho}_n : S_n \to \mathbb{R}$ be the function defined by

$$\hat{\rho}_n := \int_{G_{p(n)}} g^* \rho_n \ \mathrm{d}\mu(g) = \int_{G_{p(n)}} \rho_n \circ \theta_g \ \mathrm{d}\mu(g)$$

(cf. Remark 2.3.12). By Remark 2.3.13, $\hat{\rho}_n$ is smooth. Since μ is bi-invariant, $\hat{\rho}_n$ is $G_{p(n)}$-invariant. Clearly $\hat{\rho}_n \geq 0$. Because $\rho_n \circ \theta_e(p) = 1 \quad \forall\, p \in K_n$, the function $\hat{\rho}_n$ is strictly positive on K_n (by continuity). In addition, the support of $\hat{\rho}_n$ is contained in L_n (because $\operatorname{supp}(\rho_n) \subseteq V_n$ and V_n is $G_{p(n)}$-invariat), so it is compact.
Now define

$$\tau_n : M \to \mathbb{R}\,,\ p \mapsto \begin{cases} \hat{\rho}_n(s) & , \text{if } p = g \cdot s\,,\, g \in G\,,\, s \in S_n \\ 0 & , \text{if } p \notin G \cdot S_n \end{cases}$$

This is well-defined because, if $p = g \cdot s = \tilde{g} \cdot \tilde{s}$, then $\tilde{g}^{-1} g \cdot s = \tilde{s}$. Since S_n is a slice at $p(n)$ (hence $((S, S)) = G_{p(n)}$ by Theorem 2.3.6 (c)) and $\hat{\rho}_n$ is $G_{p(n)}$-invariant, it follows that $\hat{\rho}_n(\tilde{s}) = \hat{\rho}_n(\tilde{g}^{-1} g \cdot s) = \hat{\rho}_n(s)$.

The function τ_n is smooth on $G \cdot S_n$: Given $p = g_0 \cdot s_0 \in G \cdot S_n$. Then by Theorem 2.3.15 (i) the map τ_n is smooth in a neighborhood of p if and only if the function $\tau_n \circ F_{n,g_0}$ is smooth. By definition we have

$$\tau_n \circ F_{n,g_0}(u, s) = \tau_n(g_0\, \chi_n(g_0^{-1} \cdot u) \cdot s) = \hat{\rho}_n(s)$$

and $\hat{\rho}_n$ is smooth. So, $\tau_n|_{G \cdot S_n}$ is smooth.
Now, given an arbitrary boundary point $p \in \partial(G \cdot S_n)$ of $G \cdot S_n$. We will show that there is an open neighborhood $\mathcal{U}(p)$ of p such that $\tau_n \equiv 0$ on $\mathcal{U}(p)$. This proves the smoothness of τ_n on all of M.
Assume by contradiction that for every open neihborhood $\mathcal{U}$ of p there is a point $q \in \mathcal{U}$ such that $\tau_n(q) \neq 0$. Choose a series $(q_m)_{m \in \mathbb{N}}$ with $q_m \to p$ and $\tau_n(q_m) \neq 0 \quad \forall\, m \in \mathbb{N}$. Put $q_m =: g_m \cdot s_m$, $g_m \in G$, $s_m \in S_n$. Then $s_m \in \operatorname{supp}(\hat{\rho}_n) \subseteq L_n$. Since L_n is compact, by passing to a subsequence we can assume that $s_m \to s \in L_n \subseteq S_n$. By Proposition 2.1.10 (iii) there is a subsequence $(g_{m_k})_k$ of $(g_m)_m$ and a $g \in G$ such that $g_{m_k} \to g$. Therefore, $g \cdot s = \lim_k g_{m_k} \cdot s_{m_k} = \lim_k q_{m_k} = p$. Thus, $p \in G \cdot S_n \cap \partial(G \cdot S_n) \neq \varnothing$, which is a contradiction because $G \cdot S_n$ is open in M.

By definition τ_n is obviously G-invariant: $\tau_n(g \cdot p) = \tau_n(p) \quad \forall\, p \in M\,,\, g \in G$.

If $\tau_n(p) \neq 0$, $p \in M$, then $p = g \cdot s$ for some $g \in G$, $s \in S_n$. Thus,

$$0 \neq \tau_n(p) = \hat{\rho}_n(s) = \int_{G_{p(n)}} \rho(h \cdot s)\ \mathrm{d}\mu(h)$$

and there exists an $h \in G_{p(n)}$ such that $\rho_n(h \cdot s) \neq 0$. It follows $h \cdot s \in \operatorname{supp}(\rho_n) \subseteq V_n$, so, since V_n is $G_{p(n)}$-invariant,

$$s = \underbrace{h^{-1}}_{\in G_{p(n)}} \cdot \underbrace{(h \cdot s)}_{\in V_n} \in V_n \ .$$

Hence $p = g \cdot s \in G \cdot V_n \subseteq G \cdot \Pi_M^{-1}(\Omega_n) = \Pi_M^{-1}(\Omega_n) \subseteq \Pi_M^{-1}(\overline{\Omega_n})$. This however shows $\operatorname{supp}(\tau_n) \subseteq \Pi_M^{-1}(\overline{\Omega_n})$. Since $(\overline{\Omega_n})_n$ is locally finite, it follows that $(\Pi_M^{-1}(\overline{\Omega_n}))_n$ and thus $(\operatorname{supp}(\tau_n))_n$ are locally finite too. Also, $\{p \in M \mid \tau_n(p) \neq 0\} \subseteq G \cdot V_n \subseteq G \cdot S_n$. As we already have shown above, for each boundary point $p \in \partial(G \cdot S_n)$ there is a neighborhood $\mathcal{U}(p)$ of p such that $\tau_n \equiv 0$ on $\mathcal{U}(p)$. Therefore, the support of τ_n is contained in $G \cdot S_n$.

Since $\hat{\rho}_n$ is strictly positive on K_n and the collection $(G \cdot K_n)_n$ is a covering of M, the sum

$$\sum_{n \in N} \tau_n > 0$$

is strictly positive on M and a smooth function.

Then define the searched functions f_n , $n \in N$, as

$$f_n := \Big(\sum_{m \in N} \tau_m\Big)^{-1} \tau_n \qquad , \ n \in N \ .$$

□

2.4 Quotient Manifold Theorem

In this section we will give a brief proof of a strong version of the Quotient Manifold Theorem. To avoid confusion, some preliminary remarks should be made:
There are various ways to prove the Quotient Manifold Theorem, differing strongly in the applied techniques, though they all require elaborate preparation. A version, which is based

only on the theory of distributions and integral manifolds, is given in [Lee13, Theorem 21.10] and is completely independent from this thesis. It is however slightly weaker than the version we will present, in the sense that it does not make any statements about G-principality of the canonical projection. The proof in [Lee13] cannot easily be adapted, so that it would cover our, stronger version. As pointed out in the previous sections, we have already used the quotient manifold theorem in the examination of slices and their existence plenty of times: We have applied, that for any closed subgroup $H \leq G$ the factor group G/H carries a natural smooth structure such that the canonical projection $G \twoheadrightarrow G/H$ is a smooth submersion.
This is in fact the dilemma we face: The proof of the strong version of the Quotient Manifold Theorem, which we will give below, maybe seems like an enormous circular argument since it requires a version of it for Lie groups.
Therefore, it may be best to just take the strong version of the Quotient Manifold Theorem for granted and accept that it has been proven numerously already. Then we have provided a strict argumentation for the existence of slices and G-invariant partitions of unity, the second one will become significant later.
Another possible point of view is, to assume the weak version of the Quotient Manifold Theorem (with a proof that uses different techniques than we do, e.g. as in [Lee13]). In this case, we have added a strong global statement about quotient manifolds, namely that they are principal bundles.
The third way of interpreting this long chain of arguments is minimalistic: There are several proofs, specified on Lie groups only, that tell us that $G \twoheadrightarrow G/H$ is a submersion in a unique way for $H \leq G$ closed. With this point of view the current chapter additionally contains a complete proof of the Quotient Manifold Theorem for arbitrary manifolds.
In either way, the content of the following section is surely not necessary for the the rest of the thesis, provided that the Quotient Manifold Theorem is already familiar. We rather present the proof of the Quotient Manifold Theorem because we already have developed strong techniques with the purpose of showing the existence of a G-invariant partition of unity, so it would be a waste of effort not to present the proof, which uses exactly these techniques, here.

Definition 2.4.1. We say that the continuous G-action $\theta : G \times X \to X$ on X is ***free*** if the stabilizer G_x of x is trivial for every $x \in X$.

Definition 2.4.2. Let B be a topological space and X a G-space. Let $\pi : X \to B$ be a continuous G-invariant map (i.e. π is G-equivariant if we consider the trivial action on B). We shall say that (X, π) is a ***G-principal bundle*** if B has a covering $\{U_i\}_{i \in I}$ by open sets such that for every i there is a G-equivariant homeomorphism $\phi_i : \pi^{-1}(U_i) \to G \times U_i$ with the property that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[d, "\pi"'] & G \times U_i \arrow[dl, "\mathrm{pr}_2"] \\ U_i & \end{tikzcd}$$

Here pr_2 denotes the projection onto the second factor and $G \times U_i$ is endowed with the natural G-action $g_1 \cdot (g_2, u) := (g_1 g_2, u)$ (i.e. $G \times U_i$ carries the product action of the G-spaces G and U_i).
If B and X are smooth manifolds, π is smooth and ϕ_i is a diffeomorphism for $i \in I$, then we call (X, π) a ***smooth G-principal bundle***.

Remark 2.4.3. By definition every G-principal bundle (X, π) is a fiber bundle with fiber G, so π is an open quotient map. Furthermore, G has to act freely on X and transitively on each fiber. π factors uniquely through the orbit space via $\bar{\pi} : X/G \to B$ and $\bar{\pi}$ is a homeomorphism (respectively diffeomorphism in the smooth case). This implies that we can generally omit the base space B and consider X/G instead. Informally spoken, a (smooth) G-space X is a (smooth) G-principal bundle if and only if it is locally the product of an open subset of X/G with G itself (in the category of (smooth) G-spaces).

Theorem 2.4.4 (Quotient Manifold Theorem)**.** *Let $\theta : G \times M \to M$ be a smooth, free and proper action of the Lie group G on the manifold M. Then the orbit space M/G is a topological manifold of dimension $\dim(M/G) = \dim(M) - \dim(G)$ and has a unique smooth structure such that the canonical projection $\Pi_M : M \twoheadrightarrow M/G$ is a smooth submersion. Furthermore, with this smooth structure (M, Π_M) is a smooth G-principal bundle.*

Proof. Put $m := \dim(M)$, $k := \dim(G)$ and $n := m - k$.
Let us first show the uniqueness of the smooth structure on the orbit space. Assume $\mathcal{A}_1, \mathcal{A}_2$ are two smooth structures on M/G such that Π_M is a surjective submersion with respect to $(M/G, \mathcal{A}_1)$ respectively $(M/G, \mathcal{A}_2)$. Then

$$\Pi_M = \mathrm{id}_{M/G} \circ \Pi_M : M \to (M/G, \mathcal{A}_1) \to (M/G, \mathcal{A}_2)$$

is smooth and by the characteristic property of surjective submersions the map $\mathrm{id}_{M/G}$: $(M/G, \mathcal{A}_1) \to (M/G, \mathcal{A}_2)$ is smooth. This argument is symmetric; thus the identity map $\mathrm{id}_{M/G}$ is a diffeomorphism from $(M/G, \mathcal{A}_1)$ to $(M/G, \mathcal{A}_2)$. Therefore, the smooth structures coincide: $\mathcal{A}_1 = \mathcal{A}_2$.

Now, let us show that M/G is indeed a topological manifold:
By Proposition 2.1.9 the orbit space M/G is Hausdorff. It is also second-countable because M is second-countable and Π_M is an open map by Lemma 2.1.5. To show that the quotient space is locally Euclidean of dimension n, we construct appropriate charts. Let $x \in M/G$ be arbitrary. Choose $p \in M$ with $\Pi_M(p) = G \cdot p = x$. Choose a slice S at p as in Theorem 2.3.15 (the action is proper, so by Corollary 2.2.7 M is a Cartan G-space).
Put $V := \Pi_M(S) = \Pi_M(G \cdot S)$, then V is an open neighborhood of x. Consider the map

$$\begin{aligned} \eta : S &\to V \\ s &\mapsto \Pi_M(s) \end{aligned} \tag{2.16}$$

Clearly, η is continuous and surjective. Suppose $\eta(s_1) = \eta(s_2)$, then there is some $g \in G$ such that $g \cdot s_1 = s_2 \in (g \cdot S) \cap S$. By Theorem 2.3.6 (c) it follows that $g \in G_p = \{e\}$, so $s_1 = s_2$. Thus η is injective and therefore bijective.
Now let $W \subseteq S$ be an arbitrary open subset in S. By Corollary 2.3.16 the set $G \cdot W$ is open in M. Hence, $\eta(W) = \Pi_M(W) = \Pi_M(G \cdot W)$ is open in M/G. This shows the openness of η. Overall, we have proven that η is in fact a homeomorphism.
By our choice, S is a submanifold of M (because S is a slice as in Theorem 2.3.15) of dimension $\dim(S) = m - k + 0 = n$ (see Remark 2.3.17). Now choose a smooth chart (W, φ) for S, containing p. Then the composition $\varphi \circ \eta^{-1} : \eta(W) \to \varphi(W) \subseteq \mathbb{R}^n$ is a chart of M/G around x. Since $x \in M/G$ was arbitrary, M/G is a topological manifold of dimension n.

Now let $p \in M$ be an arbitrary point and S a slice at p as in Theorem 2.3.15. We claim that the multiplication map

$$\begin{aligned} F : G \times S &\to G \cdot S \\ (g, s) &\mapsto g \cdot s \end{aligned} \tag{2.17}$$

is a diffeomorphism. For obvious reasons, F is smooth and surjective. If $g \cdot s = \tilde{g} \cdot \tilde{s}$, then by Theorem 2.3.6 we obtain $\tilde{g}^{-1} g \in G_p$. Since the action is free, it follows that $g = \tilde{g}$, and thus $s = \tilde{s}$. So F is a bijection. It remains to show that for any given point $g_0 \cdot s_0 \in G \cdot S$ there is an open neighborhood on which F^{-1} is smooth. Reusing Theorem 2.3.15, there is a local cross section $\chi : U \to G$ in G/G_p such that the map $F_{g_0} : U \times S \to \mathcal{U}(g_0 \cdot S)$, defined by $F_{g_0}(u, s) = g_0 \chi(g_0^{-1} \cdot u) \cdot s$, is a diffeomorphism onto an open neighborhood $\mathcal{U}(g_0 \cdot S)$

of $g_0 \cdot S$. By assumption the action is free, so $G_p = \{e\}$. Hence, the natural projection $\mathrm{pr} : G \to G/G_p = G/\{e\}$ is bijective. Since it is a submersion, it has constant rank, thus by the Global Rank Theorem it is a diffeomorphism. Let $\tilde{U}$ be the preimage of $U \subseteq G/\{e\}$ under the canonical projection $\mathrm{pr} : G \to G/\{e\}$. Using the fact that χ is a local cross section, we obtain $\chi(\{g\}) = g$ for all $g \in \tilde{U}$. Thus, $F_{g_0}(\{g\}, s) = g \cdot s$ and the restriction $F|_{\tilde{U}\times S} = F_{g_0} \circ (\mathrm{pr}|_{\tilde{U}} \times \mathrm{id}_S)$ is a diffeomorphism onto the open neighborhood $\mathcal{U}(g_0 \cdot S)$ of $g_0 \cdot s_0$.

Now let us deal with the existence of a smooth structure on M/G. We just have to show that any two charts as constructed above are smoothly compatible.
Given arbitrary points $p_1, p_2 \in M$, let S_1 , S_2 be slices at p_1 respectively p_2 as in Theorem 2.3.15 and put $V_i := \Pi_M(S_i)$, $i = 1, 2$. For $i = 1, 2$, let $\eta_i := \Pi_M|_{S_i}^{V_i}$ and φ_i be smooth charts for S_i, containing p_i. Assume $V_1 \cap V_2 \neq \varnothing$. We need to show the smoothness of the transition map $\varphi_2 \circ \eta_2^{-1} \circ \eta_1 \circ \varphi_1^{-1} : (\varphi_1 \circ \eta_1^{-1})(V_1 \cap V_2) \to (\varphi_2 \circ \eta_2^{-1})(V_1 \cap V_2)$. It obviously suffices to show that

$$\begin{aligned} &\eta_2^{-1} \circ \eta_1 : \\ &\eta_1^{-1}(V_1 \cap V_2) = G \cdot S_1 \cap G \cdot S_2 \cap S_1 \quad \to \quad \eta_2^{-1}(V_1 \cap V_2) = G \cdot S_1 \cap G \cdot S_2 \cap S_2 \end{aligned}$$

is smooth. For $s_1 \in S_1$ the image $\eta_2^{-1} \circ \eta_1(s_1)$ is the only element $s_2 \in S_2$ such that $\exists\, g \in G : \ s_1 = g \cdot s_2$. Let $F_i : G \times S_i \to G \cdot S_i$, $i = 1, 2$, denote the multiplication map (defined in equation (2.17)), it is a diffeomorphism as we have proven above. Then we obtain

$$\eta_2^{-1} \circ \eta_1\,(s_1) = \mathrm{pr}_2 \circ F_2^{-1} \circ F_1\ (e, s_1)\ ,$$

where $\mathrm{pr}_2 : G \times S_2 \to S_2$ is the projection onto the second factor. Thus, the map $\eta_2^{-1} \circ \eta_1$ is smooth and we have proven that the defined atlas on M/G is a smooth one.

Clearly, if (M, Π_M) is a G-principal bundle, then the projection Π_M is a smooth submersion. Thus, it only remains to show that M is indeed a G-principal bundle.
M is covered by sets $G \cdot S$ for a slice S at (some) $p \in M$. For such a slice S the map $\eta := \Pi_M|_S^V$, where $V := \Pi_M(S)$, is a diffeomorphism by definition. In addition the multiplication map $F : G \times S \to G \cdot S$ (see equation (2.17)) is a diffeomorphism. Thus, the composition $\phi := (\,\mathrm{id}_G \times \eta\,) \circ F^{-1}$ is a diffeomorphism and, because both F and $(\,\mathrm{id}_G \times \eta\,)$ are G-equivariant (where $G{\cdot}S$ carries the natural G-action, S , V carry the trivial one and products are endowed with the product action), ϕ is G-equivariant. For $g \cdot s \in G \cdot S = \Pi_M^{-1}(V)$ with $g \in G$ and $s \in S$ we have

$$\mathrm{pr}_2\,(\phi(g \cdot s)) = \mathrm{pr}_2\,(\,g, \eta(s)\,) = \Pi_M(s) = \Pi_M(g \cdot s)\ ,$$

where $\mathrm{pr}_2 : G \times V \twoheadrightarrow V$ denotes the projection onto the second factor. Thus, the defining diagram commutes: $\Pi_M\,|_{G\cdot S} = \mathrm{pr}_2 \circ \phi$. This shows that M/G is a G-principal bundle and finishes the proof. □

3 Topological Cuts

3.1 Motivation

In this chapter we will introduce the concept of topological cuts. Contrary to what one might think, this does not mean that we will only study certain topological spaces without any further structures. In fact, the basic construction is a cut along a certain level set of a manifold M, endowed with a smooth S^1-action, so we are not working in the category of topological spaces but of smooth manifolds. Instead, the name refers to the first step of the construction: Given a fixed level set $f^{-1}(a)$ of an S^1-invariant function $f : M \to \mathbb{R}$ we want to consider the open subset $f^{-1}(\,(a,\infty)\,)$ 'above' $f^{-1}(a)$ and leave it as it is and the level set $f^{-1}(a)$ itself and identify points in the same S^1-orbit, thus collapsing orbits in the level set of a. In other words, we consider the quotient space $X := f^{-1}(\,[a,\infty)\,)\,/\!\sim$ with an equivalence relation $\sim$ so that two distinct points are in relation if and only if they are in $f^{-1}(a)$ and in the same S^1-orbit. We call this quotient space X the cut with respect to f. The natural topology on X is of course the quotient topology. However, it is rather non-intuitive how to endow X with a smooth structure and, even, why we should do so. This is the reason we speak of 'topological cuts' and suppress the smoothness, at least in the name.

As an example one might consider $M = \mathbb{R}^3 \setminus (\mathbb{R} \cdot (0,0,1))$ with the smooth function

$$f(x,y,z) := \gamma \cdot (x^2 + y^2) - z$$

for some parameter $\gamma > 0$ and $a = 0$. Let S^1 act on M by $\lambda \cdot (w,z) := (\lambda \cdot w, z)$ for $\lambda \in S^1$, $w \in \mathbb{C}\backslash\{0\}$, $z \in \mathbb{R}$. Figure 3.1 pictures the hypersurface $f^{-1}(a)$. Each plotted orbit will collapse to one point in X.

The key observation in Lerman's paper *Symplectic Cuts* [Ler95] is the close connection of cutting to (symplectic) blow-ups. This justifies giving X a smooth structure that is natural if one is interested in blow-ups in the symplectic category. It turns out that symplectic cuts are a useful general tool in symplectic geometry to achieve a variety of interesting results. Consequently, in the subsequent paper [Ler01], the main source for this thesis, Lerman introduces contact cuts, the analog in the contact category.

However, our focus is not the application of this construction but to give detailed and rigorous proofs of the basic propositions in [Ler01] since these are not presented at full length in the original paper.

3.2 The Basic Construction

Recall from Chapter 2 that any continuous action of a compact Lie group G on some manifold M is proper (see Corollary 2.1.11). In particular every continuous action of the circle S^1 on a manifold M is proper. By the quotient manifold theorem (cf. Theorem 2.4.4), the quotient space M/S^1 carries a canonical smooth structure if the action is smooth and free.

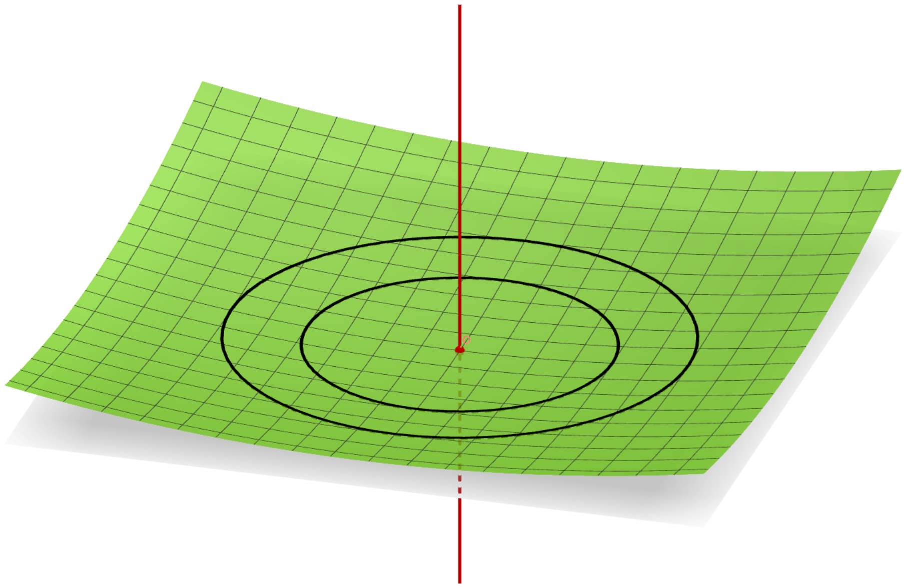

Figure 3.1: The above example with parameter $\gamma = 0.01$ visualized with the help of GeoGebra.

But before we deal with the main proposition of this chapter, we should come to an agreement regarding notation, which we will also use beyond this chapter:

Given a chart (U, φ) of the smooth manifold M (of dimension $\dim(M) =: m$) at p, we denote by $\alpha_{\varphi,p} : T_pM \to \mathbb{R}^m$ the coordinates of the tangent space at p, induced by φ, i.e. if $v = \dot{\gamma}(0) \in T_pM$, where $\gamma : I \to M$ is a smooth path through p, then

$$\alpha_{\varphi,p}(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi \circ \gamma\,(t) = d_p\varphi \cdot v\,.$$

$\alpha_{\varphi,p}$ is a linear isomorphism.
Let $F : M \to N$ be a smooth chart between two smooth manifolds, we denote by $d_pF : T_pM \to T_{F(p)}N$ the differential of F at $p \in M$.
We will write $\mathrm{J}_f(p)$ for the Jacobian matrix of a differentiable function $f : \mathbb{R}^m \underset{\text{open}}{\supseteq} U \to \mathbb{R}^n$ at $p \in U$:

$$\mathrm{J}_f(p) = \left(\frac{\partial f_i}{\partial x^j}(p)\right)_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \in \mathbb{R}^{n\times m}$$

The (real) differential of f ap p is denoted by $D_pf : \mathbb{R}^m \to \mathbb{R}^n$. For a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ we write $\mathrm{Mat}(L) \in \mathbb{R}^{n\times m}$ for the matrix associated with L (with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$). Thus $\mathrm{Mat}(D_pf) = \mathrm{J}_f(p)$.
Let $F : M \to N$ be a smooth map between two manifolds, $m := \dim(M)$, $n := \dim(N)$, $p \in M$ and (U, φ) and (W, ψ) charts around p respectively $F(p)$. Then it should be known that the following diagram commutes:

$$\begin{tikzcd} T_pM \arrow[r, "d_pF"] \arrow[d, "\alpha_{\varphi,p}"', "\wr"] & T_{F(p)}N \arrow[d, "\alpha_{\psi,F(p)}", "\wr"'] \\ \mathbb{R}^m \arrow[r, "D_{\varphi(p)}(\psi\circ F\circ\varphi^{-1})"'] & \mathbb{R}^n \end{tikzcd} \tag{3.1}$$

Proposition 3.2.1. *Let S^1 act smoothly on the manifold M via $\theta : S^1 \times M \to M$, $(\lambda, m) \mapsto \lambda \cdot m$. Suppose $a \in \mathbb{R}$ is a regular value of the smooth and S^1-invariant function $f : M \to \mathbb{R}$. Assume that the restricted smooth action of S^1 to the submanifold $f^{-1}(a)$ is free. Define an equivalence relation $\sim$ on the preimage $f^{-1}(\,[a,\infty)\,)$ by:*
$\forall\ m \neq m' : \quad m \sim m' \quad$ *if and only if:*

(i) $\quad m, m' \in f^{-1}(a) \quad$ *and*

(ii) $\quad \exists\, \lambda \in S^1 : \quad m = \lambda \cdot m'$

Then the quotient space

$$M_{[a,\infty)} := f^{-1}(\,[a,\infty)\,)\,/ \sim \quad ,$$

endowed with the quotient topology, is a topological manifold of dimension $\dim(M)$ *and carries a canonical smooth structure.*
Let $\Pi : f^{-1}(\,[a,\infty)\,) \twoheadrightarrow M_{[a,\infty)}$ denote the projection. Then the set $\Pi(\,f^{-1}(\,(a,\infty)\,)\,)$ is open and dense in $M_{[a,\infty)}$ and diffeomorphic to $f^{-1}(\,(a,\infty)\,)$ via Π. The difference $M_{[a,\infty)} \setminus \Pi(\,f^{-1}(\,(a,\infty)\,)\,)$ is a submanifold of $M_{[a,\infty)}$ and diffeomorphic to $f^{-1}(a)\,/\,S^1$.

Proof. First note that, since a is a regular value, the level set $f^{-1}(a)$ is a submanifold of codimension 1. Because f is S^1-invariant, the action restricts to level sets of f. Using the characteristic property of embeddings, we conclude that the restriction $\theta|_{S^1\times f^{-1}(a)}$: $S^1 \times f^{-1}(a) \to f^{-1}(a)$ is a smooth action too. By assumption the action is free, so by the Quotient Manifold Theorem 2.4.4 the orbit space $f^{-1}(a)/S^1$ is a smooth manifold.

Consider the action of S^1 on the product manifold $M \times \mathbb{C}$,

$$\begin{aligned} \Theta : S^1 \times (M \times \mathbb{C}) &\longrightarrow (M \times \mathbb{C}) \\ (\,\lambda, (m,z)\,) &\longmapsto \lambda \cdot (m,z) := (\lambda \cdot m, \lambda^{-1} z) \end{aligned} \tag{3.2}$$

The action is smooth since $\mathrm{pr}_M \circ \Theta$ and $\mathrm{pr}_{\mathbb{C}} \circ \Theta$ are smooth, where $\mathrm{pr}_M : M \times \mathbb{C} \to M$ and $\mathrm{pr}_{\mathbb{C}} : M \times \mathbb{C} \to \mathbb{C}$ are the projection maps. Define the function

$$\begin{aligned} \Psi : M \times \mathbb{C} &\longrightarrow \mathbb{R} \\ (m,z) &\longmapsto f(m) - |z|^2 \end{aligned} \tag{3.3}$$

Since f is smooth and S^1-invariant, Ψ is smooth (compose Ψ with product charts of $M \times \mathbb{C}$) and S^1-invariant too: $\Psi(\lambda \cdot (m,z)) = f(\lambda \cdot m) - |\lambda^{-1} z|^2 = f(m) - |z|^2$, $\lambda \in S^1, (m,z) \in M \times \mathbb{C}$.
Put

$$\begin{aligned} N_1 &:= \Big\{ (m,z) \in M \times \mathbb{C} \mid f(m) > a\,,\ |z| = \sqrt{f(m) - a} \Big\} \\ N_2 &:= \Big\{ (m,0) \in M \times \mathbb{C} \mid f(m) = a \Big\}\,. \end{aligned}$$

Clearly, we have the disjoint decomposition $\Psi^{-1}(a) = N_1 \sqcup N_2$. Because we have $N_2 = f^{-1}(a) \times \{0\}$, the subset N_2 is a closed submanifold of $M \times \mathbb{C}$ (cf. Appendix Lemma A.2.1) of codimension 3. Thus, $N_1 = \Psi^{-1}(a) \setminus N_2$ is open in $\Psi^{-1}(a)$.
We argue that every point in $\Psi^{-1}(a)$ is a regular point with respect to Ψ ; hence a is a regular value of Ψ and the level set $\Psi^{-1}(a)$ is a submanifold of $M \times \mathbb{C}$ of codimension 1.
Let $(m_0, z_0) \in \Psi^{-1}(a)$ be arbitrary. Choose a chart φ_M around m_0. Then $\varphi_M \times \mathrm{id}_{\mathbb{C}}$ is a chart containing (m_0, z_0). By diagram (3.1) it suffices to show that the Jacobian matrix $\mathrm{J}_{\Psi\circ(\varphi_M^{-1}\times \mathrm{id}_{\mathbb{C}})}(\varphi_M(m_0)\,,\, z_0)$ is not the zero matrix (then it has full rank). Let $z_0 =: x_0 + i\,y_0 \in \mathbb{C}$. We compute:

$$\mathrm{J}_{\Psi\circ(\varphi_M^{-1}\times \mathrm{id}_{\mathbb{C}})}(\varphi_M(m_0)\,,\, z_0) = \big(\nabla_{\varphi_M(m_0)}(\,f \circ \varphi_M^{-1}\,(\cdot) - |z_0|^2\,)^T \ , -2x_0\,,\, -2y_0\big)$$

If $(m_0, z_0) \in N_1$, then $z_0 \neq 0$ and the Jacobian matrix does not vanish. Now assume $(m_0, z_0) \in N_2$, so $z_0 = 0$ and $m_0 \in f^{-1}(a)$. Now, again applying diagram (3.1), we obtain:

$$\begin{aligned} \nabla_{\varphi_M(m_0)}\big(f \circ \varphi_M^{-1}(\cdot) - |z_0|^2 \big)^T &= \nabla_{\varphi_M(m_0)}\big(f \circ \varphi_M^{-1} \big)^T \\ &= \mathrm{Mat}\big(\alpha_{\mathrm{id}_{\mathbb{R}}, f(m_0)} \circ d_{m_0} f \circ (\alpha_{\varphi_M, m_0})^{-1} \big) \end{aligned}$$

By assumption, a is a regular value of f, so the differential $d_{m_0} f$ has full rank. Hence, the above matrix does not vanish, and therefore

$$\mathrm{J}_{\Psi \circ (\varphi_M^{-1} \times \mathrm{id}_{\mathbb{C}})}(\varphi_M(m_0)\,,\, z_0) \neq 0\,.$$

This proves that a is a regular value of Ψ.

Since $\Psi^{-1}(a)$ is a submanifold and Ψ is S^1-invariant, the action Θ restricts smoothly to $\Psi^{-1}(a)$ (argue as above). Additionally, if $\lambda \cdot (m, z) = (m, z)$, then $\lambda^{-1} z = z$ and $\lambda \cdot m = m$. If $z \neq 0$, then $\lambda = 1$. Assume $z = 0$, then $m \in f^{-1}(a)$ and by assumption (the restricted action on $f^{-1}(a)$ is free) we obtain $\lambda = 1$. In summary, the action on $\Psi^{-1}(a)$ is free, so the orbit space $\Psi^{-1}(a)/S^1$ is a manifold (of dimension $\dim(M)$) by the Quotient Manifold Theorem.

Now consider the map

$$\sigma : f^{-1}(\,[a, \infty)\,) \to \Psi^{-1}(a)\,, \qquad \sigma(m) := \big(m\,,\, \sqrt{f(m) - a} \big) \tag{3.4}$$

It is continuous because it is the restriction of the map $f^{-1}(\,[a, \infty)\,) \to M \times \mathbb{C}$, which itself is continuous by the universal property of the product topology.
Note that, if $m \neq m' \in f^{-1}(\,[a, \infty)\,)$ and $m \sim m'$, then $f(m) = f(m') = a$ and there is a $\lambda \in S^1$ such that $m = \lambda \cdot m'$. Thus, $\lambda \cdot \sigma(m') = \lambda \cdot (m', 0) = (m, 0) = \sigma(m)$. This implies that, if $\pi : \Psi^{-1}(a) \twoheadrightarrow \Psi^{-1}(a)/S^1$ is the orbit projection, the composition $\pi \circ \sigma$ factors through $\bar{\sigma} : M_{[a,\infty)} \to \Psi^{-1}(a)/S^1$:

$$\begin{tikzcd} f^{-1}(\,[a, \infty)\,) \arrow[r, "\sigma"] \arrow[d, "\Pi"'] & \Psi^{-1}(a) \arrow[d, "\pi"] \\ M_{[a,\infty)} \arrow[r, "\bar{\sigma}"'] & \Psi^{-1}(a)/S^1 \end{tikzcd} \tag{3.5}$$

Since $M_{[a,\infty)}$ carries the quotient topology and the composition $\pi \circ \sigma$ is continuous, the map $\bar{\sigma}$ is continuous. Now let us prove that in addition it is a homeomorphism: We shall write $[m] := \Pi(m)$ for the equivalence class of $m \in f^{-1}(\,[a, \infty)\,)$.
Consider the map

$$\begin{aligned} \tau : \Psi^{-1}(a)/S^1 &\to M_{[a,\infty)} \\ S^1 \cdot (m, z) &\mapsto \begin{cases} [e^{i\,\mathrm{Arg}(z)} \cdot m] = [\frac{z}{|z|} \cdot m] & \text{, if } z \neq 0 \\ [m] & \text{, if } z = 0 \end{cases} \end{aligned} \tag{3.6}$$

We have to show that this definition is independent of choice of representatives. Suppose $(m, z) = \lambda \cdot (m', z')$ for some $\lambda \in S^1$. Then $z = \lambda^{-1} z'$, so $|z| = |z'|$, in particular $z = 0 \Leftrightarrow z' = 0$.

Case 1: $\quad z \neq 0$
We have $z = \lambda^{-1} z'$, so $|z| = |z'| > 0$, and $\lambda \cdot m' = m$. Thus,

$$\frac{z}{|z|} \cdot m = \frac{\lambda^{-1} z'}{|z'|} \cdot (\lambda \cdot m') = (\frac{z'}{|z'|}\, \lambda^{-1}\, \lambda) \cdot m' = \frac{z'}{|z'|} \cdot m'.$$

In particular $[\,\frac{z}{|z|} \cdot m\,] = [\,\frac{z'}{|z'|} \cdot m'\,]$.

Case 2: $z = 0$
Because $z = z' = 0$ and (m, z), $(m', z') \in \Psi^{-1}(a)$, we have $m, m' \in f^{-1}(a)$. Because we also have $\lambda \cdot m' = m$, by definition of the equivalence relation $\sim$ we obtain $[m] = [m']$.

Thus, the map τ is well-defined. We state that $\tau = \bar{\sigma}^{-1}$.

(i) $\tau \circ \bar{\sigma} = \mathrm{id}_{M_{[a,\infty)}}$
Given an arbitrary element $x = [m] \in M_{[a,\infty)}$.
Case 1: $m \in f^{-1}(a)$
In this case we have

$$\bar{\sigma}([m]) = S^1 \cdot \sigma(m) = S^1 \cdot (m\,,\ \sqrt{f(m) - a}) = S^1 \cdot (m, 0)\,,$$

and thus $\tau(\bar{\sigma}(x)) = \tau(S^1 \cdot (m, 0)) = [m] = x$.
Case 2: $m \notin f^{-1}(a)$

$$\tau(\,\bar{\sigma}(x)\,) = \tau\big(\,S^1 \cdot \big(m\,,\ \underbrace{\sqrt{f(m) - a}}_{\neq 0}\big)\,\big) = \Big[\frac{\sqrt{f(m) - a}}{|\sqrt{f(m) - a}|} \cdot m\Big] = [m] = x$$

(ii) $\bar{\sigma} \circ \tau = \mathrm{id}_{\Psi^{-1}(a)/S^1}$
Let $y = S^1 \cdot (m, z) \in \Psi^{-1}(a)/S^1$ be arbitrary.
Case 1: $z = 0$

$$\bar{\sigma}(\,\tau(y)\,) = \bar{\sigma}([m]) = S^1 \cdot \big(m, \sqrt{f(m) - a}\big) = S^1 \cdot (m, 0) = S^1 \cdot (m, z) = y$$

Case 2: $z \neq 0$

$$\begin{aligned}
\bar{\sigma}(\,\tau(y)\,) \;&=\; \bar{\sigma}\Big(\Big[\frac{z}{|z|} \cdot m\Big]\Big) \;=\; S^1 \cdot \sigma\Big(\frac{z}{|z|} \cdot m\Big) \\
&=\; S^1 \cdot \Big(\,\frac{z}{|z|} \cdot m\,,\ \sqrt{f\Big(\frac{z}{|z|} \cdot m\Big) - a}\,\Big) \;\overset{(*)}{=}\; S^1 \cdot \Big(\,\frac{z}{|z|} \cdot m\,,\ \sqrt{f(m) - a}\,\Big) \\
&=\; S^1 \cdot \Big(\Big(\frac{z}{|z|}\Big)^{-1} \cdot \Big(\,\frac{z}{|z|} \cdot m\,,\ \sqrt{f(m) - a}\,\Big)\Big) \\
&=\; S^1 \cdot \Big(\,m\,,\ \frac{z}{|z|}\sqrt{f(m) - a}\,\Big) \;\overset{(**)}{=}\; S^1 \cdot (m, z) \;=\; y
\end{aligned}$$

We have used the S^1-invariance of f for $(*)$. Equation $(**)$ follows from $|z| = \sqrt{f(m) - a}$ (because $(m, z) \in \Psi^{-1}(a)$).
Hence, we have proven that $\bar{\sigma}$ is bijective with inverse function τ. To show that $\bar{\sigma}$ is, in fact, a homeomorphism, it remains to give a proof of the continuity of τ. By the universal property of quotient spaces, τ is continuous if and only if $\tau \circ \pi$ is so. Consider the map

$$\begin{aligned}
H : S^1 \times f^{-1}(\,[a, \infty)\,) &\twoheadrightarrow \Psi^{-1}(a) \\
(\lambda, m) &\mapsto \big(\,m\,,\ \lambda\sqrt{f(m) - a}\,\big)
\end{aligned}$$

Clearly, H is well-defined (i.e. its image is contained in $\Psi^{-1}(a)$), continuous (it is the restriction of the map $S^1 \times f^{-1}(\,[a, \infty)\,) \to M \times \mathbb{C}$, which is continuous by the universal property of product spaces) and surjective. In addition, $H(1, m) = \sigma(m) \quad \forall\, m \in f^{-1}(\,[a, \infty)\,)$. Define the continuous map $F : S^1 \times f^{-1}(\,[a, \infty)\,) \to M_{[a,\infty)}$ by $F := \Pi \circ \theta$, i.e. $F(\lambda, m) := [\lambda \cdot m]$. Then we obtain

$$\begin{aligned}
\bar{\sigma} \circ F(\lambda, m) &= \pi(\sigma(\lambda \cdot m)) \overset{(*)}{=} \pi\Big(\lambda \cdot m\,,\ \sqrt{f(m) - a}\Big) \\
&= \pi\Big(\lambda^{-1} \cdot \Big(\lambda \cdot m\,,\ \sqrt{f(m) - a}\Big)\Big) \\
&= \pi \circ H\,(\lambda, m)\,,
\end{aligned}$$

where we used the S^1-invariance of f for $(*)$. Thus, the following diagram commutes

$$\begin{tikzcd} S^1 \times f^{-1}([a,\infty)) \arrow[rr, two heads, "H"] \arrow[d, "F"'] & & \Psi^{-1}(a) \arrow[d, two heads, "\pi"] \arrow[dll, "\tau\circ\pi"'] \\ M_{[a,\infty)} \arrow[rr, shift left, "\bar{\sigma}"] & & \Psi^{-1}(a)/S^1 \arrow[ll, shift left, "\tau=\bar{\sigma}^{-1}"] \end{tikzcd} \tag{3.7}$$

The restriction of H to the open subset $S^1 \times f^{-1}(\,(a,\infty)\,)$ onto the open subset $\Psi^{-1}(a)\backslash\{z = 0\} = N_1$ of $\Psi^{-1}(a)$ is a homoemorphism with (continuous) inverse function

$$\begin{aligned} (H|_{S^1\times f^{-1}((a,\infty))})^{-1} : N_1 &\to S^1 \times f^{-1}(\,(a,\infty)\,) \\ (m,z) &\mapsto \Big(\frac{z}{|z|}\,,\,m\Big) \end{aligned}$$

Our goal is to prove that H is a quotient map (for the definition of quotient map see A.1). Then $\tau \circ \pi$ is continuous because $(\tau \circ \pi) \circ H = F$ is continuous. By Lemma A.2.3, if for any open subset $U \subseteq S^1 \times f^{-1}(\,[a,\infty)\,)$ with $H^{-1}(H(U)) = U$ (such a subset is called saturated) the image $H(U)$ is open in $\Psi^{-1}(a)$, then H is indeed a quotient map. So, let $U \subseteq S^1 \times f^{-1}(\,[a,\infty)\,)$ be an arbitrary saturated open subset. Obviously, it suffices to show that, given $(\lambda,m) \in U$, there exists an open neighborhood $(\lambda,m) \in W \subseteq U$ such that $H(W)$ is open in $\Psi^{-1}(a)$. If $(\lambda,m) \in S^1 \times f^{-1}(\,(a,\infty)\,)$, then there clearly exists such a neighborhood W (since the restriction of H is a homeomorphism from the open subset $S^1 \times f^{-1}(\,(a,\infty)\,)$ onto the open subset N_1). So we may assume that $(\lambda,m) \in S^1 \times f^{-1}(a)$. For any $\mu \in S^1$ we have $H(\mu,m) = (m,0) = H(\lambda,m)$. Since U is saturated, we obtain $S^1 \times \{m\} \subseteq U$. Now, let us assume the following proposition, which we will prove further below:

Proposition: There exists an open neighborhood $\hat{W} \subseteq M$ of m in M such that

$$S^1 \times \big(\,\hat{W} \cap f^{-1}([a,\infty))\,\big) \subseteq U\,.$$

Then put

$$W := \big(\,S^1 \times f^{-1}([a,\infty))\,\big) \cap \big(\,S^1 \times \hat{W}\,\big) = S^1 \times \big(\,\hat{W} \cap f^{-1}([a,\infty))\,\big) \subseteq U\,,$$

clearly, W is open in $S^1 \times f^{-1}([a,\infty))$ and contains (λ,m). We have

$$H(W) = \Psi^{-1}(a) \cap (\hat{W} \times \mathbb{C})\,.$$

"$\subseteq$": trivial

"$\supseteq$": Let $(m',z') \in \Psi^{-1}(a) \cap (\hat{W} \times \mathbb{C})$ be arbitrary. If $z' \neq 0$, then $(H|_{S^1\times f^{-1}((a,\infty))})^{-1}((m',z')) = \big(\,\frac{z'}{|z'|}, m'\,\big) \in W$.
If $z' = 0$, then $H(1,m') = \sigma(m') = (m',0) = (m',z')$ and $(1,m') \in W$.

So, $H(W)$ is open in $\Psi^{-1}(a)$.

Proof of the Proposition: Assume by contradiction that the (above) Proposition is false. Then there is an open neighborhood basis $(W_n)_{n\in\mathbb{N}}$, $W_{n+1} \subseteq W_n \subseteq M$, of m in M and a sequence $(\mu_n,m_n)_n$ in $S^1 \times f^{-1}(\,[a,\infty)\,)$ with $m_n \in W_n$, $(\mu_n,m_n) \notin U$, for all n. Thus $m_n \to m$ and, because S^1 is compact, by passing to a subsequence, we can assume that $\mu_n \to \mu$ for some $\mu \in S^1$. Therefore $(\mu_n,m_n) \to (\mu,m)$ in $S^1 \times f^{-1}(\,[a,\infty)\,)$. Since U is open in $S^1 \times f^{-1}(\,[a,\infty)\,)$ and $(\mu,m) \in S^1 \times \{m\} \subseteq U$, there exists an $n_0 \in \mathbb{N}$ such that $(\mu_n,m_n) \in U$ for $n \geq n_0$. This contradicts $(\mu_n,m_n) \notin U$ for all $n \in \mathbb{N}$.

(Proof of the Proposition)

In total we have shown that $H(U)$ is open in $\Psi^{-1}(a)$. Thus, by above argumentation, H is a quotient map and $\tau \circ \pi$ and $\tau = \bar{\sigma}^{-1}$ are continuous. So, $\bar{\sigma}$ is a homeomorphism.

Because $\Psi^{-1}(a)/S^1$ is a manifold of dimension $\dim(M)$ and homeomorphic to $M_{[a,\infty)}$, the latter is a topological manifold of dimension $\dim(M)$ too. The canonical smooth structure on $M_{[a,\infty)}$ is the unique one induced by $\bar{\sigma}$, that is, $\bar{\sigma}$ is a diffeomorphism with respect to this smooth structure and the one on $\Psi^{-1}(a)/S^1$.

By definition of the quotient topology, the set $\Pi(\, f^{-1}((a,\infty))\,)$ is open (because $f^{-1}((a,\infty))$ is saturated and open).
The restriction $\Pi|_{f^{-1}((a,\infty))} : f^{-1}((a,\infty)) \to \Pi(\, f^{-1}((a,\infty))\,)$ is a bijection between open subsets. In addition, clearly $\sigma|_{f^{-1}((a,\infty))} : f^{-1}((a,\infty) \to N_1$ and $\pi|_{N_1} : N_1 \to \pi(N_1)$ are smooth maps between the open subsets $f^{-1}((a,\infty))$ and N_1 respectivele N_1 and $\pi(N_1)$. Thus, the restriction

$$\Pi|_{f^{-1}((a,\infty))} = \bar{\sigma}^{-1}|_{\pi(N_1)} \circ \pi|_{N_1} \circ \sigma|_{f^{-1}((a,\infty))}$$

(cf. diagram (3.5)) is smooth as composition of smooth maps. To show, that $\Pi|_{f^{-1}((a,\infty))}$ is even a diffeomorphism, it suffices to prove that $(\,\pi|_{N_1} \circ \sigma|_{f^{-1}((a,\infty))}\,)^{-1}$ is smooth. This follows from the universal property of surjecive submersions and the fact that

$$\begin{aligned}(\,\pi|_{N_1} \circ \sigma|_{f^{-1}((a,\infty))}\,)^{-1} \circ \pi|_{N_1} : \quad & N_1 \to f^{-1}((a,\infty)) \\ & (m,z) \mapsto \frac{z}{|z|} \cdot m\end{aligned}$$

is smooth for obvious reasons (compare the calculation for $\bar{\sigma} \circ \tau = \mathrm{id}$, case 2, to understand, why $(\,\pi|_{N_1} \circ \sigma|_{f^{-1}((a,\infty))}\,)^{-1} \circ \pi|_{N_1}\,(m,z) = \frac{z}{|z|} \cdot m$ holds). Hence, we have proven that $\Pi|_{f^{-1}((a,\infty))} : f^{-1}((a,\infty)) \to \Pi(\, f^{-1}((a,\infty))\,)$ is in fact a diffeomorphism between open subsets.

We now want to show that $M_{[a,\infty)} \backslash \Pi(\, f^{-1}((a,\infty))\,)$ is a submanifold of $M_{[a,\infty)}$, diffeomorphic to $f^{-1}(a)/S^1$. As we pointed out earlier, $N_2 = f^{-1}(a) \times \{0\}$ is a closed and S^1-invariant submanifold of $\Psi^{-1}(a)$ (see also Lemma A.2.2). By Proposition A.2.7, this implies that $(f^{-1}(a) \times \{0\})/S^1$, endowed with the quotient topology and the canonical smooth structure by the Quotient Manifold Theorem, is a submanifold of $\Psi^{-1}(a)/S^1$. Since $\bar{\sigma}(\, M_{[a,\infty)} \backslash \Pi(\, f^{-1}((a,\infty))\,)\,) = (\, f^{-1}(a) \times \{0\}\,)/S^1$, this means that the difference $M_{[a,\infty)} \backslash \Pi(\, f^{-1}((a,\infty))\,)$ is homeomorphic to $(\, f^{-1} \times \{0\}\,)/S^1$ via $\bar{\sigma}$, and thus carries a unique smooth structure such that the restriction of $\bar{\sigma}$ becomes a diffeomorpism. Clearly, then it follows that $M_{[a,\infty)} \backslash \Pi(\, f^{-1}((a,\infty))\,)$ is a submanifold of $M_{[a,\infty)}$ of dimension $\dim(M) - 2$. Because $f^{-1}(a) \times \{0\}$ is diffeomorphic to $f^{-1}(a)$, using the characteristic property of surjective submersions, the quotient manifold $(\, f^{-1}(a) \times \{0\}\,)/S^1$ is diffeomorphic to $f^{-1}(a)/S^1$.

Because $M_{[a,\infty)} \backslash \Pi(\, f^{-1}((a,\infty))\,)$ is a submanifold of $M_{[a,\infty)}$ of codimension ≥ 1, its complement $\Pi(\, f^{-1}((a,\infty))\,)$ is dense in $M_{[a,\infty)}$. □

Remark 3.2.2. We call $M_{[a,\infty)}$ the ***cut of*** M ***with respect to the ray*** $[a,\infty)$. From now on, justified by the preceding Proposition, we will also write $f^{-1}((a,\infty))$ instead of $\Pi(\, f^{-1}((a,\infty))\,)$ and $f^{-1}(a)/S^1$ instead of $M_{[a\infty)} \backslash \Pi(\, f^{-1}((a,\infty))\,)$.
Completely analogously it is possible to introduce the ***cut of*** M ***with respect to the ray*** $(-\infty,a]$: $M_{(-\infty,a]} := f^{-1}((-\infty,a])/\backsim$, where for $m \neq m'$:

$$m \backsim m' \iff m, m' \in f^{-1}(a) \text{ and } S^1 \cdot m = S^1 \cdot m' \;.$$

Example 3.2.3. As it turns out, even the most basic example is not at all trivial:
Let S^1 act on $M = \mathbb{C}$ via multiplication. Let $a > 0$ be arbitrary and $f(z) = |z|^2$ for $z \in \mathbb{C}$. We claim that $M_{[a,\infty)} \cong \mathbb{C}$. Let $\Psi : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be the map from (3.3), so

$\Psi(w,z) = |w|^2 - |z|^2$. We have to show $\Psi^{-1}(a)/S^1 \cong \mathbb{C}$. Note that $S^1 \cdot (w,z) = S^1 \cdot (w',z')$ if and only if $wz = w'z'$. Set $r(w) := -\frac{a}{2} + \sqrt{\frac{a^2}{4} + |w|^2}$ for $w \in \mathbb{C}$, so that $r(w) = 0$ if and only if $w = 0$. Now consider the maps

$$\Psi^{-1}(a)/S^1 \longrightarrow \mathbb{C} \ , \quad S^1 \cdot (\alpha, \beta) \longmapsto \alpha\, \beta$$

and

$$\begin{aligned} \mathbb{C} \longrightarrow \Psi^{-1}(a)/S^1 \ , \quad w \longmapsto & \ S^1 \cdot (\alpha, \beta) \\ \alpha = & \sqrt{a + r(w)} \\ \beta = & \begin{cases} \frac{w}{|w|} \sqrt{r(w)} & \text{, if } w \neq 0 \\ 0 & \text{, if } w = 0 \end{cases} . \end{aligned}$$

Clearly, both maps are well-defined and inverse and $\Psi^{-1}(a)/S^1 \to \mathbb{C}$ is smooth. The smoothness of $\mathbb{C} \to \Psi^{-1}(a)/S^1$ follows because $\alpha : \mathbb{C} \to \mathbb{R}$ and $\beta : \mathbb{C} \to \mathbb{R}$ are smooth, although the author has to admit that he has not checked this precisely for β (so this example could actually be false).

Proposition 3.2.4 (cf. [Ler01, Remark 2.2])**.** *The S^1-action on M defines a natural smooth S^1-action on the cut $M_{[a,\infty)}$ and the restriction $f|_{f^{-1}([a\infty))}$ descends to an S^1-invariant smooth function $\bar{f} : M_{[a,\infty)} \to \mathbb{R}$.*
More generally, if a Lie group G acts smoothly on M via $(g,m) \mapsto g \cdot_G m$, preserving f and commuting with the given S^1-action on M, then the G-action defined by $g \bar{\cdot}_G [m] := [g \cdot_G m]$ is a smooth G-action on $M_{[a,\infty)}$, preserving $\bar{f}$ and commuting with the S^1-action on the cut.

Proof. Let us start by showing that the action induced by an arbitrary Lie group G is well-defined. If $f(m) \geq a$, then $f(g \cdot_G m) \geq a$. If $[m] = [m']$ and $m \neq m'$, then $m, m' \in f^{-1}(a)$, so $f(g \cdot_G m) = f(m) = a = f(m') = f(g \cdot_G m')$. In addition, there is a $\lambda \in S^1$ such that $m = \lambda \cdot m'$, and thus $g \cdot_G m = g \cdot_G (\lambda \cdot m') = \lambda \cdot (g \cdot_G m')$, applying that the actions commute. This shows $[g \cdot_G m] = [g \cdot_G m']$.
Obviously, the defined map is a group action. Now let us continue with proving the smoothness of this action:
By definition of the smooth structure on the cut, the action of G on $M_{[a,\infty)}$ is smooth if and only if the map

$$\begin{aligned} & G \times \Psi^{-1}(a)/S^1 \to \Psi^{-1}(a)/S^1 \\ & (g\, ,\, S^1 \cdot (m,z)) \mapsto \begin{cases} S^1 \cdot \sigma\big(g \cdot_G \big(\frac{z}{|z|} \cdot m\big)\big) & \text{, if } z \neq 0 \\ S^1 \cdot \sigma(g \cdot_G m) & \text{, if } z = 0 \end{cases} \end{aligned}$$

is smooth (here we are using the definition of $\tau = \bar{\sigma}^{-1}$, see equation (3.6)). This, on the other hand, is equivalent to the smoothness of the map

$$\begin{aligned} & H : G \times \Psi^{-1}(a) \to \Psi^{-1}(a)/S^1 \\ & (g,(m,z)) \mapsto \begin{cases} S^1 \cdot \sigma\big(g \cdot_G \big(\frac{z}{|z|} \cdot m\big)\big) & \text{, if } z \neq 0 \\ S^1 \cdot \sigma(g \cdot_G m) & \text{, if } z = 0 \end{cases} \end{aligned}$$

(using that $\mathrm{id}_G \times \pi$ is a surjective submersion). Let $g \in G$, $(m,z) \in \Psi^{-1}(a)$ be arbitrary. Assume $z = 0$. Then $H(g\, ,\, (m,z)) = S^1 \cdot (g \cdot_G m, 0) = S^1 \cdot (g \cdot_G m, z)$. Now suppose $z \neq 0$.

Then

$$\begin{aligned} H(g,(m,z)) &= S^1 \cdot \sigma\big(g \cdot_G \Big(\frac{z}{|z|} \cdot m\Big)\big) \\ &= S^1 \cdot \big(g \cdot_G \Big(\frac{z}{|z|} \cdot m\Big),\ \sqrt{f(m)-a}\,\big) \\ &= S^1 \cdot \frac{z}{|z|} \cdot \big(g \cdot_G m,\ \frac{z}{|z|}\sqrt{f(m)-a}\,\big) \\ &= S^1 \cdot \big(g \cdot_G m,\ \frac{z}{|z|}\sqrt{f(m)-a}\,\big) \\ &= S^1 \cdot (g \cdot_G m,\ z) \end{aligned}$$

Thus, if we define the smooth map $\tilde{H} : G \times \Psi^{-1}(a) \to \Psi^{-1}(a)$ by $\tilde{H}(g,\,(m,z)) := (g \cdot_G m,\ z)$, we obviously have $H = \pi \circ \tilde{H}$, so H is smooth. In conclusion, the map $(g,\bar{m}) \mapsto g \,\bar{\cdot}_G\, \bar{m}$ is a smooth group action of G on the cut $M_{[a,\infty)}$.

In particular, since S^1 is an abelian Lie group, we obtain the natural smooth S^1-action $\bar{\cdot}_{S^1}$ on the cut.
Clearly, the function $\bar{f}$ defined by $\bar{f}([m]) := f(m)$ is unambiguous. We have

$$\bar{f} \circ \bar{\sigma}^{-1}\left(S^1 \cdot (m,z)\right) = \begin{Bmatrix} \bar{f}\big(\,[\frac{z}{|z|} \cdot m]\,\big) & \text{, if } z \neq 0 \\ \bar{f}([m]) & \text{, if } z = 0 \end{Bmatrix} = f(m)$$

Thus, the composition $\bar{f} \circ \bar{\sigma}^{-1} \circ \pi$ is smooth and, therefore, $\bar{f} \circ \bar{\sigma}^{-1}$ is smooth as well, which proves the smoothness of $\bar{f}$. The rest of the proposition is trivial. □

3.3 Cutting Manifolds with Boundary

Before we generalize the first proposition of this chapter, we should recapitulate some prevalent notation:
Let X and Y be topological spaces and $A \subseteq X, B \subseteq Y$ subsets of X respectively Y. Given a bijection $f : A \xrightarrow{\sim} B$, we can define the equivalence relation $\sim$ on the disjoint union $X \sqcup Y = \{(x,0) \,|\, x \in X\} \cup \{(y,1) \,|\, y \in Y\}$ by:

$$(x,0) \sim (y,1) \;\Leftrightarrow\; (y,1) \sim (x,0) \;\Leftrightarrow\; x \in A\,,\, y \in B \text{ and } f(x) = y$$

for $x \in X$ and $y \in Y$.
$(x,0) \sim (x',0)$ if and only if $x = x'$, for $x,x' \in X$. $(y,1) \sim (y',1)$ if and only if $y = y'$, for $y,y' \in Y$.
The quotient space $X \cup_f Y := (X \sqcup Y)/\!\sim$ is endowed with the quotient topology.
We will often just write x instead of $(x,0)$ and y instead off $(y,1)$ if it is clear from context that we mean an element of $X \sqcup Y$. Furthermore, we will also informally denote the elements of $X \cup_f Y$ with x respectively y when we mean the equivalence classes of x respectively y.
Let us now prove a useful lemma:

Lemma 3.3.1. *Let X, Y, Z be topological spaces, $\pi : X \to Y$ a quotient map, $U \subseteq Z$, $W \subseteq X$ open subsets and $\rho : U \xrightarrow{\sim} W$ a homeomorphism. Suppose further that $\pi|_W : W \xrightarrow{\sim} \pi(W)$ is bijective and W is saturated (i.e. $\pi^{-1}(\pi(W)) = W$). Let $\hat{\rho} := \pi|_W \circ \rho : U \xrightarrow{\sim} \pi(W)$.
Then $\hat{\pi} : Z \cup_\rho X \twoheadrightarrow Z \cup_{\hat{\rho}} Y$ defined by $\hat{\pi}(z) = z$ for $z \in Z$ and $\hat{\pi}(x) = \pi(x)$ for $x \in X$ is a well-defined quotient map.*

Proof. By Lemma A.2.4, $\pi|_W : W \to \pi(W)$ is a quotient map. Because it is bijective by assumption, we conclude that $\pi|_W$ is a homeomorphism onto the open subset $\pi(W) \subseteq Y$. Thus, $\hat{\rho}$ is a homeomorphism too.

Suppose $z \in U$, $x \in W$ and $x = \rho(z)$. Then $\hat{\rho}(z) = \pi(\rho(z)) = \pi(x)$, so $\hat{\pi}$ is well-defined. By the universal property of the coproduct, the map $Z \sqcup X \to Z \sqcup Y$, $z \mapsto z$, $x \mapsto \pi(x)$ is continuous, thus the composition $Z \sqcup X \to Z \sqcup Y \to (Z \sqcup Y)/\sim_Y$ is continuous as well, where $\sim_X$, $\sim_Y$ are the equivalence relations on $Z \sqcup X$ respectively $Z \sqcup Y$. Since $\hat{\pi}$ is well-defined, the above map factors through the continuous map $\hat{\pi} : (Z \sqcup X)/\sim_X \longrightarrow (Z \sqcup Y)/\sim_Y$. Because $Z \sqcup X \to Z \sqcup Y \to (Z \sqcup Y)/\sim_Y$ is surjective, $\hat{\pi}$ is also surjective.

Let $V \subseteq Z \cup_\rho X = (Z \sqcup X)/\sim_X$ be an arbitrary open subset, saturated with respect to $\hat{\pi}$. We need to show that its image $\hat{\pi}(V)$ is open. Then by Lemma A.2.3 we can conclude that $\hat{\pi}$ is a quotient map. Let

$$\iota_{Z,X} : Z \hookrightarrow Z \sqcup X \qquad \iota_{Z,Y} : Z \hookrightarrow Z \sqcup Y$$
$$\iota_X : X \hookrightarrow Z \sqcup X \qquad \iota_Y : Y \hookrightarrow Z \sqcup Y$$

denote the inclusion maps and

$$\pi_X : Z \sqcup X \twoheadrightarrow (Z \sqcup X)/\sim_X$$
$$\pi_Y : Z \sqcup Y \twoheadrightarrow (Z \sqcup Y)/\sim_Y$$

the projections. Then, by applying the definitions of the corresponding topologies, we get:

$$\begin{aligned}\hat{\pi}(V) \text{ is open } &\iff \pi_Y^{-1}(\hat{\pi}(V)) \text{ is open}\\ &\iff \iota_{Z,Y}^{-1}(\pi_Y^{-1}(\hat{\pi}(V))) \text{ is open and}\\ &\qquad \iota_Y^{-1}(\pi_Y^{-1}(\hat{\pi}(V))) \text{ is open}\end{aligned}$$

We state that

$$\iota_{Z,Y}^{-1}(\pi_Y^{-1}(\hat{\pi}(V))) = \iota_{Z,X}^{-1}(\pi_X^{-1}(V)) \ . \tag{3.8}$$

To show this, suppose $z \in Z$ is arbitrary. Then we clearly have

$$\begin{aligned} z \in \iota_{Z,Y}^{-1}(\pi_Y^{-1}(\hat{\pi}(V))) &\iff \pi_Y(z,0) \in \hat{\pi}(V) \iff \hat{\pi}(\pi_X(z,0)) \in \hat{\pi}(V)\\ &\iff \pi_X(z,0) \in \hat{\pi}^{-1}(\hat{\pi}(V)) = V \iff z \in \iota_{Z,X}^{-1}(\pi_X^{-1}(V))\end{aligned}$$

This shows equation (3.8), and thus $\iota_{Z,Y}^{-1}(\pi_Y^{-1}(\hat{\pi}(V)))$ is open. Next we claim

$$\iota_Y^{-1}(\pi_Y^{-1}(\hat{\pi}(V))) = \pi(\iota_X^{-1}(\pi_X^{-1}(V))) \ . \tag{3.9}$$

First suppose that $y \in \iota_Y^{-1}(\pi_Y^{-1}(\hat{\pi}(V)))$ is arbitrary. Thus $\pi_Y(y,1) \in \hat{\pi}(V)$, say $\pi_Y(y,1) = \hat{\pi}(v)$ for $v \in V$.
Assume that $v \notin \pi_X(\iota_X(X))$. Then $v \in \pi_X(\iota_{Z,X}(Z))$, say $v = \pi_X \circ \iota_{Z,X}(z)$ for $z \in Z$. Thus $\pi_Y(z,0) = \hat{\pi}(v) = \pi_Y(y,1)$ and $y = \hat{\rho}(z) = \pi(\rho(z))$, $z \in U$. Hence $v = \pi_X(z,0) = \pi_X(\rho(z),1) \in \pi_X(\iota_X(X))$, which is a contradiction. Thus $v \in \pi_X(\iota_X(X))$, say $v = \pi_X(x,1)$ for $x \in X$. We conclude that $\pi_Y(y,1) = \hat{\pi}(v) = \hat{\pi}(\pi_X(x,1)) = \pi_Y(\pi(x),1)$. This implies $y = \pi(x) \in \pi(\iota_X^{-1}(\pi_X^{-1}(V)))$.
To prove the converse inclusion, let $y \in \pi(\iota_X^{-1}(\pi_X^{-1}(V)))$ be arbitrary. Choose $x \in X$ with $y = \pi(x)$ and $\pi_X(x,1) \in V$. Thus $\pi_Y(y,1) = \pi_Y(\pi(x),1) = \hat{\pi}(\pi_X(x,1)) \in \hat{\pi}(V)$. This shows $y \in \iota_Y^{-1}(\pi_Y^{-1}(\hat{\pi}(V)))$ and therefore equation (3.9).
Since V is open, so is $\iota_X^{-1}(\pi_X^{-1}(V))$. If $\iota_X^{-1}(\pi_X^{-1}(V))$ is also saturated, its image under π is open because π is a quotient map. So, assume $\pi(x) \in \pi(\iota_X^{-1}(\pi_X^{-1}(V)))$; we have to show that $x \in \iota_X^{-1}(\pi_X^{-1}(V))$. Let $\pi(x) = \pi(x')$, where $\pi_X(x',1) \in V$. Then $\hat{\pi}(\pi_X(x,1)) = \hat{\pi}(\pi_X(x',1)) \in \hat{\pi}(V)$. Because V is saturated with respect to $\hat{\pi}$ by assumption, this implies $\pi_X(x,1) \in V$.
Thus, $\iota_Y^{-1}(\pi_Y^{-1}(\hat{\pi}(V))) = \pi(\iota_X^{-1}(\pi_X^{-1}(V)))$ is open in Y, which finishes the proof.

□

Remark 3.3.2. Given two manifolds M and N with open subsets $U \subseteq M$, $V \subseteq N$ and a diffeomorphism $F : U \to V$, the topological space $M \cup_F N$ is a topological manifold of dimension $\dim(M) = \dim(N)$ and carries a unique smooth structure such that the inclusion maps $\iota_M : M \hookrightarrow M \cup_F N$ and $\iota_N : N \hookrightarrow M \cup_F N$ are smooth embeddings. If $\mathcal{A}_M = (U_i, \varphi_i)_i$ and $\mathcal{A}_N = (V_j, \psi_j)_j$ are atlases for M respectively N, then an atlas for $M \cup_F N$ is given by $\mathcal{A}_{M \cup_F N} := (\tilde{U_i}, \tilde{\varphi_i})_i \cup (\tilde{V_j}, \tilde{\psi_j})_j$, where $\tilde{U_i} = \iota_M(U_i)$ and $\tilde{\varphi_i} \circ \iota_M|_{U_i} = \varphi_i$ and $(\tilde{V_j}, \tilde{\psi_j})$ is defined analogously (it is easy to see that this is indeed an atlas and that the equivalence class of $\mathcal{A}_{M \cup_F N}$ does not depend on the choice of representatives $\mathcal{A}_M$ and $\mathcal{A}_N$).
This construction obviously still works if M and N are manifolds with boundary such that $U \cap \partial M = \varnothing = V \cap \partial N$ (then $M \cup_F N$ is a smooth manifold with boundary).

Proposition 3.3.3. *Let M be a manifold with boundary. Suppose the boundary $P := \partial M$ is endowed with a smooth and free S^1-action. Put $X := M/\sim$, with $\sim$ the equivalence relation on M such that for all $m \neq m'$: $m \sim m'$ if and only if the following hold:*

(i) $m, m' \in P$

(ii) $m = \lambda \cdot m'$ *for some* $\lambda \in S^1$.

Then X is a topological manifold of dimension $\dim(M)$ *and we can endow X with a smooth structure, depending on a choice, so that P/S^1 is a submanifold of X and $M \backslash P$ is diffeomorphic to $X \backslash (P/S^1)$ via the projection $\Pi : M \twoheadrightarrow X$.*

Proof. By the Collar Neighborhood Theorem there exists a diffeomorphism $\Phi : \mathcal{U}(P) \xrightarrow{\sim} P \times [0,\infty)$, where $\mathcal{U}(P)$ is an open neighborhood of the boundary P, such that $\Phi(q) = (q,0) \quad \forall q \in P$. The same theorem also states that the canonical map $\mathrm{id} : M \xrightarrow{\sim} M \backslash P \cup_{\Phi|_{\mathcal{U}(P)\backslash P}} (P \times [0,\infty))$ is a diffeomorphism.
Consider the projection onto the second factor $f : P \times \mathbb{R} \to \mathbb{R}$. It is S^1-invariant with respect to the trivial action on $\mathbb{R}$ and the product action on $P \times \mathbb{R}$. Furthermore, 0 is a regular value since $d_0 f : T_p P \times T_0\mathbb{R} \to T_0\mathbb{R}$ is the projection onto the second factor, thus $\neq 0$, for an arbitrary $p \in P$. By assumption the restricted S^1-action on $f^{-1}(0) = P \times \{0\}$ is free. We endow $Y := (P \times \mathbb{R})_{[0,\infty)}$ with the smooth structure given by Proposition 3.2.1. Again, applying Proposition 3.2.1, we obtain that $\mathcal{U}(P)\backslash P$ is diffeomorphic to $P \times (0,\infty) = f^{-1}((0,\infty)) \subseteq Y$ (the notation refers to Remark 3.2.2) via $F := \pi|_{P \times (0,\infty)} \circ \Phi|_{\mathcal{U}(P)\backslash P}$, where $\pi : P \times [0,\infty) = f^{-1}([0,\infty)) \twoheadrightarrow Y$ is the projection. Therefore, using the preceding Remark 3.3.2, $N := M\backslash P \cup_F Y$ is a smooth manifold. By Lemma 3.3.1 the induced map

$$\hat{\pi} : M\backslash P \cup_{\Phi|_{\mathcal{U}(P)\backslash P}} (P \times [0,\infty)) \longrightarrow N = M\backslash P \cup_F (P \times \mathbb{R})_{[0,\infty)}$$

is a quotient map. Since Π and $\hat{\pi}$ are quotient maps and id is a homeomorphism, the map $\overline{\mathrm{id}}$, defined by the following commuting diagram, is a homeomorphism.

$$\begin{array}{ccc} M & \xrightarrow{\quad \mathrm{id} \quad} & M\backslash P \cup_{\Phi|_{\mathcal{U}(P)\backslash P}} (P \times [0,\infty)) \\ {\scriptstyle \Pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \hat{\pi}} \\ X = M/\sim & \xrightarrow{\quad \overline{\mathrm{id}} \quad} & N = M\backslash P \cup_F (P \times \mathbb{R})_{[0,\infty)} \end{array}$$

The smooth structure on X is the one that turns $\overline{\mathrm{id}}$ into a diffeomorphism. Then the described properties of X follow simply because Y fulfills these due to Proposition 3.2.1. □

Remark 3.3.4. Notice that the only choice we made in the proof above was the collar neighborhood $\Phi : \mathcal{U}(P) \to P \times [0,\infty)$.

Remark 3.3.5. For an arbitrary manifold N, on which S^1 acts smoothly and freely, the diffeomorphism $N \times \mathbb{R} \xrightarrow{\sim} N \times \mathbb{R}$, $(n,t) \mapsto (n,-t)$ induces a natural diffeomorphism $(N \times \mathbb{R})_{[0,\infty)} \cong (N \times \mathbb{R})_{(-\infty,0]}$ of the cuts with respect to the projection onte the second factor. Thus, if we choose a diffeomorphism Φ of a neighborhood of $P := \partial M$ onto $P \times (-\infty, 0]$

that fixes P, we have a natural diffeomorphism Φ' of this neighborhood onto $P \times [0,\infty)$ and $X = M/\!\sim\; \cong M\backslash P \cup_{\Phi'} (P \times \mathbb{R})_{[0,\infty)} \cong M\backslash P \cup_{\Phi} (P \times \mathbb{R})_{(-\infty,0]}$ (notice that the notation '$\cup_\Phi$ ' is of course slightly imprecise).

Remark 3.3.6. Now consider a collar neighborhood $\mathcal{U} \cong P \times [0,\infty)$ of $P = \partial M$. We can embed M into the manifold without boundary $\widetilde{M} := M \cup_{\mathcal{U}\backslash P} (P \times \mathbb{R})$. Let $\Phi : W \to \Phi(W) \subseteq P \times \mathbb{R}$ be an arbitrary diffeomorphism of an open neighborhood W of P in $\widetilde{M}$ onto the open subset $\Phi(W)$ in $P \times \mathbb{R}$ such that $\Phi|_P = \mathrm{id}$. Consider the connected components P_n of P. They are S^1-invariant. For each n choose an open collar neighborhood $\mathcal{U}_n \subseteq \mathcal{U} \cap W$ of P_n in M such that the $\mathcal{U}_n$ are pairwise disjoint. Then $\mathcal{U}_n$ and $\mathcal{U}_n\backslash P_n$ are connected and $\Phi(\mathcal{U}_n) \subseteq P_n \times \mathbb{R}$. For all n we have either $\Phi(\mathcal{U}_n) \subseteq P_n \times [0,\infty)$ open or $\Phi(\mathcal{U}_n) \subseteq P_n \times (-\infty,0]$ open. Let N_+ denote the set of the n with $\Phi(\mathcal{U}_n) \subseteq P_n \times [0,\infty)$ and N_- denote the set of the n with $\Phi(\mathcal{U}_n) \subseteq P_n \times (-\infty,0]$. Then we can define the smooth structure on X through

$$\begin{aligned} X &= M/\!\sim \qquad (3.10)\\ &\cong M\backslash P \;\cup_{n\in N_+} \big(\cup_{\Phi|_{\mathcal{U}_n\backslash P_n}} (P_n \times \mathbb{R})_{[0,\infty)}\big) \;\cup_{n\in N_-} \big(\cup_{\Phi|_{\mathcal{U}_n\backslash P_n}} (P_n \times \mathbb{R})_{(-\infty,0]}\big) \end{aligned}$$

with the convention that we glue each $(P_n \times \mathbb{R})_{[0,\infty)}$ respectively $(P_n \times \mathbb{R})_{(-\infty,0]}$ separately to $M\backslash P$. Notice that, strictly speaking, we should write $\Phi(\mathcal{U}_n)/\!\sim$ for $n \in N_+$ (the open image of $\Phi(\mathcal{U}_n)$ in $(P_n \times \mathbb{R})_{[0,\infty)}$) instead of the entire cut $(P_n \times \mathbb{R})_{[0,\infty)}$ and similarly for $(P_n \times \mathbb{R})_{(\infty,0]}$.
Let us additionally assume that $N_- = \varnothing$. Clearly, $\mathcal{V} = \bigsqcup_n \mathcal{U}_n \subseteq W$ is a collar neighborhood of P in M. Then $X \cong M\backslash P \cup_{\Phi|_{\mathcal{V}\backslash P}} (P \times \mathbb{R})_{[0,\infty)}$ (again, to be precise one should write $\Phi(\mathcal{V})/\!\sim$ instead of the entire cut), so the smooth structures on X defined by equation (3.10) and the construction in Proposition 3.3.3 coincide. This is a consequence of the fact that we can cut each connected component of a manifold separately and consider the disjoint union of these cuts which is exactly the same as cutting the whole manifold at once.

Example 3.3.7. Consider $M = \mathbb{T}^1 \times [0,1] = \mathbb{R}/(2\pi\mathbb{Z}) \times [0,1]$ and the S^1-action on $\partial M = \mathbb{T}^1 \times \{0,1\}$ defined by:

$$\lambda \cdot (x,j) = (\lambda + x, j) \quad , \lambda, x \in \mathbb{R}/(2\pi\mathbb{Z})\,,\; j = 0,1$$

Then $X = S^2$ for geometric reasons (there is an obvious choice for a homeomorphism).

Example 3.3.8. Analogously to the previous example, we consider the 3-dimensional case: Let $M = \mathbb{T}^2 \times [0,1]$ with the smooth and free S^1-action on the boundary $\partial M = \mathbb{T}^2 \times \{0,1\}$ defined by

$$\lambda \cdot ((x,y)\,,\,j) := \begin{cases} ((x+\lambda, y)\,,\,j) & ,\text{ if } j = 0 \\ ((x, y+\lambda)\,,\,j) & ,\text{ if } j = 1 \end{cases}$$

for $\lambda, x, y \in \mathbb{T}^1 = S^1$. We claim that $X = S^3$.

Let $S^2_+ := S^2 \cap \big\{ (x_1,x_2,x_3) \in \mathbb{R}^3 \,\big|\, x_1 \geq 0 \big\}$. Consider the spherical coordinates

$$\begin{aligned} \text{sp.coord} : S^1 \times [0,1] &\longrightarrow S^2_+ \subseteq \mathbb{R}^3 \\ (\varphi, \vartheta) &\longmapsto \big(\sin\!\big(\tfrac{\pi}{2}\,\vartheta\big),\, \cos\!\big(\tfrac{\pi}{2}\,\vartheta\big)\cos(\varphi),\, \cos\!\big(\tfrac{\pi}{2}\,\vartheta\big)\sin(\varphi)\big) \end{aligned}$$

on S^2. The map is obviously continuous and surjective. Furthermore, for $\varphi \in S^1$ define

$$\Phi_\varphi : \mathbb{R}^3 \hookrightarrow \mathbb{R}^4 = \mathbb{C} \times \mathbb{R}^2 \quad , \quad \Phi_\varphi(x,y,z) := (x \cdot e^{i\varphi}, y, z)\ .$$

This linear map is a smooth embedding that preserves the distance. Then the map

$$\begin{aligned} \Psi : M = \mathbb{T}^2 \times [0,1] &\longrightarrow S^3 \\ (x,y,t) &\longmapsto \Phi_x(\text{sp.coord}(y,t)) \end{aligned}$$

is continuous and surjective and factors through $\overline{\Psi} : M/\!\sim\, = X \to S^3$. $\overline{\Psi}$ is continuous by the universal property of the quotient topology and injective by definition. Since M is compact, so is $X = M/\!\sim$. Because S^3 is Hausdorff, $\overline{\Psi}$ is closed and therefore a homeomorphism. Thus, we can equip X with the smooth structure given in Proposition 3.3.3 as well as with the smooth structure induced by $\overline{\Psi}$. It is a well-known fact that every 3-manifold carries a unique smooth structure (see [aut21a]). Therefore, these two smooth structures on X must coincide and $\overline{\Psi}$ is in fact a diffeomorphism if we endow X with the canonical smooth structure from Proposition 3.3.3.

For a final result in this chapter let us develop a criterion to decide whether two smooth structures on $X = M/\!\sim$ from Proposition 3.3.3, with respect to the choice of two collar neighborhoods $\Phi_j : \mathcal{U}_j \to \partial M \times [0,\infty)$, $j = 1,2$, are the same. For this we need the following lemma:

Lemma 3.3.9. *Let P be a manifold with a free S^1-action and $V_1, V_2 \subseteq P \times \mathbb{R}$ be two open neighborhoods of $P \times \{0\}$. Let $W_i := V_i \cap (P \times [0,\infty))$, $i = 1,2$, and $G : W_1 \to W_2$ a diffeomorphism of manifolds with boundary. Put $f = \mathrm{pr}_2 : P \times \mathbb{R} \twoheadrightarrow \mathbb{R}$ the projection onto the second factor and $\Pi : P \times \mathbb{R} \twoheadrightarrow (P \times \mathbb{R})_{[0,\infty)}$ the projection onto the cut with respect to f and the S^1-action $\lambda \cdot (p\,,\, t) := (\lambda \cdot p\,,\, t)$. Assume the following conditions hold:*

(i) $G|_{P\times\{0\}} = \mathrm{id}$

(ii) $f \circ G = f$ *near the boundary* $P \times \{0\}$ *in* $P \times [0,\infty)$.

(iii) G *is* S^1*-equivariant near the boundary* $P \times \{0\}$.

Then $\overline{G} : \Pi(W_1) \to \Pi(W_2)$, $[p,t] \mapsto [G(p,t)]$ *is a diffeomorphism too.*

Proof. Clearly $\overline{G}$ is well-defined and bijective. Furthermore, W_i is saturated with respect to Π, so $\Pi(W_i)$ is open in $(P \times \mathbb{R})_{[0,\infty)}$. By symmetry, it suffices to prove that $\overline{G}$ is smooth. We only need to show that $\overline{G}$ is smooth on $(P \times \{0\})/\!\sim$.
As in the proof of Proposition 3.2.1 let $\bar{\sigma} : (P\times\mathbb{R})_{[0,\infty)} \to \Psi^{-1}(0)/S^1$ denote the homeomorphism $\bar{\sigma}([p,t]) = S^1 \cdot \big(p, t, \sqrt{f(p,t)}\,\big) = \big[\,p, t, \sqrt{f(p,t)}\,\big]$. By definition $\bar{\sigma}$ is a diffeomorphism, so $\overline{G}$ is smooth iff $\overline{H} := \bar{\sigma} \circ \overline{G} \circ \bar{\sigma}^{-1}$ is smooth. For $[p,t,z] \in \bar{\sigma}(\Pi(W_1))$ near $f^{-1}(0)/S^1$ we have

$$
\begin{aligned}
\overline{H}([p,t,z]) &= \bar{\sigma} \circ \overline{G} \circ \bar{\sigma}^{-1}\,(\,[p,t,z]\,) \\
&\overset{(3.6)}{=} \begin{cases} \bar{\sigma} \circ \overline{G}\,(\,[\frac{z}{|z|}p, t]\,) & \text{, if } z \neq 0 \\ \bar{\sigma} \circ \overline{G}\,(\,[p,t]\,) & \text{, if } z = 0 \end{cases} \\
&= \begin{cases} \bar{\sigma}\,(\,[\,G(\frac{z}{|z|}p, t)\,]\,) & \text{, if } z \neq 0 \\ \bar{\sigma}\,(\,[\,G(p,t)\,]\,) & \text{, if } z = 0 \end{cases} \\
&= \begin{cases} \big[\,G(\frac{z}{|z|}p,t)\,,\, \sqrt{f(G(\frac{z}{|z|}p,t))}\,\big] & \text{, if } z \neq 0 \\ \big[\,G(p,t),\, \sqrt{f(G(p,t))}\,\big] & \text{, if } z = 0 \end{cases} \\
&\overset{(ii)}{=} \begin{cases} \big[\,G(\frac{z}{|z|}p,t)\,,\, \sqrt{f(\frac{z}{|z|}p,t)}\,\big] & \text{, if } z \neq 0 \\ \big[\,G(p,t)\,,\, \sqrt{f(p,t)}\,\big] & \text{, if } z = 0 \end{cases} \\
&\overset{(iii)}{=} \begin{cases} \big[\,\frac{z}{|z|}\,G(p,t)\,,\, \sqrt{f(p,t)}\,\big] & \text{, if } z \neq 0 \\ \big[\,G(p,t)\,,\, \sqrt{f(p,t)}\,\big] & \text{, if } z = 0 \end{cases} \\
&= \begin{cases} \big[\,G(p,t)\,,\, \frac{z}{|z|}\,\sqrt{f(p,t)}\,\big] & \text{, if } z \neq 0 \\ \big[\,G(p,t)\,,\, \sqrt{f(p,t)}\,\big] & \text{, if } z = 0 \end{cases} \\
&= [\,G(p,t)\,,\, z\,]
\end{aligned}
$$

Notice that we could apply (iii) because $(\frac{z}{|z|}p\,,\, t)$ lies in a neighborhood of $P \times \{0\}$ as in (iii) and thus $G(p\,,\, t) = G((\frac{z}{|z|})^{-1}\frac{z}{|z|}\,p\,,\, t) = (\frac{z}{|z|})^{-1}\,G(\frac{z}{|z|}\,p\,,\, t)$. Because G is a diffeomorphism of manifolds with boundary, we can extend G to a smooth map on an open neighborhood W_1' of W_1 in $P\times\mathbb{R}$. Then the map $(p,t,z)\mapsto(G(p,t),z)$, defined on $W_1'\times\mathbb{C}\subseteq P\times\mathbb{R}\times\mathbb{C}$, is smooth. Thus, $\overline{H}$ is smooth near $f^{-1}(0)/S^1$ and $\overline{G}$ is smooth near $(P\times\{0\})/\!\sim$. $\square$

Proposition 3.3.10. *In the situation of Proposition 3.3.3 let $\Phi_i:\mathcal{U}_i\to P\times[0,\infty)$, $i=1,2$, be two collar neighborhoods of the boundary $P=\partial M$. If $G:=\Phi_2\circ\Phi_1^{-1}:W_1:=\Phi_1(\mathcal{U}_1\cap\mathcal{U}_2)\to\Phi_2(\mathcal{U}_1\cap\mathcal{U}_2)=:W_2$ satisfies conditions (ii) and (iii) from Lemma 3.3.9, then the smooth structures on $X=M/\!\sim$, induced by Φ_1 and Φ_2, coincide.*

Proof. Adopt the notation of the proof of Proposition 3.3.3, i.e. $\mathrm{id}_j:M\to M\backslash P\cup_{\Phi_j}(P\times[0,\infty))$, $j=1,2$, is the induced diffeomorphism by Φ_j and $\overline{\mathrm{id}}_j:X\to N_j:=M\backslash P\cup_{\Phi_j}(P\times\mathbb{R})_{[0,\infty)}$ is the descended homeomorphism, defining the smooth structure $\mathcal{A}_j$ on X with respect to Φ_j. Then

$$\begin{aligned}\mathcal{A}_1=\mathcal{A}_2 &\iff \text{The identity } \mathrm{id}_X:(X,\mathcal{A}_1)\to(X,\mathcal{A}_2) \text{ is a diffeomorphism.}\\ &\iff \overline{\mathrm{id}}_2\circ\overline{\mathrm{id}}_1{}^{-1}:N_1\to N_2 \text{ is a diffeomorphism.}\end{aligned}$$

By symmetry, it suffices to show that $\overline{\mathrm{id}}_2\circ\overline{\mathrm{id}}_1{}^{-1}$ is smooth. Notice that $\overline{\mathrm{id}}_2\circ\overline{\mathrm{id}}_1{}^{-1}$ is just the identity on the open subset $M\backslash P\subseteq N_j$, $j=1,2$. On the open subset $\Pi(W_1)\subseteq N_1$ the map $\overline{\mathrm{id}}_2\circ\overline{\mathrm{id}}_1{}^{-1}$ is just the map $\overline{G}:\Pi(W_1)\to\Pi(W_2)$, $[p,t]\mapsto[G(p,t)]$, which is smooth by Lemma 3.3.9. Because $M\backslash P$ and $\Pi(W_1)$ cover N_1, the map $\overline{\mathrm{id}}_2\circ\overline{\mathrm{id}}_1{}^{-1}$ is indeed smooth. $\square$

4 Symplectic and Contact Reduction

4.1 Preliminaries

This section will contain a short recapitulation of some advanced definitions occurring in the broader context of group actions on manifolds as well as some useful lemmata which we will apply in the two main proofs of this chapter.
We assume, the reader is familiar with the concepts of Lie algebras, one-parameter subgroups of Lie groups and exponential maps, otherwise Lee gives a solid introduction in chapters 8 and 20 of [Lee13], which is also recommended for a more detailed approach to the following definitions regarding Lie groups.

Definition 4.1.1. Given a smooth (left) action $\theta : G \times M \to M$ of the Lie group G on the manifold M, we define the ***infinitesimal generator of*** θ as the map

$$\begin{aligned} \widehat{\theta} : \mathrm{Lie}(G) = \mathfrak{g} &\longrightarrow \mathfrak{X}(M) = \big\{\text{vector fields on } M\big\} \\ X &\longmapsto \widehat{X} \text{ with } \widehat{X}_p \text{ defined as} \\ \widehat{X}_p &:= \frac{d}{dt}\bigg|_{t=0} (\exp(tX) \cdot p) = d_e\theta^{(p)} \cdot X_e \end{aligned}$$

Facts 4.1.2.

(a) *$\widehat{X}$ is smooth, so $\widehat{\theta}$ is well-defined.*

(b) *$\widehat{\theta}$ is a Lie algebra antihomomorphism, i.e. it is linear and satisfies $\widehat{\theta}([X,Y]) = -[\widehat{\theta}(X), \widehat{\theta}(Y)]$.*

Notation 4.1.3. Let $F : G \to H$ be a Lie group homomorphism. Then we denote by $F_* : \mathrm{Lie}(G) = \mathfrak{g} \to \mathrm{Lie}(H) = \mathfrak{h}$ the induced Lie algebra homomorphism, i.e. $F_*(X) = Y$ is the unique left invariant vector field with $d_eF \cdot X_e = Y_e$.

Definition 4.1.4. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

(a) For $g \in G$ put $\mathrm{Ad}_g := (\mathrm{int}_g)_* : \mathfrak{g} \to \mathfrak{g}$, where $\mathrm{int}_g : G \to G$ denotes the conjugation by g: $\mathrm{int}_g(h) := ghg^{-1}$.

(b) The map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ is called the ***adjoint representation of G***.

(c) For $g \in G$ put $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$, $\mu \mapsto \mu \circ \mathrm{Ad}_{g^{-1}}$.

(d) The map $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$, $g \mapsto \mathrm{Ad}^*_g$ is called the ***coadjoint representation of G***.

Facts 4.1.5.

(a) Ad *is a Lie group representation, i.e. it is a Lie group homomorphism into* $\mathrm{Aut}(V)$ *for some finite dimensional vector space V, in particular* Ad *is smooth.*

(b) Ad^* *is a Lie group representation.*

(c) *It follows from (a) resp. (b) that* Ad *resp.* Ad^* *induces a natural (smooth) group action of G on the finite-dimensional vector space $\mathfrak{g}$ resp. $\mathfrak{g}^*$ via $g \cdot X := \mathrm{Ad}_g(X)$ resp. $g \cdot \mu := \mathrm{Ad}^*_g(\mu)$ for $g \in G$, $X \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$.*

Now let us dedicate ourselves to the announced lemmata.

Lemma 4.1.6. *Let the Lie group G act on M, $p \in M$. For all $v \in T_pM$ the following holds:*

$$v \in T_p(G \cdot p) \quad \Longleftrightarrow \quad \begin{array}{l} \text{There is a smooth path } \gamma : I := (-\varepsilon, \varepsilon) \to G \\ \text{such that } \gamma(0) = e \text{ and } v = \left.\dfrac{d}{dt}\right|_{t=0} \big(\gamma(t) \cdot p\big) \ . \end{array}$$

Proof. Assume $v \in T_p(G \cdot p)$. Choose a smooth path $\delta : I \to G \cdot p \subseteq M$ with $\delta(0) = p$, $\dot\delta(0) = v$. By Proposition 2.1.13 the canonical bijection $\Phi : G \cdot p \to G/G_p$ is a diffeomorphism. Now choose a local cross section $\chi : U \to G$ of G/G_p such that U contains $\Phi \circ \delta(I)$ (if necessary, make I smaller). Then $\gamma := \chi \circ \Phi \circ \delta$ is smooth, $\gamma(0) = \chi(G_p) = e$ and for $t \in I$ we have $\delta(t) = g \cdot p$ for some $g \in G$ and $\chi(g\, G_p) = gh$ for some $h \in G_p$. Then

$$\gamma(t) \cdot p = gh \cdot p = g \cdot p = \delta(t)$$

For "$\Leftarrow$" note that we have the smooth sequence

$$I \xrightarrow{\ \gamma\ } G \xrightarrow{\ \mathrm{pr}\ } G/G_p \xrightarrow{\ \Phi^{-1}\ } G \cdot p \quad ,$$

that maps $t \in I$ onto $\gamma(t) \cdot p$. □

Remark 4.1.7. Notice that the condition $\gamma(0) = e$ is not necessary for "$\Leftarrow$". It suffices to demand $\gamma(0) \in G_p$.

Lemma 4.1.8. *Let $\theta : G \times M \to M$ be a smooth G-action. Suppose $\gamma_1, \gamma_2 : I = (t_0 - \varepsilon, t_0 + \varepsilon) \to G$ are two paths in G with $\gamma_1(t_0) = \gamma_2(t_0)$ and $\dot\gamma_1(t_0) = \dot\gamma_2(t_0)$. Then*

$$\left.\frac{d}{dt}\right|_{t=t_0} \Big(\gamma_1(t) \cdot p\Big) = \left.\frac{d}{dt}\right|_{t=t_0} \Big(\gamma_2(t) \cdot p\Big) \quad .$$

Proof. Let $g = \gamma_1(t_0) = \gamma_2(t_0)$. For $i = 1, 2$ we compute

$$\left.\frac{d}{dt}\right|_{t=t_0} \Big(\gamma_i(t) \cdot p\Big) = \left.\frac{d}{dt}\right|_{t=t_0} \Big(\theta^{(p)}(\gamma_i(t))\Big) = d_g\theta^{(p)} \cdot \dot\gamma_i(t_0) \quad .$$

□

Lemma 4.1.9. *Suppose $\theta : G \times M \to M$ is a smooth G-action and $p \in M$. Let $\widehat{\theta} : \mathfrak{g} \to \mathfrak{X}(M)$ be the infinitesimal generator of θ as in Definition 4.1.1 and $\mathrm{ev}_p : \mathfrak{X}(M) \to T_pM$ the evaluation function. Their composition is linear with image*

$$\mathrm{im}\big(\,\mathrm{ev}_p \circ \widehat{\theta}\,\big) = T_p(G \cdot p) \quad .$$

In particular, for every vector x tangent to the orbit through p there exists an $X \in \mathfrak{g}$ with $x = \widehat{X}_p$.

Proof. ev_p and $\widehat{\theta}$ are both linear, so the composition is so too. The inclusion "$\subseteq$" follows directly from Lemma 4.1.6 and the definition of $\widehat{X}_p$. Now consider $x \in T_p(G \cdot p)$. Using Lemma 4.1.6 again, we see that there is a path $\gamma : I \to G$ through e such that $\frac{d}{dt}\big|_{t=0} (\gamma(t) \cdot p) = x$. Let X be the unique vector field in $\mathfrak{g}$ with $X_e = \dot\gamma(0)$. Then $\dot\gamma(0) = \frac{d}{dt}\big|_{t=0} \exp(tX)$ and the statement follows from Lemma 4.1.8. □

Proposition 4.1.10 (Tangent space of quotient manifold)**.** *Given a smooth, free and proper G-action $\theta : G \times M \to M$ on M. For $p \in M$ we have the commuting diagram*

$$\begin{tikzcd}
 & T_pM \arrow[dl, two heads, "\mathrm{pr}_p"'] \arrow[dr, two heads, "d_p\pi"] & \\
T_pM \,/\, T_p(G\cdot p) \arrow[rr, "\cong", "\Psi_p"'] & & T_{G\cdot p}\, M/G
\end{tikzcd} \tag{4.1}$$

where $\pi : M \twoheadrightarrow M/G$ and $\mathrm{pr}_p : T_pM \twoheadrightarrow T_pM \,/\, T_p(G\cdot p)$ are the natural projections and Ψ_p, defined by

$$\Psi_p(x + T_p(G\cdot p)) := d_p\pi \cdot x \ , \tag{4.2}$$

is a linear isomorphism.

Proof. By the Quotient Manifold Theorem, M/G is indeed a manifold. To show that Ψ_p is well-defined it suffices to prove $T_p(G\cdot p) \subseteq \ker(d_p\pi)$. This follows directly from Lemma 4.1.6 and $\pi(g\cdot p) = \pi(p)\ \forall g \in G$. Ψ_p is linear and surjective because $d_p\pi$ is so. By Proposition 2.1.12, $G\cdot p$ is a submanifold of M, diffeomorphic to G, so

$$\dim(T_pM \,/\, T_p(G\cdot p)) = \dim(M) - \dim(G) = \dim(T_{G\cdot p}\, M/G) \ .$$

Therefore, Ψ_p is in fact an isomorphism. □

Remark 4.1.11. Notice that Ψ_p could also be defined solely by the commutativity of diagram (4.1).

Proposition 4.1.12. *Let $\theta : G \times M \to M$ be a smooth, free and proper G-action on the manifold M. Let $\pi : M \to M/G$ denote the projection. Then*

$$\ker\, d_p\pi = T_p(G\cdot p) \quad \forall\, p \in M \ .$$

Proof. For $v \in T_p(G\cdot p)$ there exists a smooth path $\gamma : I \to G\cdot p$ with $v = \frac{d}{dt}\big|_{t=0}(i \circ \gamma\,(t))$, where $i : G\cdot p \hookrightarrow M$ is the inclusion. Then $d_p\pi \cdot v = \frac{d}{dt}\big|_{t=0}(\pi \circ i \circ \gamma\,(t)) = \frac{d}{dt}\big|_{t=0}(\pi(p)) = 0$. For the other inclusion note that $\dim \ker\, d_p\pi = \dim(M) - (\dim(M) - \dim(G)) = \dim(G) = \dim\, T_p(G\cdot p)$ because $G \cong G\cdot p$. □

Let us complete this section with the basic definitions of symplectic and contact geometry.

Definition 4.1.13 (Symplectic bilinear form)**.**

(a) Given a linear space V, a 2-linear form $\omega : V \times V \to \mathbb{R}$ is called a ***symplectic (bilinear) form*** if ω is nondegenerate.

(b) A ***symplectic vector space*** is a pair (V, ω), where ω is a symplectic bilinear form on V.

(c) The ***symplectic orthogonal complement*** of a linear subspace U of a symplectic vector space V is the linear subspace

$$U^\omega := \left\{\, v \in V \mid \omega(v, u) = 0 \ \ \forall u \in U \,\right\} \ .$$

(d) A linear subspace $U \subseteq V$ of a symplectic vector space is called ***isotropic*** if $U \subseteq U^\omega$.

(e) A linear map $F : (V_1, \omega_1) \to (V_2, \omega_2)$ between two symplectic vector spaces is called ***symplectic*** if $F^*\omega_2 = \omega_1$, where F^* denotes the linear pullback with respect to F.

Facts 4.1.14. *Let (V, ω) be a symplectic vector space. The following hold:*

(a) $\dim(U) + \dim(U^\omega) = \dim(V)$

(b) $(U^\omega)^\omega = U$

(c) *V has a basis of the form $e_1, f_1, \ldots, e_n, f_n$ such that*

$$\omega = e_1^* \wedge f_1^* + \ldots + e_n^* \wedge f_n^* \ ,$$

where $e_1^*, f_1^*, \ldots, e_n^*, f_n^*$ is the dual basis of $e_1, f_1, \ldots, e_n, f_n$. Such a basis is called ***symplectic basis****. In particular, any symplectic vector space has even dimension.*

(d) *A subspace U is isotropic if and only if $\omega|_U := \omega|_{U\times U} = 0$.*

Facts 4.1.15. *For a vector space V of dimension $2n$, endowed with an alternating bilinear form ω, the following are equivalent:*

(i) *ω is nondegenerate, i.e. the induced linear map $\Omega : V \to V^*$, $v \mapsto \omega(v, \cdot)$ is an isomorphism.*

(ii) *For every $0 \neq x \in V$ there exists a $y \in V$ such that $\omega(x, y) \neq 0$.*

(iii) $\omega^{\wedge n} = \underbrace{\omega \wedge \ldots \wedge \omega}_{n\text{-times}} \neq 0\,.$

Definition 4.1.16 (Symplectic manifold)**.**

(a) A ***symplectic manifold*** (M, ω) consists of a manifold M and a (differential) 2-form $\omega \in \Omega^2(M)$ such that the following hold:

  (i) ω_p is a symplectic bilinear form for all $p \in M$.

  (ii) ω is closed, i.e. $d\omega = 0$.

(b) A ***symplectomorphism*** is a diffeomorphism $F : (M_1, \omega_1) \to (M_2, \omega_2)$ between two symplectic manifolds with $F^*\omega_2 = \omega_1$.

Definition 4.1.17 (Contact manifold)**.**

(a) A ***contact manifold*** (M, ξ) consists of a manifold M of dimension $2n + 1$ together with a smooth cooriented distribution $\ker(\alpha) = \xi \subseteq TM$ for some $\alpha \in \Omega^1(M)$ such that $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$. Then α is called a ***contact form*** and ξ its ***contact distribution***.

(b) By a ***strict contact manifold*** we mean a tuple (M, α) of a manifold M together with a fixed contact form α.

(c) A ***contactomorphism*** is a diffeomorphism $F : (M_1, \xi_1) \to (M_2, \xi_2)$ with $dF(\xi_1) = \xi_2$ (equivalently: $F^*\alpha_2 = f\alpha_1$ for some $f \in C^\infty(M, \mathbb{R}\backslash\{0\})$ if we write $\xi_i = \ker(\alpha_i)$, $i = 1, 2$).

Remark 4.1.18.

(a) The condition $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$ does not depend on the choice of α, so the term 'contact manifold' is indeed well-defined.

(b) With the above Facts 4.1.15 it is easy to see that $\alpha \wedge (d\alpha)^{\wedge n} \neq 0$ is equivalent to the nondegenerecy of $d\alpha|_\xi$ at every point if we write $\xi = \ker(\alpha)$.

4.2 Symplectic Reduction

Our goal in this section will be, to prove the well-known Marsden-Weinstein-Meyer Theorem, which gives a natural construction of a symplectic structure on the quotient space obtained by a free and proper hamiltonian action of a Lie group on a symplectic manifold. The structure of the proof we will present below is inspired by [Sil08].

Definition 4.2.1. Let (M, ω) be a symplectic manifold and $\theta : G \times M \to M$ a smooth G-action on M. θ is a ***symplectic action*** if θ_g is a symplectomorphism for all $g \in G$.

Now consider the case where (M, ω) is a symplectic manifold, $\theta : G \times M \to M$ a symplectic G-action and $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G with dual vector space $\mathfrak{g}^*$.

Definition 4.2.2. A smooth map $\mu : M \to \mathfrak{g}^*$ is called a ***moment map*** if the following two conditions hold:

(i) μ is G-equivariant (with the natural action on $\mathfrak{g}^*$ by Ad^*, see Facts 4.1.5).

(ii) For $X \in \mathfrak{g}$ we define

$$\mu^X : M \to \mathbb{R}\,, \ \ \mu^X(p) := \langle \mu(p), X \rangle = \mu(p)(X) \ \ ,$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the canonical bilinear form. Then

$$d\mu^X = -\iota_{\widehat{X}}\omega \quad \forall\, X \in \mathfrak{g} \tag{4.3}$$

with $\widehat{X} = \widehat{\theta}(X)$ the induced vector field by X.

Then (M, ω, G, μ) is called a ***hamiltonian G-space*** and θ is a ***hamiltonian action***.

Remark 4.2.3.

(a) A simple calculation shows that $\mu : M \to \mathfrak{g}^*$ is smooth if and only if μ^X is smooth for all $X \in \mathfrak{g}$. In particular, '$d\mu^X$' is a well-defined expression.

(b) Recall that for any real n-dimensional vector space V there is a natural identification $V \xrightarrow{\sim} T_xV$ of V with its tangent space at $x \in V$ so that for an arbitrary isomorphism $\Phi : \mathbb{R}^n \to V$ the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\Phi"] \arrow[d, "\mathrm{can}"'] & V \arrow[d] \\ T_{\Phi^{-1}(x)}\mathbb{R}^n \arrow[r, "d_{\Phi^{-1}(x)}\Phi"'] & T_xV \end{tikzcd}$$

Now let $F : M \to V^*$ be a smooth function, where M is a manifold containing p and V a finite-dimensional vector space. Under the above identification $V^* \cong T_{F(p)}V^*$ and $\mathbb{R} \cong T_{F^X(p)}\mathbb{R}$, we obtain

$$\langle d_pF \cdot v\,,\, X \rangle = d_pF^X \cdot v \quad \forall\, v \in T_pM\,,\ X \in V$$

by a basic computation.

(c) Import special cases are $G = \mathbb{R}$ and $G = S^1$. Note that, since G is abelian, Ad^* is trivial, so condition (i) is equivalent to the G-invariance of μ, and, since G has dimension 1, condition (ii) is equivalent to $d\mu^{\partial_t} = -\iota_{\widehat{\partial_t}}\omega$ for the standard vector field ∂_t on G. Therefore, one often speaks of a moment map $\mu : M \to \mathbb{R}$, referring to μ^{∂_t}. Notice that the integral curve of $\widehat{\partial_t}$ through p is $\exp(t\,\partial_t) \cdot p = t \cdot p$, so $\widehat{\partial_t}$ is the vector field generated by the G-action. This means we search for a (hamiltonian) map $\mu : M \to \mathbb{R}$ such that $d\mu = -\iota_Y\omega$, where $Y := \widehat{\partial_t}$ is the vector field generated by the action. This hopefully explains why we call such an action hamiltonian. If we have $d\mu = -\iota_Y\omega$, then the action is already symplectic because

$$\mathcal{L}_Y\omega \overset{\text{Cartan}}{=\!=\!=} d(\iota_Y\omega) = -d(d\mu) = 0 \ \ ,$$

thus $\rho_t^*\omega = \omega$, where $\rho_t = \theta_t$ is the flow of $Y = \widehat{\partial_t}$ (if θ denotes the action). Furthermore, μ is even G-invariant because $\frac{d}{dt}(\mu(t \cdot p)) = d\mu \cdot Y = 0$. Thus μ induces a moment map $\tilde{\mu}$ with $\tilde{\mu}^{\partial_t} = \mu$.

Theorem 4.2.4 (Marsden-Weinstein-Meyer). *Let (M, ω, G, μ) be a hamiltonian G-space and $i : N := \mu^{-1}(0) \hookrightarrow M$ denote the inclusion.*

(a) If G acts freely on N, then 0 is a regular value of μ and N is a closed submanifold of M.

(b) If in addition the restricted action of G on N is proper, then the quotient manifold $M_{\text{red}} := N/G$ carries a unique 2-form ω_{red} such that $\pi^*\omega_{\text{red}} = i^*\omega$, where $\pi : N \twoheadrightarrow M_{\text{red}}$ is the projection. ω_{red} is a symplectic form on M_{red}.

Remark 4.2.5.

(a_1) G acts on $\mu^{-1}(0)$ since $\mu(g \cdot p) = \mathrm{Ad}_g^*(\mu(p)) = \mu(p) \circ \mathrm{Ad}_{g^{-1}} = 0$ for $p \in \mu^{-1}(0)$.

(a_2) We will show something even stronger: 0 is a regular value of μ if and only if G acts locally freely on N.

(b) The condition in (b) is fulfilled if G is compact or (more generally) if G acts properly on M (cf. Proposition 2.1.10).

Definition 4.2.6. We call the pair $(M_{\text{red}}, \omega_{\text{red}})$ the ***symplectic quotient*** or the ***reduction*** of (M, ω) with respect to (G, μ).

Lemma 4.2.7. *Let (M, ω, G, μ) be a hamiltonian G-space and $\mathfrak{g}_p$ denote the Lie algebra of the stabilizer of $p \in M$. Let $\mathfrak{g}_p^0 := \{\, \zeta \in \mathfrak{g}^* \mid \langle \zeta, X \rangle = 0 \;\; \forall\, X \in \mathfrak{g}_p \,\}$ be the annihilator of $\mathfrak{g}_p$. Considering $d_p\mu : T_pM \to \mathfrak{g}^*$ we obtain:*

$$\begin{aligned} \ker\, d_p\mu &= (T_p(G \cdot p))^{\omega_p} \\ \mathrm{im}\; d_p\mu &= \mathfrak{g}_p^0 \end{aligned}$$

Proof. Notice that G_p is a closed subgroup of G, so it is a Lie group and its Lie algebra $\mathfrak{g}_p$ is embedded in $\mathfrak{g}$. By equation (4.3) and Remark 4.2.3 (b) we get

$$\langle d_p\mu \cdot v, X \rangle = d_p\mu^X \cdot v = \omega_p(v, \widehat{X}_p) \quad \forall\, v \in T_pM\,,\ X \in \mathfrak{g}\ . \tag{4.4}$$

Consider $v \in \ker\, d_p\mu$ and let $x \in T_p(G \cdot p)$ be arbitrary. By Lemma 4.1.9 we can choose an $X \in \mathfrak{g}$ with $\widehat{X}_p = x$. Then $\omega_p(v, x) = 0$ by equation (4.4). Thus, v lies in the symplectic orthogonal complement to $T_p(G \cdot p)$. If, on the other hand, $v \in (T_p(G \cdot p))^{\omega_p}$, then $\langle d_p\mu \cdot v, X \rangle = 0$ for all $X \in \mathfrak{g}$ by Lemma 4.1.9 and equation (4.4). Hence, $d_p\mu \cdot v = 0$.
For $\zeta = d_p\mu \cdot v \in \mathrm{im}\; d_p\mu$ we have to show $\langle \zeta, X \rangle = 0 \;\; \forall\, X \in \mathfrak{g}_p$. If $X \in \mathfrak{g}_p$, then this means nothing else than $X_e \in T_eG_p$, so the complete integral curve $\exp(tX)$ lies in G_p. Thus, by definition of $\widehat{X}$, we obtain $\widehat{X}_p = 0$ and, by reusing equation (4.4), $\langle d_p\mu \cdot v, X \rangle = 0$. The final inclusion $\mathfrak{g}_p^0 \subseteq \mathrm{im}\; d_p\mu$ is a purely dimensional argument:

$$\begin{aligned} &\dim(\mathfrak{g}_p^0) \\ &= \dim(\mathfrak{g}^*) - \dim(\mathfrak{g}_p^*) \\ &= \dim(G) - \dim(G_p) \\ &= \dim(G/G_p) \end{aligned}
\qquad
\begin{aligned} &\dim(\mathrm{im}\; d_p\mu) \\ &= \dim(T_pM) - \dim(\ker\, d_p\mu) \\ &= \dim(T_pM) - \dim(\, (T_p(G \cdot p))^{\omega_p}\,) \\ &= \dim(T_pM) - [\, \dim(T_pM) - \dim(T_p(G \cdot p))\,] \\ &= \dim(G \cdot p) \\ &= \dim(G/G_p) \end{aligned}$$

□

Corollary 4.2.8. *Given a hamiltonian G-space (M, ω, G, μ). For $p \in M$ we have:*

$$G \text{ acts locally freely at } p \quad \Longleftrightarrow \quad p \text{ is a regular point of } \mu$$

In particular, we have proven (a) in Theorem 4.2.4.

Proof.

$$\begin{aligned} G \text{ acts locally freely at } p &\iff \exists U \subseteq G \text{ open neighborhood of } e : G_p \cap U = \{e\} \\ &\iff G_p \text{ is discrete} \\ &\iff \mathfrak{g}_p = \{0\} \\ &\overset{4.2.7}{\iff} d_p\mu \text{ is surjective} \end{aligned}$$

□

Lemma 4.2.9. *Let (V, ω) be a symplectic vector space with isotropic subspace U. Then U^ω/U carries a canonical symplectic form $\bar{\omega}$ defined by $\bar{\omega}(x+U, y+U) := \omega(x,y)$.*

Proof. $\bar{\omega}$ is well-defined: We have to show $\omega(x+s, y+t) = \omega(x,y)$ for all $x, y \in U^\omega$, $s, t \in U$.

$$\omega(x+s, y+t) = \omega(x,y) + \underbrace{\omega(x,t)}_{=0} + \underbrace{\omega(s,y)}_{=0} + \underbrace{\omega(s,t)}_{=0} = \omega(x,y) \ ,$$

where $\omega(s,t) = 0$ because $U \subseteq U^\omega$. Obviously, $\bar{\omega}$ is an alternating bilinear form. To show nondegeneracy, assume that for $x \in U^\omega$ we have $\omega(x,y) = 0 \ \forall y \in U^w$. Then $x \in (U^\omega)^\omega = U$, so $x + U = 0 \in U^\omega/U$. □

Proof of Theorem 4.2.4 (b). Since N is a submanifold of M, the restricted action of G on N is smooth. By the Quotient Manifold Theorem $M_{\text{red}} = N/G$ is a smooth manifold. Because π is a surjective submersion, the pullback π^* is injective, thus there is at most one 2-form ω_{red} satisfying $\pi^*\omega_{\text{red}} = i^*\omega$. To prove existence, recall that the tangent space of N/G at $\pi(p) = G\cdot p$ can be identified with $T_pN/T_p(G\cdot p)$ via Ψ_p as in Proposition 4.1.10. By Lemma 4.2.7 we have $T_p(G\cdot p) \subseteq T_pN = \ker d_p\mu = (T_p(G\cdot p))^{\omega_p}$, so the tangent space of the orbit through p is isotropic. Consider the symplectic form $\overline{\omega_p}$ on $T_p(G\cdot p)^{\omega_p}/T_p(G\cdot p)$ from Lemma 4.2.9. Let us define

$$(\omega_{\text{red}})_{\pi(p)} := (\Psi_p^{-1})^*\overline{\omega_p} \ .$$

This is indeed well-defined: Assume $q = g\cdot p$. We have to show

$$\overline{\omega_p}(\Psi_p^{-1}(\bar{x}_1), \Psi_p^{-1}(\bar{x}_2)) = \overline{\omega_q}(\Psi_q^{-1}(\bar{x}_1), \Psi_q^{-1}(\bar{x}_2)) \quad \forall \bar{x}_1, \bar{x}_2 \in T_{\pi(p)}\, N/G \ .$$

Choose $x_1, x_2 \in T_pN$ with $d_p\pi \cdot x_k = \bar{x}_k$, $k = 1, 2$. Then $\Psi_p^{-1}(\bar{x}_k) = x_k + T_p(G\cdot p)$ and $\Psi_q^{-1}(\bar{x}_k) = d_p\theta_g \cdot x_k + T_q(G\cdot q)$, where θ denotes the action of G on M. It remains to show that $\omega_p(x_1, x_2) = \omega_q(d_p\theta_g \cdot x_1, d_p\theta_g \cdot x_2)$. This however follows directly since θ is a symplectic action by assumption.
An immediate consequence from the definition of ω_{red} is

$$(\omega_{\text{red}})_{\pi(p)}(d_p\pi\cdot x_1, d_p\pi\cdot x_2) = \omega_p(x_1, x_2) \quad \forall\, p \in N,\, x_1, x_2 \in T_pN \ ,$$

which implies $\pi^*\omega_{\text{red}} = i^*\omega$ if ω_{red} is smooth. To prove the smoothness, assume $\bar{X}_1, \bar{X}_2$ are arbitrary vector fields on an open subset $V \subseteq M_{\text{red}}$. We want to show that $\omega_{\text{red}}(\bar{X}_1, \bar{X}_2)$ is a smooth function around $\pi(p) \in V$. Since $\pi|_{\pi^{-1}(V)}$ is a submersion, we can find an open neighborhood U of p in $\pi^{-1}(V) \subseteq N$ and vector fields X_1, X_2 on U that are $\pi|_U$-related to $\bar{X}_1, \bar{X}_2$. Then $\omega_{\text{red}}(\bar{X}_1, \bar{X}_2) \circ \pi|_U = i^*\omega\,|_U\,(X_1, X_2)$ which is smooth. $\pi|_U$ is a submersion onto the open neighborhood $\pi(U)$ of $\pi(p)$, hence $\omega_{\text{red}}(\bar{X}_1, \bar{X}_2)$ is smooth around $\pi(p)$.
It remains to show that ω_{red} is a symplectic form. Since $\overline{\omega_p}$ is a symplectic bilinear form and Ψ_p is an isomorphism, $(\omega_{\text{red}})_{\pi(p)}$ inherits this property. In addition, $\pi^*d\omega_{\text{red}} = d(\pi^*\omega_{\text{red}}) = d(i^*\omega) = i^*d\omega = 0$ and π^* is injective, thus $d\omega_{\text{red}} = 0$. □

Remark 4.2.10. Suppose, additionally we have the moment map $\nu : M \to \mathfrak{h}^*$ of another hamiltonian action $\vartheta : H \times M \to M$ of the Lie group H on M. If the actions of G and H commute (i.e. $\theta_g \circ \vartheta_h = \vartheta_h \circ \theta_g \quad \forall\, g \in G,\, h \in H$) and if μ is H- and ν is G-invariant, then the unique map $\nu_{\text{red}} : M_{\text{red}} \to \mathfrak{h}^*$ with $\nu_{\text{red}} \circ \pi = \nu \circ i$ is a moment map for the hamiltonian H-space $(M_{\text{red}}, \omega_{\text{red}}, H, \nu_{\text{red}})$, where H acts on M_{red} via ϑ_{red}, $\vartheta_{\text{red}}(h, [p]) := [\vartheta(h,p)]$.
In particular, if G is commutative, then M_{red} inherits a reduced hamiltonian G-action for which the constant zero function is a moment map.

Proof. Notice that we require the G-invariance of ν to prove that ν_{red} is well-defined and the H-invariance of μ and commutativity to show that ϑ_{red} is well-defined. That ϑ_{red} is a smooth symplectic action and ν_{red} is H-equivariant can be checked via simple calculations. For $X \in \mathfrak{h}$ let $\widehat{X}$ denote the induced vector field on M and $\widehat{X}_{\mathrm{red}}$ the induced vector field on M_{red}. For $p \in N$ the vector $\widehat{X}_p$ lies in T_pN and

$$d_p\pi \cdot \widehat{X}_p = d_p\pi \cdot d_e\vartheta^{(p)} \cdot X_e = d_e(\pi \circ \vartheta^{(p)}) \cdot X_e = d_e\vartheta^{(\pi(p))}_{\mathrm{red}} \cdot X_e = (\widehat{X}_{\mathrm{red}})_{\pi(p)}$$

and thus

$$\begin{aligned} \pi^* d\nu^X_{\mathrm{red}} &= d(\pi^*\nu^X_{\mathrm{red}}) = d(i^*\nu^X) = i^*d\nu^X = -i^*\iota_{\widehat{X}}\omega = -(\omega_{\mathrm{red}})_{\pi(\cdot)}(d.\pi\ \widehat{X}.\,,\, d.\pi\,\cdot) \\ &= \pi^*(-\iota_{\widehat{X}_{\mathrm{red}}}\omega_{\mathrm{red}})\ , \end{aligned}$$

which implies $d\nu^X_{\mathrm{red}} = -\iota_{\widehat{X}_{\mathrm{red}}}\omega_{\mathrm{red}}$.
If G is abelian, then μ is G-invariant since Ad^* becomes trivial. □

Remark 4.2.11. The only reason, why whe cannot generalize this construction to other level sets than the zero level set, is that normally other level sets are not G-invariant. However, if G is abelian, than all level sets are G-invariant and we can endow $\mu^{-1}(c)/G$ with a natural reduced symplectic form.

4.3 Contact Reduction

Inspired by the symplectic case, we now want to develop a contact analog.

Definition 4.3.1. A G-action θ on M ***preserves*** a contact distribution $\xi \subseteq TM$ if θ_g is a contactomorphism for all $g \in G$. A G-action θ, which preserves $\xi := \ker\alpha$, ***preserves the coorientation of the contact form*** α if for all $g \in G$, $p \in M$ and $v \in T_pM$

$$\alpha_p(v) > 0 \implies (\theta_g^*\,\alpha)_p\,(v) > 0\ .$$

Remark 4.3.2. If G is connected and the action preserves $\xi = \ker\alpha$, then it also preserves the coorientation of α since the continuous function $G \to \mathbb{R}\backslash\{0\}$, $g \mapsto (\theta_g^*\,\alpha)_p\,(v)$ is strictly positive for any $v \in T_pM$ with $\alpha_p(v) > 0$.

Theorem 4.3.3. *Let (M, α) be a strict contact manifold on which the Lie group G acts smoothly and properly via $\theta : G \times M \to M$. Suppose θ preserves $\xi := \ker\,\alpha$ and the coorientation of α. Then there exists a G-invariant contact form $\bar\alpha$ for ξ, i.e. $\xi = \ker\,\bar\alpha$ and $\theta_g^*\,\bar\alpha = \bar\alpha \ \ \forall\, g \in G$.*

To prove this result, we finally need the extensive tools developed in section 2.3. However, we still need one more ingredient, namely a generalization of Remark 2.3.13.

Proposition 4.3.4. *Let M be a smooth G-space, $H \le G$ a compact subgroup and $S \subseteq M$ an H-invariant submanifold with inclusion map $i : S \hookrightarrow M$. If $\sigma \in \Gamma(i^*(T^*M^{\otimes k}))$, $k \in \mathbb{N}_0$, where $i^*(T^*M^{\otimes k})$ denotes the pullback bundle of $T^*M^{\otimes k}$, and μ is the normalized Haar measure on H, then*

$$\bar\sigma := \int_H h^*\sigma\ \mathrm{d}\mu(h)$$

is a smooth H-invariant section of $i^*(T^*M^{\otimes k})$.

Remark 4.3.5. If $\varphi : M \to M$ is a smooth map with $\varphi(S) \subseteq S$, then $\varphi^*\sigma \in \Gamma(i^*(T^*M^{\otimes k}))$. In particular, since S is H-invariant, $h^*\sigma$ is a smooth section of the pullback bundle for all $h \in H$.

Proof of Remark 4.3.5. We want to show that $(\varphi^*\sigma)(X_1 \circ i, \ldots, X_k \circ i)$ is smooth for arbitrary local sections $X_1, \ldots, X_k$ of TM on $U \subseteq M$ open. The composition $\sigma \circ \varphi|_S^S$ is a smooth section of the bundle

$$(\varphi|_S^S)^*(i^*(T^*M^{\otimes k})) \underset{\text{can}}{\cong} (\varphi|_S)^*(T^*M^{\otimes k}) \underset{\text{can}}{\cong} i^*(\varphi^*(T^*M^{\otimes k})) \underset{\text{can}}{\cong} i^*((\varphi^*(TM))^{*\,\otimes k})$$

and $d\varphi \cdot X_j$ is a smooth section of $\varphi^*(TM)$.
Hence, $(\sigma \circ \varphi|_S^S)((d\varphi \cdot X_1) \circ i, \ldots, (d\varphi \cdot X_k) \circ i) = (\varphi^*\sigma)(X_1 \circ i, \ldots, X_k \circ i)$ is indeed smooth. □

Lemma 4.3.6 (Extension Lemma for Vector Bundles, cf. [Lee13, p. 257])**.** *Let $\pi : E \twoheadrightarrow M$ be a vector bundle and $A \subseteq M$ a closed subset contained in an open subset $U \subseteq M$. If $\sigma : A \to E$ is a (not necessarily continuous) section of π and around every point of A there exists an open neighborhood and a smooth section on it, which is an extension of σ, then there exists a global smooth section $\tilde{\sigma}$ of π with support in U, which restricts to σ on A.*

Proof. Use an appropriate partition of unity. □

Proof of Proposition 4.3.4. We have to prove that the integral exists and that $\bar{\sigma}$ varies smoothly on S. Given an arbitrary $p \in S$ we can choose a chart $(V, \varphi = (x_1, \ldots, x_m))$ of M, containing p, and write

$$\sigma = \sum_{(i_1,\ldots,i_k)} \lambda_{(i_1,\ldots,i_k)}\ dx^{i_1} \otimes \ldots \otimes dx^{i_k} = \sum_I \lambda_I\ dx^I$$

for smooth functions $\lambda_I : S \cap V \to \mathbb{R}$, $I = (i_1, \ldots, i_k)$. Since i is an immersion, we can find smooth extensions $\kappa_I : V \to \mathbb{R}$ of λ_I (if necessary, make V smaller). Then

$$\rho := \sum_I \kappa_I\ dx^I \ \in \Gamma(T^*M^{\otimes k}\big|_V)$$

is a smooth extension of σ onto an open neighborhood of p in M. Because S is a submanifold of M, it is locally closed, so we can find an open set $S \subseteq U \subseteq M$ in which S is closed. By the Extension Lemma for Vector Bundles 4.3.6 we can find a smooth section $\tilde{\sigma}$ on U, such that $\tilde{\sigma}|_S = \sigma$.
Now let $p \in S$ be arbitrary. Since S is H-invariant and H is compact, using Corollary A.1.10, we obtain an open H-invariant set $V \subseteq U$ that contains p. By Remark 2.3.13 the section $\int_H h^*\tilde{\sigma}|_V\ d\mu(h)$ is smooth. Thus, $\bar{\sigma}$ is well-defined and smooth on $V \cap S$. This finishes the proof since $p \in S$ was arbitrary. □

Proof of Theorem 4.3.3. Let us choose families $(f_n)_{n\in N}$, $(S_n)_{n\in N}$, $N \subseteq \mathbb{N}$, as in Proposition 2.3.18, so that S_n is a slice at $p(n) \in M$. For $n \in N$ let us define the $G_{p(n)}$-invariant smooth tensor field

$$\alpha'_n := \int_{G_{p(n)}} h^*\left(\alpha\,|_{S_n}\right)\ \mathrm{d}\mu_n(h)\ \in\ \Gamma(\,i_n^*(T^*M)\,)\ ,$$

where μ_n denotes the normalized Haar measure on $G_{p(n)}$ and $i_n : S_n \hookrightarrow M$ is the inclusion. Notice that $G_{p(n)}$ is indeed compact since G acts properly on M. Because S_n is a slice at $p(n)$, S_n is $G_{p(n)}$-invariant and $G{\cdot}S_n$ is open in M. Extend α'_n onto $G{\cdot}S_n$ via $\bar{\alpha}_n \in \Omega^1(G{\cdot}S_n)$ with

$$(\bar{\alpha}_n)_{g\cdot s}(v) := (\alpha'_n)_s(d_{g\cdot s}\theta_{g^{-1}} \cdot v) \quad \forall\, g \in G,\ s \in S_n,\ v \in T_{g\cdot s}M\ .$$

This is well-defined: Suppose $p = g \cdot s = \tilde{g} \cdot \tilde{s}$ for $g, \tilde{g} \in G$, $s, \tilde{s} \in S_n$ and $v \in T_pM$. It follows from Theorem 2.3.6 (c) that $\tilde{g}^{-1}g \in G_{p(n)}$ and therefore

$$\begin{aligned}
(\alpha'_n)_{\tilde{s}}\,(\,d_p\theta_{\tilde{g}^{-1}} \cdot v\,) &= (\alpha'_n)_{\tilde{g}^{-1}g\cdot s}\,(\,d_p\theta_{\tilde{g}^{-1}} \cdot v\,) \\
&= (\alpha'_n)_{\theta_{\tilde{g}^{-1}g}(s)}\,(\,d_s\theta_{\tilde{g}^{-1}g} \cdot d_{g\cdot s}\theta_{g^{-1}} \cdot v\,) \\
&= ((\theta_{\tilde{g}^{-1}g})^*\alpha'_n)_s\,(\,d_p\theta_{g^{-1}} \cdot v) \\
&= (\alpha'_n)_s(d_p\theta_{g^{-1}} \cdot v\,)\ .
\end{aligned}$$

Now let us prove the smoothness of $\bar{\alpha}_n$. First, note that $\cdot : G \times S_n \twoheadrightarrow G \cdot S_n$ is a surjective submersion: For arbitrary (g_0, s_0) consider a local cross section $\chi : U \to G$ in $G/G_{p(n)}$ and the corresponding diffeomorphism $F_{g_0} : (g_0 \cdot U) \times S_n \xrightarrow{\sim} \mathcal{U}(g_0 \cdot S_n)$ onto an open neighborhood $\mathcal{U}(g_0 \cdot S_n)$ of $g_0 \cdot S_n$ from Theorem 2.3.15. Then the composition

$$\mathcal{U}(g_0 \cdot S_n) \xrightarrow{F_{g_0}^{-1}} g_0 \cdot U \times S_n \xrightarrow{g_0^{-1} \cdot \times \mathrm{id}} U \times S_n \xrightarrow{\chi \times \mathrm{id}} G \times S_n \xrightarrow{g_0 \cdot \times \mathrm{id}} G \times S_n$$

is a smooth local section of $\cdot : G \times S_n \twoheadrightarrow G \cdot S_n$ through (g_0, s_0). Thus $\cdot : G \times S_n \twoheadrightarrow G \cdot S_n$ is a submersion.
Given an arbitrary vector field X on an open subset of $G \cdot S_n$, we have to show that $\bar{\alpha}_n(X)$ is smooth. Since $\cdot : G \times S_n \twoheadrightarrow G \cdot S_n$ is a surjective submersion, it suffices to prove that $(g, s) \mapsto (\bar{\alpha}_n)_{g \cdot s}(X_{g \cdot s})$ is smooth. By Proposition 4.3.4 α'_n is a smooth section of $i_n^*(T^*M)$, so we might as well consider it as a (smooth) bundle homomorphism $i_n^*(TM) \to S_n \times \mathbb{R}$ over S_n. Because we have $(\bar{\alpha}_n)_{g \cdot s}(X_{g \cdot s}) = \alpha'_n \circ d\theta \; (0^G_{g^{-1}}, X_{g \cdot s})$ with the zero vector field 0^G on G, we have indeed proven the smoothness of $(g, s) \mapsto (\bar{\alpha}_n)_{g \cdot s}(X_{g \cdot s})$ and thus of $\bar{\alpha}_n$.
Now we can finally define $\bar{\alpha}$ as

$$\bar{\alpha} := \sum_{n \in N} f_n \, \bar{\alpha}_n \quad \in \Omega^1(M) \ .$$

By construction $\bar{\alpha}_n$ is G-invariant and by choice f_n is so too, so $\bar{\alpha}$ is G-invariant. It remains to show $\xi = \ker \alpha = \ker \bar{\alpha} =: \bar{\xi}$. For $v \in \xi_p$, $p \in G \cdot S_n$, using that the action preserves ξ, we obtain $(\bar{\alpha}_n)_p(v) = 0$ and thus $\xi \subseteq \bar{\xi}$. For the converse inclusion it suffices to show that $\bar{\alpha}_p \neq 0$. Choose $v \in T_pM$ with $\alpha_p(v) > 0$. By assumption $(g^*\alpha)_p\,(v) > 0 \; \forall \, g \in G$. So, if $p = g \cdot s \in G \cdot S_n$, we compute

$$\begin{aligned} (\bar{\alpha}_n)_p(v) &= \int_{G_{p(n)}} (h^*\alpha)_s(d_p\theta_{g^{-1}} \cdot v) \;\; \mathrm{d}\mu_n(h) \\ &= \int_{G_{p(n)}} \underbrace{((hg^{-1})^*\alpha)_p\,(v)}_{>0} \;\; \mathrm{d}\mu_n(h) \;\; > \;0 \ . \end{aligned}$$

Thus, $\bar{\alpha}_p(v) > 0$, which finishes the proof. □

Our next goal is, to prove the contact version of Marsden-Weinstein-Meyer. As it is frequently the case in contact geometry, the result is slightly weaker than a symplectic analog and with some constraints but is proven similarly with additional assumptions. Let us start by defining a contact moment map.

For the rest of this section let (M, α) be a strict contact manifold with kernel $\xi := \ker \alpha$. Assume, G acts smoothly on M via $\theta : G \times M \to M$ and α is G-invariant (notice that if G is connected and θ acts properly on M, then by Remark 4.3.2 and Theorem 4.3.3 such a contact form exists for any given contact distribution ξ, preserved by θ; in particular this holds for $G = S^1$). Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ its dual vector space.

Definition 4.3.7. Define the ***(contact) moment map*** $\mu : M \to \mathfrak{g}^*$ by

$$\mu(p)(X) := \mu^X(p) := \alpha_p(\widehat{X}_p) \quad \forall \, p \in M \, , \, X \in \mathfrak{g} \ ,$$

where $\widehat{X} = \widehat{\theta}(X)$ is the generated vector field on M.

Lemma 4.3.8. *μ is G-equivariant with respect to the G-action on $\mathfrak{g}^*$ induced by* Ad^*.

Proof.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(\mathrm{int}_{g^{-1}})_*} & \mathfrak{g} \\ \mathrm{exp}\downarrow & \circlearrowleft & \downarrow\mathrm{exp} \\ G & \xrightarrow[\mathrm{int}_{g^{-1}}]{} & G \end{array} \quad \Longrightarrow \quad \exp(\, t \ (\mathrm{int}_{g^{-1}})_*(X)\,) = \mathrm{int}_{g^{-1}}(\exp(tX))$$

and thus

$$\begin{aligned} \widehat{X}_{g\cdot p} &= \frac{d}{dt}\Big|_{t=0}\Big(\exp(tX)\cdot(g\cdot p)\Big) \\ &= \frac{d}{dt}\Big|_{t=0}\Big(gg^{-1}\exp(tX)\, g\cdot p\Big) \\ &= d_p\theta_g \cdot \frac{d}{dt}\Big|_{t=0}\Big(g^{-1}\exp(tX)\, g\cdot p\Big) \\ &= d_p\theta_g \cdot \frac{d}{dt}\Big|_{t=0}\Big(\mathrm{int}_{g^{-1}}(\exp(tX))\cdot p\Big) \\ &= d_p\theta_g \cdot \frac{d}{dt}\Big|_{t=0}\Big(\exp(\, t \ (\mathrm{int}_{g^{-1}})_*(X)\,)\cdot p\Big) \\ &= d_p\theta_g \cdot \widehat{\theta}(\mathrm{Ad}_{g^{-1}}(X))_p \end{aligned}$$

Therefore,

$$\begin{aligned} (\mu(g\cdot p))\,(X) &= \alpha_{g\cdot p}\,(\widehat{X}_{g\cdot p}) \\ &= \alpha_{g\cdot p}\,\big(\,d_p\theta_g\cdot\widehat{\theta}(\mathrm{Ad}_{g^{-1}}(X))_p\,\big) \\ &= (\theta_g^*\,\alpha)_p\,\big(\,\widehat{\theta}(\mathrm{Ad}_{g^{-1}}(X))_p\,\big) \\ &= \alpha_p\,\big(\,\widehat{\theta}(\mathrm{Ad}_{g^{-1}}(X))_p\,\big) \\ &= (\mu(p))\,(\,\mathrm{Ad}_{g^{-1}}(X)\,) \\ &= \mu(p)\circ\mathrm{Ad}_{g^{-1}}\ (X) \\ &= (g\cdot\mu(p))\,(X) \end{aligned}$$

□

In particular the zero level set $N := \mu^{-1}(0)$ is G-invariant.
By applying Lemma 4.1.9 we obtain

$$p\in N=\mu^{-1}(0) \quad\Longleftrightarrow\quad \alpha_p\big|_{T_p(G\cdot p)}=0 \quad\Longleftrightarrow\quad T_p(G\cdot p)\subseteq \xi_p=\ker\,\alpha_p \tag{4.5}$$

Lemma 4.3.9. *Let $\mathfrak{g}_p$ be the Lie algebra of the stabilizer G_p of p and $\mathfrak{g}_p^0$ its annihilator. Then the following equations hold for $p\in N=\mu^{-1}(0)$:*

(1) $d\mu^X = -\iota_{\widehat{X}}\,d\alpha \quad \forall\, X\in\mathfrak{g}$

(2) $\ker\, d_p\mu\ \cap\ \xi_p\ =\ (T_p(G\cdot p))^{(d\alpha|_\xi)_p}$

(3) $\mathrm{im}\ d_p\mu\ =\ \mathfrak{g}_p^0\ =\ \{\,\zeta\in\mathfrak{g}^*\mid\langle\zeta,X\rangle=0\ \ \forall\, X\in\mathfrak{g}_p\,\}$

Proof.

(1) Let Φ_Y^t denote the flow of some vector field $Y\in\mathfrak{X}(M)$ at time t. Then $\Phi_{\widehat{X}}^t=\theta_{\exp(tX)}$. Since α is G-invariant, this means $(\Phi_{\widehat{X}}^t)^*\alpha=\alpha\ \forall\, t\in\mathbb{R}$, so $\mathcal{L}_{\widehat{X}}\alpha=0$. By Cartan's magic formula we get

$$d\mu^X = d(\alpha(\widehat{X})) = \mathcal{L}_{\widehat{X}}\alpha - \iota_{\widehat{X}}(d\alpha) = -\iota_{\widehat{X}}(d\alpha)\ .$$

(2) Use equation (1) and argue as in Lemma 4.2.7.

(3) For "$\subseteq$" use equation (1) and argue as in Lemma 4.2.7. For the converse inclusion note that $\ker d_p\mu \not\subseteq \xi_p$: Otherwise we would have

$$\begin{aligned}\dim(M) &= \dim \ker d_p\mu + \dim \operatorname{im} d_p\mu \\ &\leq \dim (T_p(G\cdot p))^{(d\alpha|_\xi)_p} + \dim \mathfrak{g}_p^0 \\ &= \dim \xi_p - \dim(G/G_p) + \dim(G/G_p) = \dim(M) - 1 ,\end{aligned}$$

which is a contradiction. Thus, $\ker d_p\mu + \xi_p = T_pM$ and $\dim \ker d_p\mu = 1 + \dim (T_p(G\cdot p))^{(d\alpha|_\xi)_p} = \dim(M) - \dim(G) + \dim(G_p)$, which implies $\dim \operatorname{im} d_p\mu = \dim(G) - \dim(G_p) = \dim \mathfrak{g}_p^0$.

□

Remark 4.3.10. In the proof above we have shown that $\ker d_p\mu \not\subseteq \xi_p$.

Corollary 4.3.11. *For $p \in N = \mu^{-1}(0)$ we have*

$$G \text{ acts locally freely at } p \quad \Longleftrightarrow \quad p \text{ is a regular point of } \mu$$

Proof. As in Corollary 4.2.8. □

Now assume that G acts freely on N. By the preceding corollary every point of N is a regular point of 0, so $N = \mu^{-1}(0)$ is a closed submanifold of codimension $\dim(G)$. Then a significant consequence of Lemma 4.3.9 is, that for $p \in N$ we have

$$T_p(G\cdot p) \underset{(4.5)}{\subseteq} \xi_p \cap T_pN = \xi_p \cap \ker d_p\mu = (T_p(G\cdot p))^{(d\alpha|_\xi)_p} \subseteq \xi_p ,$$

so the tangent space of the orbit is isotropic with respect to $d\alpha_p|_{\xi_p}$.

Theorem 4.3.12 (Geiges, [Gei97, Theorem 6])**.** *Let (M,α) be a strict contact manifold with contact distribution ξ and $\theta : G\times M \to M$ be a smooth action on M, preserving the contact form α. Let μ denote the moment map with respect to the given action.*

(a) If G acts freely on $N := \mu^{-1}(0)$, then 0 is a regular value of μ and N is a closed submanifold in M of codimension $\dim(G)$.

(b) If in addition the restricted action of G on N is proper, then there exists a unique 1-form α_{red} on the quotient manifold $M_{\text{red}} := N/G$ such that $\pi^*\alpha_{\text{red}} = i^*\alpha$, where $\pi : N \twoheadrightarrow N/G$ denotes the projection and $i : N \hookrightarrow M$ is the inclusion. α_{red} is a contact form on M_{red}.

(c) Let β be another G-invariant contact form with kernel ξ and moment map ν. Then $\mu^{-1}(0) = \nu^{-1}(0)$ and, under the conditions of (b), $\ker\alpha_{\text{red}} = \ker\beta_{\text{red}}$ *, so ξ defines a reduced contact structure $\xi_{\text{red}} := \ker \alpha_{\text{red}}$ on N/G.*

Remark 4.3.13.

(a_1) We have already proven (a) above, including the G-invariance of N.

(a_2) Corollary 4.3.11 shows that 0 is a regular value if and only if the action of G on N is locally free.

(b) If G acts properly on M (e.g. if G is compact), then the restricted action on N is also proper.

Definition 4.3.14. We call the pair $(M_{\text{red}}, \alpha_{\text{red}})$ the ***contact quotient*** or the ***reduction*** of (M,α) with respect to the action θ.

Lemma 4.3.15. *Let V be a vector space, $\alpha \in \bigwedge^1 V^*$ a 1-linear form, $\xi := \ker\alpha$ and $U \subseteq N \subseteq V$ linear subspaces with $U \subseteq \xi$. Let $\pi : N \twoheadrightarrow N/U$ denote the projection and $i : N \hookrightarrow V$ the inclusion.*
Then there exists a unique 1-linear form $\bar{\alpha}$ on N/U with $\pi^*\bar{\alpha} = i^*\alpha$. Its kernel is $\bar{\xi} := \ker\bar{\alpha} = (N \cap \xi)/U$.

Proof. π^* is injective, so there exists at most one such form $\bar{\alpha}$. Now define $\bar{\alpha}(x+U) := \alpha(x)$, which is well-defined because $U \subseteq \xi = \ker\alpha$. Obviously, we have $\bar{\xi} = \ker\bar{\alpha} = (N\cap\xi)/U$. □

Lemma 4.3.16. *Let V be a vector space, α a 1-linear form with kernel ξ, ω a 2-linear form on V, such that $\omega|_\xi$ is nondegenerate, and $U \subseteq \xi \subseteq V$, $N \subseteq V$ linear subspaces with $U \subseteq U^{\omega|_\xi} = \xi \cap N \subseteq N$. Let $\pi : N \twoheadrightarrow N/U =: V_{\text{red}}$ denote the projection and $i : N \hookrightarrow V$ the inclusion. Let $\bar{\omega}$ be an arbitrary 2-linear form on V_{red} with $\pi^*\bar{\omega} = i^*\omega$.*
Then $\bar{\omega}|_{\bar{\xi}}$ is a symplectic form, where $\bar{\xi} := \ker\bar{\alpha}$ and $\bar{\alpha} \in \bigwedge^1 V_{\text{red}}^*$ is determined by $\pi^*\bar{\alpha} = i^*\alpha$.

Proof. By assumption, $\omega|_\xi$ is a symplectic form and U is an isotropic subspace with respect to $\omega|_\xi$. By Lemma 4.2.9 this gives us a reduced symplectic form $\omega_{\text{red}} \in \bigwedge^2(U^{\omega|_\xi}/U)^*$ defined by $\omega_{\text{red}}(x+U, y+U) = \omega(x,y)$. Notice that for $x, y \in U^{\omega|_\xi} = \xi \cap N$ we have

$$\omega_{\text{red}}(x+U, y+U) = \omega(x,y) = i^*\omega\,(x,y) = \pi^*\bar{\omega}\,(x,y) = \bar{\omega}(x+U, y+U) \ ,$$

so $\bar{\omega}$ and ω_{red} coincide on $(\xi \cap N)/U = \bar{\xi}$. □

Proof of Theorem 4.3.12. π^* is injective, so α_{red} is uniquely determined. Now let us deal with the existence. For all $p \in N$, $\bar{v} \in T_{\pi(p)}M_{\text{red}}$ put

$$(\alpha_{\text{red}})_{\pi(p)}(\bar{v}) := \alpha_p(v) \text{ for some } v \in T_pN \text{ with } d_p\pi \cdot v = \bar{v}$$

This definition is independent of the choice of representatives: Assume $p' = g \cdot p$ with $p, p' \in N$ and $v \in T_pN$, $v' \in T_{p'}N$ with $\bar{v} = d_p\pi \cdot v = d_{p'}\pi \cdot v'$.

$$\begin{aligned} \alpha_p(v) &= (\theta_g^*\,\alpha)_p(v) = \alpha_{p'}(d_p\theta_g \cdot v) = \alpha_{p'}(d_p\big(\theta_g|_N^N\big) \cdot v) \\ d_{p'}\pi \cdot d_p\big(\theta_g|_N^N\big) \cdot v &= d_p\big(\pi \circ \theta_g|_N^N\big) \cdot v = d_p\pi \cdot v = \bar{v} = d_{p'}\pi \cdot v' \end{aligned}$$

Thus, $d_p\big(\theta_g|_N^N\big) \cdot v - v' \in \ker d_{p'}\pi = T_{p'}(G \cdot p') \subseteq \xi_{p'} = \ker\alpha_{p'}$, whereby we have used Proposition 4.1.12. This shows $\alpha_p(v) = \alpha_{p'}(d_p\big(\theta_g|_N^N\big) \cdot v) = \alpha_{p'}(v')$.

By definition of α_{red} we obtain $(\alpha_{\text{red}})_{\pi(p)}(d_p\pi \cdot v) = (i^*\alpha)_p(v)$ for $p \in N$ and $v \in T_pN$. Now the smoothness of α_{red} follows completely analogously as in the proof of the Marsden-Weinstein-Meyer Theorem 4.2.4 and thus $\pi^*\alpha_{\text{red}} = i^*\alpha$.

It remains to show that α_{red} is indeed a contact form. M has odd dimension, so $\dim(M_{\text{red}}) = \dim(M) - 2\dim(G)$ is odd too. Put $\xi_{\text{red}} := \ker\alpha_{\text{red}}$ and $[p] := \pi(p)$. By Remark 4.1.18 (b) it suffices to prove that $(d\,\alpha_{\text{red}})|_{\xi_{\text{red}}}$ is nondegenerate. Consider an arbitrary fixed $p \in N$. Let $\Psi_p : T_pN\,/\,T_p(G\cdot p) \xrightarrow{\sim} T_{[p]}M_{\text{red}}$ be the canonical identification from Proposition 4.1.10 with projection $\text{pr}_p : T_pN \twoheadrightarrow T_pN\,/\,T_p(G\cdot p)$ and $\bar{\alpha} := \overline{\alpha_p}$ be the reduced 1-form on $T_pN\,/\,T_p(G\cdot p)$ from Lemma 4.3.15 with kernel $\bar{\xi}$. Notice that

$$(\alpha_{\text{red}})_{[p]}(\bar{v}) = \bar{\alpha}(\Psi_p^{-1}(\bar{v})) \quad \text{ for } \bar{v} \in T_{[p]}M_{\text{red}} \ .$$

Thus, $\Psi_p(\bar{\xi}) = (\xi_{\text{red}})_{[p]}$ and $\bar{\alpha} = \Psi_p^*((\alpha_{\text{red}})_{[p]})$. Define $\omega := (d\alpha)_p \in \bigwedge^2(T_pM)^*$ and $\bar{\omega} := (\Psi_p)^*(d\,\alpha_{\text{red}})_{[p]} \in \bigwedge^2(T_pN\,/\,T_p(G\cdot p))^*$. Then $\omega|_{\xi_p}$ is nondegenerate since α is a contact form. It follows from $\pi^*\alpha_{\text{red}} = i^*\alpha$ that $\pi^*(d\,\alpha_{\text{red}}) = i^*(d\alpha)$ and therefore

$$\text{pr}_p^*\,\bar{\omega} = (\Psi_p \circ \text{pr}_p)^*(d\,\alpha_{\text{red}})_{[p]} \overset{4.1.10}{=\!=\!=} (d_p\pi)^*(d\,\alpha_{\text{red}})_{[p]} = (d_pi)^*(d\alpha)_p = (d_pi)^*\omega \ .$$

By Lemma 4.3.16 $\bar{\omega}|_{\bar{\xi}}$ is a symplectic form. Since Ψ_p is an isomorphism, we have proven that $(d\,\alpha_{\text{red}})_{[p]}|_{\Psi_p(\bar{\xi})} = (d\,\alpha_{\text{red}})_{[p]}|_{(\xi_{\text{red}})_{[p]}}$ is nondegenerate. This finishes the proof of (b).

Now let β be another G-invariant contact form with $\ker \beta = \xi$ and moment map ν. Then

$$N = \mu^{-1}(0) = \left\{\, p \in M \mid T_p(G \cdot p) \subseteq \xi_p \,\right\} = \nu^{-1}(0)$$

and as the proof above shows, we obtain

$$\begin{aligned} \ker\,(\alpha_{\text{red}})_{[p]} \;&=\; \Psi_p^{-1}(\ker\,\overline{\alpha_p}) \overset{4.3.15}{=\!=\!=} \Psi_p^{-1}(\,(T_pN \cap \xi_p)\,/\,T_p(G \cdot p)\,) \\ &= \ldots = \ker\,(\beta_{\text{red}})_{[p]}\ . \end{aligned}$$

□

Remark 4.3.17. An analog of Remark 4.2.10 also holds in the contact case: If another Lie group H acts smoothly on M via ϑ, preserving α and commuting with the G-action θ, and if $\mu : M \to \mathfrak{g}^*$ is H-invariant and the contact moment map $\nu : M \to \mathfrak{h}^*$ with respect to the action ϑ is G-invariant, then $M_{\text{red}} = \mu^{-1}(0)/G$ admits a canonical smooth H-action ϑ_{red} for which α_{red} is H-invariant. The contact moment map $\nu_{\text{red}} : M_{\text{red}} \to \mathfrak{h}^*$ for this action is the unique map with $\nu_{\text{red}} \circ \pi = \nu \circ i$.
In particular, if G is commutative, then M_{red} inherits a reduced G-action, preserving α_{red}, for which the moment map μ_{red} is the constant zero function.

Remark 4.3.18. However, there is no contact counterpart of Remark 4.2.11 since we only have $T_p(G \cdot p) \subseteq \xi_p$ if $p \in \mu^{-1}(0)$, so Lemma 4.3.9 can only be formulated for points in the zero level set, whereas Lemma 4.2.7 holds for all points in M.

5 Symplectic and Contact Cuts

5.1 Symplectic Cuts

In this section we shall endow the topological cut of a symplectic manifold with respect to a corresponding moment map with a natural symplectic structure, so that the symplectic form remains unchanged above the relevant level set and the level set, together with the symplectic reduced form from Chapter 4, becomes a symplectic submanifold of the cut.

Lemma 5.1.1. *If* (M_1, ω_1) *and* (M_2, ω_2) *are symplectic manifolds, the* ***product symplectic manifold*** $(M_1 \times M_2, \omega)$ *is given by the symplectic form* $\omega := \omega_1 + \omega_2 := \mathrm{pr}_1^* \omega_1 + \mathrm{pr}_2^* \omega_2$ *with projections* $\mathrm{pr}_i : M_1 \times M_2 \to M_i$.

Proof. Trivial. □

Recall from 4.2.3 (c) that a moment map for an S^1-action on a symplectic manifold (M, ω) is just a smooth map $f : M \to \mathbb{R}$ satisfying $df = -\iota_{\widehat{\partial_t}} \omega$.

Theorem 5.1.2. *Let* f *be a moment map for a hamiltonian* S^1*-action on a symplectic manifold* (M, ω)*. Suppose the action is free on the level set* $f^{-1}(a)$*. Then the cut* $M_{[a,\infty)}$ *is well-defined and carries a natural symplectic form* ω_{cut}*. The canonical diffeomorphism* $M \supseteq f^{-1}((a,\infty)) \xrightarrow{\sim} f^{-1}((a,\infty)) \subseteq M_{[a,\infty)}$ *becomes a symplectomorphism and the inclusion* $(f^{-1}(a)/S^1\,,\, \omega_{\mathrm{red}}) \hookrightarrow (M_{[a,\infty)}, \omega_{\mathrm{cut}})$ *is symplectic.*

Proof. As in the proof of Proposition 3.2.1, put $\Psi : M \times \mathbb{C} \to \mathbb{R}\,,\ \Psi(m,z) := f(m) - |z|^2\,,$ and $\sigma : f^{-1}([a,\infty)) \to \Psi^{-1}(a)\,,\ \sigma(m) := (m, \sqrt{f(m) - a})$. Consider the product symplectic manifold $(M \times \mathbb{C}, \omega + 2\,dx \wedge dy)$ with the S^1-action $\lambda \cdot (m,z) := (\lambda \cdot m, \lambda^{-1} z)$. This is a free action on $\Psi^{-1}(a)$. The differential of Ψ is $d\Psi = df - 2x\,dx - 2y\,dy$. We argue that Ψ is a moment map for the S^1-action on $M \times \mathbb{C}$. Then, by Corollary 4.2.8, a is a regular value of Ψ and therefore of f. Additionally, f is S^1-invariant as moment map, thus the cut on M with respect to a is well-defined. To prove that Ψ is a moment map, we have to show that

$$df - 2x\,dx - 2y\,dy \;=\; \omega + 2\,dx \wedge dy\ (-, \widehat{\theta}^{M \times \mathbb{C}}(\partial_t))\ ,$$

where $\widehat{\theta}^{M \times \mathbb{C}}$ denotes the infinitesimal generator on $M \times \mathbb{C}$. We have

$$\begin{aligned} \widehat{\theta}^{M \times \mathbb{C}}(\partial_t)_{(m,z)} &= \frac{d}{dt}\Big|_{t=0} e^{it} \cdot (m,z) \\ &= (\widehat{\theta}^M(\partial_t)_m\,,\, -iz) \,\in\, T_{(m,z)}(M \times \mathbb{C}) \,\cong\, T_m M \times \mathbb{C} \end{aligned}$$

with infinitesimal generator $\widehat{\theta}^M$ on M. Hence,

$$\begin{aligned} \omega + 2\,dx \wedge dy\ (-, \widehat{\theta}^{M \times \mathbb{C}}(\partial_t)) &= \omega\,(-, \widehat{\theta}^M(\partial_t)) \,+\, 2\,dx \wedge dy\,(iz, -) \\ &= df - 2x\,dx - 2y\,dy\ . \end{aligned}$$

Since Ψ is a moment map and S^1 is a commutative group, by Marsden-Weinstein-Meyer for arbitrary level sets (cf. Remark 4.2.11) there is a unique symplectic form ω' on $\Psi^{-1}(a)/S^1$ with $\mathrm{pr}^* \omega' = \mathrm{incl}^*(\omega + 2\,dx \wedge dy)$. Let $(\,\omega + 2\,dx \wedge dy\,)_{\mathrm{red}}$ denote this symplectic form ω'.

As in the proof of Proposition 3.2.1 we write $\bar{\sigma} : M_{[a,\infty)} \xrightarrow{\sim} \Psi^{-1}(a)/S^1$ for the descent of σ. Then put $\omega_{\mathrm{cut}} := \bar{\sigma}^*(\,\omega \,+\, 2\,dx \wedge dy\,)_{\mathrm{red}}$, it is symplectic because $\bar{\sigma}$ is a diffeomorphism by construction.

We want to show that $(\Pi|^{f^{-1}((a,\infty))}_{f^{-1}((a,\infty))})^*\omega_{\mathrm{cut}} = \omega|_{f^{-1}((a,\infty))}$, with projection $\Pi : f^{-1}([\infty)) \to M_{[a,\infty)}$. By diagram (3.5) we just need to validate that $(\sigma|_{f^{-1}((a,\infty))})^*(\,\omega \,+\, 2\,dx \wedge dy\,) = \omega|_{f^{-1}((a,\infty))}$. However, this is trivial since $d(y \circ \mathrm{pr}_{\mathbb{C}} \circ \sigma|_{f^{-1}((a,\infty))}) = d\,(0) = 0$.

For the last statement recall that by $f^{-1}(a)/S^1 \subseteq M_{[a,\infty)}$ we mean in fact the submanifold $\bar{\sigma}^{-1}(f^{-1}(a)/S^1)$ with $f^{-1}(a)/S^1 \cong (f^{-1}(a) \times \{0\})/S^1 \subseteq \Psi^{-1}(a)/S^1$. Since a is a regular value of f and S^1 is commutative, Marsden-Weinstein-Meyer gives us a natural symplectic form ω_{red} on $f^{-1}(a)/S^1$. We would like to show that $(\bar{\sigma}^{-1}(f^{-1}(a)/S^1), \bar{\sigma}^*\omega_{\mathrm{red}}) \hookrightarrow (M_{[a,\infty)}, \omega_{\mathrm{cut}})$ is a symplectic inclusion, i.e. the pullback of ω_{cut} along the inclusion is $\bar{\sigma}^*\omega_{\mathrm{red}}$. Stare at the following commutative diagram:

$$\begin{tikzcd}
 & M \times \mathbb{C} & M \times \{0\} \arrow[l, hook', "\mathrm{incl}_4"'] \\
 & \Psi^{-1}(a) \arrow[u, hook, "\mathrm{incl}_3"] \arrow[d, two heads, "\mathrm{pr}_2"] & f^{-1}(a) \arrow[l, hook', "\mathrm{incl}_2"'] \arrow[u, hook', "\mathrm{incl}_5"'] \arrow[dd, two heads, "\mathrm{pr}_1"] \\
M_{[a,\infty)} \arrow[r, "\cong", "\bar{\sigma}"'] & \Psi^{-1}(a)/S^1 & \\
\bar{\sigma}^{-1}(f^{-1}(a)/S^1) \arrow[u, hook] \arrow[rr, "\cong", "\bar{\sigma}"'] & & f^{-1}(a)/S^1 \arrow[ul, hook', "\mathrm{incl}_1"']
\end{tikzcd} \tag{5.1}$$

Clearly, it suffices to show that the pullback of $(\,\omega \,+\, 2\,dx \wedge dy\,)_{\mathrm{red}}$ along incl_1 is ω_{red}. Put $\Omega := \omega \,+\, 2\,dx \wedge dy$. We have

$$\begin{aligned} \mathrm{pr}_1^*\,(\,\mathrm{incl}_1^*\,\Omega_{\mathrm{red}}\,) \;&=\; \mathrm{incl}_2^*\,(\,\mathrm{pr}_2^*\,\Omega_{\mathrm{red}}\,) \;=\; \mathrm{incl}_2^*\,(\,\mathrm{incl}_3^*\,\Omega\,) \\ &=\; \mathrm{incl}_5^*\,(\,\mathrm{incl}_4^*\,\Omega\,) \;=\; \mathrm{incl}_5^*\,\omega \;=\; \mathrm{pr}_1^*\,\omega_{\mathrm{red}} \end{aligned}$$

and, because pr_1^* is injective, this proves $\mathrm{incl}_1^*\,\Omega_{\mathrm{red}} = \omega_{\mathrm{red}}$. □

Proposition 5.1.3. *Let (M, ω) be a symplectic manifold with boundary $P = \partial M$. Suppose S^1 acts smoothly and freely on P (i.e. P is a smooth principal S^1-bundle) and that $\omega|_P$ is S^1-invariant with kernel $T_p(S^1 \cdot p)$. Furthermore, assume that there is a contact form α on P such that $d\alpha = \omega|_P$ and its Reeb vector field R_α generates the S^1-action on P. We can endow a smooth manifold $X = M/\!\sim$ from Remark 3.3.6 with a symplectic structure, so that $M \backslash P$ is symplectomorphic to $X \backslash (P/S^1)$ via the projection and P/S^1 is a symplectic submanifold of X.*

Proof. Let $[p] := \mathcal{O}_p := S^1 \cdot p$ denote the orbit through $p \in P$. If we identify $T_{[p]}\,P/S^1$ with $T_pP/T_p\mathcal{O}_p$, let us define the 2-form ω_{red} on P/S^1 as

$$(\omega_{\mathrm{red}})_{[p]}(v_1 + T_p\mathcal{O}_p, v_2 + T_p\mathcal{O}_p) := \omega_p(v_1, v_2) \quad \forall\, p \in P\,,\; v_1, v_2 \in T_pP \;\;.$$

We claim that $T_p\mathcal{O}_p$ is isotropic with symplectic complement T_pP. Then the definition of ω_{red} is independent of the choice of representatives and ω_{red} is symplectic by Lemma 4.2.9 (observe that we still need the S^1-invariance of $\omega|_P$ for independence of choice of base point). For $v_1 \in T_pP$ and $v_2 \in T_p\mathcal{O}_p$ we have $\omega_p(v_1, v_2) = 0$ since $\omega|_P$ has kernel $T_p\mathcal{O}_p$. Thus, v_1 lies in the symplectic complement of $T_p\mathcal{O}_p$. Because both T_pP and $(T_p\mathcal{O}_p)^{\omega_p}$ have codimension 1, they are in fact equal. The elaborate proof, that ω_{red} is smooth, is the same argument

as in the proof of Marsden-Weinstein-Meyer 4.2.4. Thus, P/S^1 carries a reduced symplectic form. Note that in fact the above argumentation is, strictly speaking, obsolete because we will see below that $\omega|_P = \tilde{\omega}|_P$ for some symplectic form $\tilde{\omega}$ for which the assumptions of Theorem 5.1.2 (with P a zero level set) hold, so it follows directly that P/S^1 is a symplectic manifold.
Let us define the following form on $P \times \mathbb{R}$

$$\widetilde{\omega} := d((t+1)\,\alpha) = dt \wedge \alpha + (t+1)\,d\alpha \ ,$$

where we have omitted the necessary pullbacks along the projections. $\tilde{\omega}$ is symplectic since $\tilde{\omega}(v+\lambda\,\partial_t\,,\,R_\alpha) \neq 0$ for any vector $v \in TP$ and any real number $\lambda \neq 0$ and $\tilde{\omega}(v+\lambda\,R_\alpha\,,\,u) \neq 0$ for any $v \in \ker(\alpha)\,,\,\lambda \in \mathbb{R}$ and some appropriate $u \in \ker(\alpha)$ depending on v ($d\alpha$ is nondegenerate on the kernel of α).
By assumption $\widetilde{\omega}|_{P\times\{0\}} = d\alpha = \omega|_P$ and, because R_α generates the S^1-action on P, we have $R_\alpha = \widehat{\partial}_t$ the infinitesimal generated vector field on P.
Let S^1 act trivially on $\mathbb{R}$ and consider the product action on $P\times\mathbb{R}$ (i.e. $\lambda\cdot(p,t) = (\lambda\cdot p, t)$). The projection $f(p,t) := t$ onto the second factor is a moment map for this action on $P\times\mathbb{R}$ because

$$-\iota_{\widehat{\partial}_t}\widetilde{\omega} = \widetilde{\omega}(-\,,\underbrace{\widehat{\partial}_t}_{\in TP} + \underbrace{0}_{\in T\mathbb{R}}) = \underbrace{\alpha(R_\alpha)}_{=1}\,dt + (t+1)\,\underbrace{d\alpha(-,R_\alpha)}_{=0} = dt = df \ .$$

A neighborhood of P has the form $P\times[0,\infty)$. Extend ω on a neighborhood of $P\times[0,\infty)$ in $P\times\mathbb{R}$. P is a hypersurface in $P\times\mathbb{R}$, and thus a coisotropic submanifold. By the Coisotropic Embedding Theorem (cf. [Sil08, Theorem 2.9]) there is a symplectomorphism Φ from a neighborhood $P\times\{0\}$ in $(P\times\mathbb{R}\,,\,\omega)$ into a neighborhood of $P\times\{0\}$ in $(P\times\mathbb{R}\,,\,\widetilde{\omega})$ such that $\Phi|_{P\times\{0\}} = \mathrm{id}$. Consider the connected components P_n of P. By equation (3.10) we can define

$$X \cong M\backslash P \ \cup_{n\in N_+} \left(\cup_{\Phi|_{\mathcal{U}_n\backslash P_n}} (P_n\times\mathbb{R})_{[0,\infty)}\right) \ \cup_{n\in N_-} \left(\cup_{\Phi|_{\mathcal{U}_n\backslash P_n}} (P_n\times\mathbb{R})_{(-\infty,0]}\right)$$

(see Remark 3.3.6 for the notation). Check that Theorem 5.1.2 has an analog for cuts with respect to the ray $(-\infty,a]$. For each $n \in N_+\cup N_-$ consider the symplectic cut $\widetilde{\omega}_{\mathrm{cut}}$ of $\widetilde{\omega}$ on $(P_n\times\mathbb{R})_{[0,\infty)}$ resp. $(P_n\times\mathbb{R})_{(-\infty,0]}$ with respect to the projection onto the second factor. Now let us finally define the desired symplectic form Ω on X:

$$\Omega := \begin{cases} \omega & \text{on } M\backslash P \\ \widetilde{\omega}_{\mathrm{cut}} & \text{on } (P_n\times\mathbb{R})_{[0,\infty)} \\ \widetilde{\omega}_{\mathrm{cut}} & \text{on } (P_n\times\mathbb{R})_{(-\infty,0]} \end{cases}$$

Again, for clarity of notation we omit necessary pullbacks in the above definition. Ω is well-defined because $\widetilde{\omega}_{\mathrm{cut}} = \widetilde{\omega} = \omega$ on $\mathcal{U}_n\backslash P_n$ since Φ is a symplectomorphism. The rest of the Proposition is an easy observation together with an application of Theorem 5.1.2. $\square$

Remark 5.1.4. It is quite likely that the assumption, that $\omega|_P = d\alpha$ for a contact form α with $R_\alpha = \widehat{\partial}_t$, is in fact a consequence of the other conditions in the above proposition. Although some uncertainties remain, at his supervisor's suggestion the author would like to recommend, giving Theorem 3 in [BW58] a try.

5.2 Contact Cuts

We now want to consider the contact case and ask the nearby question, whether we obtain results similar to the previous section.

Lemma 5.2.1. *Let (M,α) be a strict contact manifold and $(N, d\lambda)$ a symplectic manifold with 1-form λ. Then $(M\times N, \alpha+\lambda := \mathrm{pr}_M^*\,\alpha + \mathrm{pr}_N^*\,\lambda)$ is a strict contact manifold.*

Proof. Put $\dim M =: 2m+1$, $\dim N =: 2n$.

$$
\begin{aligned}
(\alpha+\lambda)\wedge d(\alpha+\lambda)^{m+n} &= (\alpha+\lambda)\wedge \sum_{j=0}^{m+n}\binom{m+n}{j}\, d\alpha^j \wedge d\lambda^{m+n-j} \\
&= \binom{m+n}{m}\,(\alpha+\lambda)\wedge d\alpha^m \wedge d\lambda^n \\
&= \binom{m+n}{m}\,\alpha \wedge d\alpha^m \wedge d\lambda^n
\end{aligned}
$$

Since $d\lambda$ is symplectic, we have $d\lambda^n \neq 0$. For arbitrary $(p,q) \in M \times N$ choose bases $v_1,\dots,v_{2m+1} \in T_pM$ with $(\alpha\wedge d\alpha^n)_p(v_1,\dots,v_{2m+1}) \neq 0$ and $w_1,\dots,w_{2n}$ with $(d\lambda^n)_q(w_1,\dots,w_{2n}) \neq 0$. Then

$$
((\alpha+\lambda)\wedge d(\alpha+\lambda)^{m+n})_{(p,q)}(v_1,\dots,v_{2m+1},w_1,\dots,w_{2n}) \neq 0 \ .
$$

□

Remark 5.2.2. Analogously to the symplectic case, we identify the moment map f for an S^1-action on a strict contact manifold, which preserves the contact form, with a real valued function, i.e. we also call f^{∂_t} the momment map for the given action. By Lemma 4.3.8 f is S^1-invariant and, thus, f^{∂_t} is so too.

Theorem 5.2.3. *Let (M,α) be a strict contact manifold on which S^1 acts smoothly, preserving α. Let $f: M \to \mathbb{R}$ denote the corresponding moment map. Suppose S^1 acts freely on $f^{-1}(0)$. Then the cut $M_{[0,\infty)}$ is well-defined and carries a natural contact form α_{cut}. The canonical diffeomorphism $M \supseteq f^{-1}((0,\infty)) \xrightarrow{\sim} f^{-1}((0,\infty)) \subseteq M_{[0,\infty)}$ becomes a contactomorphism and the inclusion $(f^{-1}(0)/S^1 , \alpha_{\mathrm{red}}) \hookrightarrow (M_{[0,\infty)}, \alpha_{\mathrm{cut}})$ is contact.*

Proof. Again, let $\Psi(m,z) := f(m) - |z|^2$ and $\sigma(m) := (m, \sqrt{f(m)-0})$. By the preceding lemma the 1-form $\beta := \alpha + (x\,dy - y\,dx)$ on $M\times\mathbb{C}$ is contact. A simple calculation shows that it is also S^1-invariant if we consider the usual action $\lambda\cdot(m,z) := (\lambda\cdot m, \lambda^{-1}z)$. Similar to the proof of Theorem 5.1.2 we will show that Ψ is a moment map with respect to β. Since the restricted action on $f^{-1}(0)$ is free, the action on $\Psi^{-1}(0)$ is free and by Corollary 4.3.11 this implies that 0 is a regular value of f and Ψ. As moment maps bot f and Ψ are S^1-invariant, so the cut of M with respect to the ray $[0,\infty)$ is well-defined.
Now let us compute

$$
\begin{aligned}
\beta_{(m_0,z_0)}(\widehat{\theta}^{M\times\mathbb{C}}(\partial_t)_{(m_0,z_0)}) &= \beta_{(m_0,z_0)}(\widehat{\theta}^{M}(\partial_t)_{m_0}, -iz_0) \\
&= \alpha_{m_0}(\widehat{\theta}^{M}(\partial_t)_{m_0}) + x_0\, dy_{z_0}(-iz_0) - y_0\, dx_{z_0}(-iz_0) \\
&= f(m_0) - x_0^2 - y_0^2 \\
&= \Psi(m_0,z_0)
\end{aligned}
$$

Hence, Ψ is indeed the moment map for β. By the Contact Reduction Theorem of Geiges 4.3.12 the quotient $\Psi^{-1}(0)/S^1$ carries a natural contact form β_{red}. Now put $\alpha_{\mathrm{cut}} := \bar{\sigma}^*\,\beta_{\mathrm{red}}$, where $\bar{\sigma}: M_{[0,\infty)} \to \Psi^{-1}(0)/S^1$ is the descendent map as in the proof of Proposition 3.2.1.

As in the symplectic case 5.1.2, it remains to show $(\sigma|_{f^{-1}((0,\infty))})^*\,\beta = \alpha|_{f^{-1}((0,\infty))}$ and $(f^{-1}(0)/S^1 \hookrightarrow \Psi^{-1}(0)/S^1)^*\beta_{\mathrm{red}} = \alpha_{\mathrm{red}}$. The first equation is a trivial observation and the second one follows from diagram (5.1) as in the proof of Theorem 5.1.2. □

There is a contact version of Proposition 5.1.3 which can be found in [Ler01], however we will not state it here because some (hidden) details of his proof remain unclear to the author and this thesis is meant to be self-contained in the sense that every statement is proven rigorously from well-known and established principles.

5.3 Combining Symplectic and Contact Cuts

5.3.1 Cuts of Hypersurfaces

In this last section we want to examine some further properties of symplectic and contact cuts with the focus on the interplay of these concepts. For our first result we will need the definitions of Liouville vector fields and surfaces of contact type:

Definition 5.3.1. Let (M, ω) be a symplectic manifold.

(a) A ***Liouville vector field*** X is a vector field on M such that $\mathcal{L}_X \omega = \omega$.

(b) A hypersurface $\Sigma \subseteq M$ is of ***contact type*** if there is some Liouville vector field X on an open neighborhood of Σ, which is transverse to Σ.

The following proposition gives a natural contact form on a hypersurface of contact type with respect to a fixed Liouville vector field:

Proposition 5.3.2. *Let Σ be a hypersurface of contact type with transverse Liouville vector field X. Then $\alpha := (\iota_X \omega)|_\Sigma$ is a contact form on Σ.*

Proof. For arbitrary differential forms $\eta_1 \in \Omega^{k_1}(M), \eta_2 \in \Omega^{k_2}(M)$ and a vector field Y we have $\iota_Y(\eta_1 \wedge \eta_2) = \iota_Y \eta_1 \wedge \eta_2 + (-1)^{k_1} \eta_1 \wedge \iota_Y \eta_2$. Using this result, we deduce $\iota_X(\omega^k) = k\,(\iota_X \omega \wedge \omega^{k-1})$ for all $k \in \mathbb{N}$ via induction. Let $\dim(M) = 2m$, then

$$\alpha \wedge (d\alpha)^{m-1} = (\iota_X \omega \wedge (d(\iota_X \omega))^{m-1})|_\Sigma \overset{(*)}{=} (\iota_X \omega \wedge \omega^{m-1})|_\Sigma = (\frac{1}{m}\, \iota_X \omega^m)|_\Sigma \ .$$

We have used Cartan's magic formula in $(*)$. Since ω is a symplectic form, $\omega^m \neq 0$. This and the fact that X is transverse to Σ imply that $\alpha \wedge (d\alpha)^{m-1} \neq 0$. □

We now intend to examine the contact cuts of hypersurfaces of contact type and their relation to the symplectic cut of the surrounding space.

Theorem 5.3.3. *Let $f : M \to \mathbb{R}$ be a moment map for a hamiltonian S^1-action on a symplectic manifold (M, ω). Suppose Σ is an S^1-invariant connected hypersurface of contact type with transverse S^1-invariant Liouville vector field X on an open S^1-invariant neighborhood U of Σ, such that Σ is closed in U. The induced contact form $\alpha := \iota_X \omega\,|_\Sigma$ is S^1-invariant and for the corresponding moment map f_Σ we have $f_\Sigma = f|_\Sigma + c$ for a constant c.*
Assume that $c = 0$, S^1 acts freely on $f^{-1}(0)$ and X is tangent to $f^{-1}(0)$. Then 0 is a regular value of both f and f_Σ and the contact quotient Σ_{red} is a hypersurface of contact type in the symplectic quotient M_{red} with respect to the descended vector field X_{red}. Furthermore, the reduced contact form α_{red} is exactly the induced form $(\iota_{X_{\mathrm{red}}} \omega_{\mathrm{red}})|_{\Sigma_{\mathrm{red}}}$ on Σ_{red}.

Proof. That α is S^1-invariant, is a simple calculation. Since Σ is connected, for the second statement it suffices to show $d\,f_\Sigma = d\,f|_\Sigma$:

$$d\,f_\Sigma = d\,(\alpha(\,\widehat{\partial_t}\,)) = \mathcal{L}_{\widehat{\partial_t}} \alpha - \iota_{\widehat{\partial_t}}\, d\alpha = 0 - \iota_{\widehat{\partial_t}}\, d\alpha = -\iota_{\widehat{\partial_t}}\, d\alpha \ ,$$

where the Lie derivative vanishes because α is S^1-invariant. On the other hand, if $i_\Sigma : \Sigma \hookrightarrow M$ denotes the inclusion, we have

$$\begin{aligned} d\,(f|_\Sigma) &= i_\Sigma^*\, df = i_\Sigma^*(-\iota_{\widehat{\partial_t}}\, \omega) = (-\iota_{\widehat{\partial_t}}\, \omega)|_\Sigma \\ d\alpha &= d\,((\iota_X \omega)|_\Sigma) = d(\iota_X \omega)|_\Sigma = (\,\mathcal{L}_X \omega\,)|_\Sigma = \omega|_\Sigma \\ \Longrightarrow d\,f_\Sigma &= -\iota_{\widehat{\partial_t}}\, \omega|_\Sigma = (-\iota_{\widehat{\partial_t}}\, \omega)|_\Sigma = d\,(f|_\Sigma) \end{aligned}$$

Now suppose $c = 0$ and that S^1 acts freely on the zero level set. By Marsen-Weinstein-Meyer 4.2.4 resp. Geiges 4.3.12, 0 is a regular value of f resp. f_Σ and both $f^{-1}(0)$ and $f_\Sigma^{-1}(0)$ are closed submanifolds in M resp. Σ. Thus, $f_\Sigma^{-1}(0) = f^{-1}(0) \cap \Sigma$ is a submanifold in $f^{-1}(0)$. Since Σ is closed in U, $f_\Sigma^{-1}(0)$ is closed in $f^{-1}(0) \cap U$, which itself is open in $f^{-1}(0)$. By Proposition A.2.7 $\Sigma_{\text{red}} = f_\Sigma^{-1}(0)/S^1$ is a submanifold of $(f^{-1}(0) \cap U)/S^1$, and $(f^{-1}(0) \cap U)/S^1$ is a submanifold of $M_{\text{red}} = f^{-1}(0)/S^1$, so Σ_{red} is a submanifold of M_{red}. Now, additionally assume that $X \in T f^{-1}(0)$ on $f^{-1}(0) \cap U$. Then X descends to a vector field X_{red} on an open subset of the symplectic quotient M_{red}. Since the projections $\text{pr} : M \twoheadrightarrow M_{\text{red}}$, $\text{pr}_\Sigma : \Sigma \twoheadrightarrow \Sigma_{\text{red}}$ are surjective submersions, their pullbacks are injective.

$$
\begin{aligned}
&\text{pr}^* \mathcal{L}_{X_{\text{red}}}\omega_{\text{red}} = \text{pr}^* d\,\iota_{X_{\text{red}}}\omega_{\text{red}} = d\,\text{pr}^*\,\iota_{X_{\text{red}}}\omega_{\text{red}} = d\,\iota_X\omega = \omega = \text{pr}^*\,\omega_{\text{red}} \\
\Longrightarrow\ & X_{\text{red}} \text{ is Liouville vector field for } \Sigma_{\text{red}} \\
&\text{pr}_\Sigma^*\,(\iota_{X_{\text{red}}}\,\omega_{\text{red}})|_{\Sigma_{\text{red}}} = \iota_X\omega\,|_\Sigma = \alpha = \text{pr}_\Sigma^*\,\alpha_{\text{red}} \\
\Longrightarrow\ & (\iota_{X_{\text{red}}}\,\omega_{\text{red}})|_{\Sigma_{\text{red}}} = \alpha_{\text{red}}
\end{aligned}
$$

It remains to prove that X_{red} is transverse to Σ_{red}. Assume by contradiction $X_{\text{red}} \in T\Sigma_{\text{red}}$ at some point $[p] \in \Sigma_{\text{red}}$. Then $d\,\text{pr} \cdot X = X_{\text{red}} = d\,\text{pr}_\Sigma \cdot Y$ for some $Y \in T\Sigma$ (we omit the necessary base point $[p]$). Thus, $X - Y \in \ker d\,\text{pr} = T_p(S^1 \cdot p) \subseteq T_p\Sigma$, which would imply $X \in T\Sigma$. ☐

In the opinion of the author, Lerman's version of this theorem (Proposition 2.17 in [Ler01]) forgets the essential condition that X is tangent to the zero level set. This however leads to a subsequent error in Corollary 2.18 since X cannot descend to the cut in general without further, very particular, assumptions. We try to fix this in our next corollary, nonetheless the statement loses much of its generality.

Corollary 5.3.4. *Given the situation of Theorem 5.3.3, together with the assumptions* $c = 0$ *and a free* S^1*-action on* $f^{-1}(0)$*, but with* $df \cdot X = f$ *on* U *instead of just* $df \cdot X = 0$ *on* $f^{-1}(0)$*. The cut* $\Sigma_{[0,\infty)}$ *is a hypersurface of contact type in the cut* $M_{[0,\infty)}$ *and we have* $\alpha_{\text{cut}} = (\iota_{\overline{X}}\,\omega_{\text{cut}})|_{\Sigma_{[0,\infty)}}$ *for a natural descended vector field* $\overline{X}$ *of* X *(defined in the proof below).*

Proof. Without loss of generality we may assume $U = M$. As it is our usual convention, let $\Psi : M \times \mathbb{C} \to \mathbb{R}$, $(m, z) \mapsto f(m) - |z|^2$ be the corresponding map to the moment map f and Ψ_Σ be the corresponding map to the moment map f_Σ, i.e. $\Psi_\Sigma = \Psi|_{\Sigma\times\mathbb{C}}$. Indeed, $\Sigma_{[0,\infty)}$ is a subset of $M_{[0,\infty)}$. Recall, how the smooth structure on cuts has been defined: It therefore suffices to prove that the inclusion $\Psi_\Sigma^{-1}(0)/S^1 \hookrightarrow \Psi^{-1}(0)/S^1$ is a hypersurface of contact type. Set $Y := \frac{1}{2}\,(x\,\partial_x + y\,\partial_y) \in \mathfrak{X}(\mathbb{C})$. Then we get $\iota_Y(2\,dx \wedge dy) = x\,dy - y\,dx$ and consequently $d\,(\iota_Y(2\,dx \wedge dy)) = 2\,dx \wedge dy$, i.e. Y is a Liouville vector field on $(\mathbb{C}\,,\,2\,dx \wedge dy)$. Hence, $X + Y$ is a Liouville vector field on $(M \times \mathbb{C}\,,\,\omega + 2\,dx \wedge dy)$, transverse to the hypersurface of contact type $\Sigma \times \mathbb{C}$. X and Y are S^1-invariant, therefore $X + Y$ is so too. Furthermore, observe that $X + Y$ is tangent to $\Psi^{-1}(0)$ because $d_{(m,z)}\Psi \cdot (X + Y)_{(m,z)} = d_m f \cdot X_m - |z|^2 = f(m) - |z|^2 = 0$, so we can apply Theorem 5.3.3 with moment map Ψ, hypersurface $\Sigma \times \mathbb{C}$ and vector field $X + Y$. Finally, let $\overline{X}$ be the descended vector field $(X + Y)_{\text{red}}$ on $\Psi^{-1}(0)/S^1$. Then $\alpha_{\text{cut}} = (\iota_{\overline{X}}\,\omega_{\text{cut}})|_{\Sigma_{[0,\infty)}}$ follows from $\iota_{X+Y}\,(\omega + 2\,dx \wedge dy)\,|_{\Sigma\times\mathbb{C}} = \alpha + (x\,dy - y\,dx)$ and Theorem 5.3.3. ☐

5.3.2 Cuts of Symplectizations

Our second, much more useful result in this chapter shows that the operations of cutting and forming the symplectization of a contact manifold commute. Let us first recapitulate the required definitions:

Given a strict contact manifold (M, α) we can consider its ***symplectization*** $(M \times \mathbb{R}, d\,(e^t\alpha))$, renouncing the required pullbacks of projections. Obviously, we have

$$
d\,(e^t\alpha) = e^t\,\big(dt \wedge \alpha + d\alpha\big) \tag{5.2}
$$

If $\dim M = 2m-1$, then $(d\,(e^t\alpha))^m = m\,e^{mt}\,dt \wedge \alpha \wedge (d\alpha)^{m-1} \neq 0$, so $\omega := d\,(e^t\alpha)$ is a symplectic form on $M \times \mathbb{R}$. Because $d\,\iota_{\partial_t}\omega = \omega$ holds, ∂_t is a Liouville vector field, transverse to the hypersurfaces of contact type $M \times \{t_0\}$. For $t_0 = 0$ the induced contact form on $M \times \{t_0\}$ is the original contact form α.

If additionally S^1 acts on M, preserving α, with corresponding moment map μ, then $\nu : M \times \mathbb{R} \to \mathbb{R}\,,\ \nu(m,t) := e^t\,\mu(m)$ is a symplectic moment map with respect to $\omega = d\,(e^t\alpha)$ and the canonical action $\lambda \cdot (m,t) := (\lambda \cdot m, t)$ on $M \times \mathbb{R}$ because for $(m,t) \in M \times \mathbb{R}$ we have

$$\begin{aligned}
&\big(-\iota_{\widehat{\partial_t}}\,\omega\,\big)_{(m,t)} \\
&= \omega_{(m,t)}\big(-, (\widehat{\partial_t}^{M}, 0)\big) \\
&\overset{(5.2)}{=} e^t\left[\,\alpha_m\big(\widehat{\partial_t}^{M}\big)\,dt + \,d\alpha_m\big(-, \widehat{\partial_t}^{M}\big)\,\right] \\
&= e^t\left[\,\mu(m)\,dt - \big(\mathcal{L}_{\widehat{\partial_t}^{M}}\,\alpha\big)_m + d\big(\iota_{\widehat{\partial_t}^{M}}\,\alpha\big)_m\,\right] \\
&= e^t\left[\,\mu(m)\,dt - 0 + d_m\mu\right] \\
&= d_{(m,t)}\nu\ .
\end{aligned}$$

Symplectization and reduction commute, as the following proposition states:

Proposition 5.3.5. *Suppose (M,α) is a strict contact manifold with an S^1-action, preserving α. Assume the action is free on the zero level set of the corresponding moment map μ. Then the map*

$$\begin{aligned}
I : \big(\,(M \times \mathbb{R})_{\mathrm{red}}\,,\,(d\,(e^t\alpha))_{\mathrm{red}}\,\big) &\xrightarrow{\cong} \big(M_{\mathrm{red}} \times \mathbb{R}\,,\,d\,(e^t\alpha_{\mathrm{red}})\big) \\
S^1 \cdot (m,t) &\longmapsto (S^1 \cdot m, t)
\end{aligned}$$

is a symplectomorphism of the reduced symplectization of M onto the symplectized reduction of M.

Proof. First of all, I is well-defined because $\mu(m) = 0$ if $(m,t) \in \nu^{-1}(0)$ and the above definition is independent of choice of representatives. The inverse function $(S^1 \cdot m, t) \mapsto S^1 \cdot (m,t)$ is also obviously well-defined. In addition, the maps

$$\begin{aligned}
\mu^{-1}(0) \times \mathbb{R} &\longrightarrow M_{\mathrm{red}} \times \mathbb{R}\,, & (m,t) &\mapsto (S^1 \cdot m, t) \\
\mu^{-1}(0) \times \mathbb{R} &\longrightarrow (M \times \mathbb{R})_{\mathrm{red}}\,, & (m,t) &\mapsto S^1 \cdot (m,t)
\end{aligned}$$

certainly are smooth. Hence, I is a diffeomorphism. Certainly, every symplectic diffeomorphism, i.e. a diffeomorphism such that its differential is linear symplectic, is already a symplectomorphism, so it suffices to prove this for I. Let $(m,t) \in \mu^{-1}(0) \times \mathbb{R}\,,\ \bar{v}_1, \bar{v}_2 \in T_{S^1\cdot(m,t)}(M \times \mathbb{R})_{\mathrm{red}}$ be arbitrary. We would like to ensure that

$$I^*\big(\,d\,(e^t\alpha_{\mathrm{red}})\,\big)_{S^1\cdot(m,t)}\,(\bar{v}_1,\bar{v}_2) = \big[\,(\,d\,(e^t\alpha)\,)_{\mathrm{red}}\,\big]_{S^1\cdot(m,t)}\,(\bar{v}_1,\bar{v}_2)$$

holds. Write $\mathrm{pr}_1 : \mu^{-1}(0) \twoheadrightarrow M_{\mathrm{red}}$ and $\mathrm{pr}_2 : \mu^{-1}(0) \times \mathbb{R} \twoheadrightarrow (M \times \mathbb{R})_{\mathrm{red}}$ for the projections and notice that $(\mathrm{pr}_1 \times \mathrm{id}_{\mathbb{R}}) = I \circ \mathrm{pr}_2$. Choose $v_i\,,\ i = 1,2\,,$ with $d_{(m,t)}\mathrm{pr}_2 \cdot v_i = \bar{v}_i$ and $x_i \in T_m\,\mu^{-1}(0)\,,\ \lambda_i \in \mathbb{R}$ such that $v_i = x_i + \lambda_i\partial_t$. Put $\bar{x}_i := \mathrm{pr}_1 \cdot x_i$. Then

$$\begin{aligned}
&I^*\big(\,d\,(e^t\alpha_{\mathrm{red}})\,\big)_{S^1\cdot(m,t)}\,(\bar{v}_1,\bar{v}_2) \\
=\ &\big[\,d\,(e^t\alpha_{\mathrm{red}})\big]_{(S^1\cdot m,t)}\,\big(\,\bar{x}_1 + \lambda_1\partial_t\,,\,\bar{x}_2 + \lambda_2\partial_t\,\big) \\
=\ &e^t\left[\,\lambda_1\,(\alpha_{\mathrm{red}})_{S^1\cdot m}(\bar{x}_2) - \lambda_2\,(\alpha_{\mathrm{red}})_{S^1\cdot m}(\bar{x}_1) + (d\alpha_{\mathrm{red}})_{S^1\cdot m}(\bar{x}_1,\bar{x}_2)\,\right] \\
=\ &e^t\left[\,\lambda_1\,\alpha_m(x_2) - \lambda_2\,\alpha_m(x_1) + (d\alpha)_m(x_1,x_2)\,\right] \\
=\ &d\,(e^t\alpha)_{(m,t)}(v_1,v_2) \\
=\ &\big[\,(\,d\,(e^t\alpha)\,)_{\mathrm{red}}\,\big]_{S^1\cdot(m,t)}\,(\bar{v}_1,\bar{v}_2)\ .
\end{aligned}$$

□

For the second main theorem of this section we will need the following preliminary lemma:

Lemma 5.3.6. *Let* $F : (M_1, \alpha_1) \xrightarrow{\sim} (M_2, \alpha_2)$ *be a contactomorphism. Then* $F \times \mathrm{id}_{\mathbb{R}} : (M_1 \times \mathbb{R},\, d\,(e^t \alpha_1)\,) \xrightarrow{\sim} (M_2 \times \mathbb{R},\, d\,(e^t \alpha_2)\,)$ *is a symplectomorphism.*

Proof. This is a straightforward computation. □

Now we can finally state the last theorem, which simply says that cutting the symplectization of a manifold is the same as constructing the symplectization of the cut.

Theorem 5.3.7. *Let* (M, α) *be a strict contact manifold on which* S^1 *acts under preservation of* α *with moment map* μ*, such that the action is free on the zero level set. Then the symplectization of the cut* $M_{[0,\infty)} \times \mathbb{R}$ *is symplectomorphic to the cut of the symplectization* $(M \times \mathbb{R})_{[0,\infty)}$ *along its moment map* $\nu(m,t) := e^t\, \mu(m)$*.*

Proof. Clearly, the restricted action on $\nu^{-1}(0)$ is free, so we can form the cut of $M \times \mathbb{R}$. Let $\Psi : M \times \mathbb{C} \longrightarrow \mathbb{R}\,,\ \Psi(m,z) := \mu(m) - |z|^2$ denote the natural moment map on $M \times \mathbb{C}$. By definition (cf. proof of Theorem 5.2.3) we have a contactomorphism

$$\big(M_{[0,\infty)}\,,\,\alpha_{\mathrm{cut}}\big) \underset{\text{contact}}{\cong} \big(\,(M \times \mathbb{C})_{\mathrm{red}} = \Psi^{-1}(0)/S^1\,,\,(\alpha + x\,dy - y\,dx)_{\mathrm{red}}\big)$$

and by the preliminary lemma and Proposition 5.3.5

$$\begin{aligned}\big(M_{[0,\infty)} \times \mathbb{R}\,,\, d\,(e^t \alpha_{\mathrm{cut}})\,\big) &\underset{\text{sympl.}}{\cong} \big(\,(M \times \mathbb{C})_{\mathrm{red}} \times \mathbb{R}\,,\, d\,(e^t\,(\alpha + x\,dy - y\,dx)_{\mathrm{red}}\,)\,\big)\\ &\underset{\text{sympl.}}{\cong} \big(\,(M \times \mathbb{C} \times \mathbb{R})_{\mathrm{red}}\,,\,(\,d\,(e^t\,(\alpha + x\,dy - y\,dx)\,)\,)_{\mathrm{red}}\,\big)\end{aligned}$$

Again, by definition (cf. proof of Theorem 5.1.2) there is a symplectomorphism

$$\big(\,(M \times \mathbb{R})_{[0,\infty)}\,,\,(d\,(e^t \alpha))_{\mathrm{cut}}\,\big) \underset{\text{sympl.}}{\cong} \big(\,(M \times \mathbb{R} \times \mathbb{C})_{\mathrm{red}}\,,\,(d\,(e^t \alpha) + 2\,dx \wedge dy)_{\mathrm{red}}\,\big)$$

Thus, to prove the theorem, it suffices to show

$$\begin{aligned}&\big(\,(M \times \mathbb{C} \times \mathbb{R})_{\mathrm{red}}\,,\,(\,d\,(e^t\,(\alpha + x\,dy - y\,dx)\,)\,)_{\mathrm{red}}\,\big)\\ \underset{\text{sympl.}}{\cong}\ &\big(\,(M \times \mathbb{R} \times \mathbb{C})_{\mathrm{red}}\,,\,(d\,(e^t \alpha) + 2\,dx \wedge dy)_{\mathrm{red}}\,\big)\ .\end{aligned} \tag{5.3}$$

The moment map on $M \times \mathbb{C} \times \mathbb{R}$ is given by

$$(m,z,t) \longmapsto e^t\,\Psi(m,z) = e^t\,(\,\mu(m) - |z|^2\,)$$

and on $M \times \mathbb{R} \times \mathbb{C}$ the moment map is

$$(m,t,z) \longmapsto (e^t\,\mu(m)) - |z|^2\ .$$

Hence,

$$\begin{aligned}(M \times \mathbb{C} \times \mathbb{R})_{\mathrm{red}} &= (\Psi^{-1}(0) \times \mathbb{R})/S^1 = \big\{\,(m,z,t) \in M \times \mathbb{C} \times \mathbb{R} \mid \mu(m) = |z|^2\,\big\}\,/S^1\\ (M \times \mathbb{R} \times \mathbb{C})_{\mathrm{red}} &= \big\{\,(m,t,z) \in M \times \mathbb{R} \times \mathbb{C} \mid e^t\,\mu(m) = |z|^2\,\big\}\,/S^1\ .\end{aligned}$$

Define the diffeomorphism

$$H : M \times \mathbb{C} \times \mathbb{R} \xrightarrow{\sim} M \times \mathbb{R} \times \mathbb{C}\,, \quad (m,z,t) \longmapsto (m,t,e^{t/2}\,z)\ .$$

Observe that H maps points (m,z,t) onto points of the form (n,s,w) with $e^s\,\mu(n) = |w|^2$ if and only if $\mu(m) = |z|^2$. Additionally, if λ is in S^1, then

$$H(\lambda \cdot (m,z,t)) = (\,\lambda \cdot m\,,\,t\,,\,e^{t/2}\,\lambda^{-1}\,z\,) = \lambda \cdot H(m,z,t)\ ,$$

so H is S^1-equivariant. Thus, H descends to a diffeomorphism

$$\bar{H} : (M \times \mathbb{C} \times \mathbb{R})_{\text{red}} \longrightarrow (M \times \mathbb{R} \times \mathbb{C})_{\text{red}} \ .$$

Put $\omega_1 := d\,(e^t\,(\alpha + x\,dy - y\,dx)\,)$ and $\omega_2 := d\,(e^t\alpha) + 2\,dx \wedge dy$. If we can show $H^*\,\omega_2 = \omega_1$, then H is a symplectomorphism, which implies that (5.3) holds via the symplectomorphism $\bar{H}$.

Let us compute the differential of H:

If $v \in T_mM$, $w \in T_z\mathbb{C} \cong \mathbb{C}$, $z = x + iy$, $\lambda \in \mathbb{R}$, then

$$d_{(m,z,t)}H \cdot (v + w + \lambda\,\partial_t) = \underbrace{v}_{\in TM} + \underbrace{\lambda\,\partial_t}_{\in T\mathbb{R}} + \underbrace{e^{t/2}\,w \,+\, \frac{\lambda}{2}\,e^{t/2}\,z}_{\in T\mathbb{C} \cong \mathbb{C}}$$

or somewhat more informally

$$d_{(m,z,t)}H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & e^{t/2} & \frac{1}{2}\,e^{t/2}\,z \end{pmatrix} \quad .$$

Suppose $v_j \in T_mM$, $\hat{z}_j \in T_z\mathbb{C} \cong \mathbb{C}$ and $\lambda_j \in \mathbb{R}$ for $j = 1, 2$ are arbitrary. To prove $(H^*\omega_2)_{(m,z,t)} = (\omega_1)_{(m,t,z)}$, it suffices to check the following equations:

$$(H^*\omega_2)_{(m,z,t)}\,(v_1, v_2) = (\omega_1)_{(m,z,t)}\,(v_1, v_2) \tag{1}$$
$$(H^*\omega_2)_{(m,z,t)}\,(\hat{z}_1, \hat{z}_2) = (\omega_1)_{(m,z,t)}\,(\hat{z}_1, \hat{z}_2) \tag{2}$$
$$(H^*\omega_2)_{(m,z,t)}\,(\lambda_1\,\partial_t, \lambda_2\,\partial_t) = (\omega_1)_{(m,z,t)}\,(\lambda_1\,\partial_t, \lambda_2\,\partial_t) \tag{3}$$
$$(H^*\omega_2)_{(m,z,t)}\,(v_1, \hat{z}_2) = (\omega_1)_{(m,z,t)}\,(v_1, \hat{z}_2) \tag{4}$$
$$(H^*\omega_2)_{(m,z,t)}\,(v_1, \lambda_2\,\partial_t) = (\omega_1)_{(m,z,t)}\,(v_1, \lambda_2\,\partial_t) \tag{5}$$
$$(H^*\omega_2)_{(m,z,t)}\,(\hat{z}_1, \lambda_2\,\partial_t) = (\omega_1)_{(m,z,t)}\,(\hat{z}_1, \lambda_2\,\partial_t) \tag{6}$$

Equations (1) and (2) are left to the reader. For (3) and (4) notice that each term is zero. For demonstration purpose we present the calculations for (5) and (6):

Proof of equation (5)*:* Put $v := v_1$ and $\lambda := \lambda_2$.

$$\begin{aligned} (H^*\omega_2)_{(m,z,t)}\,(v\,,\,\lambda\,\partial_t) &= (\omega_2)_{H(m,z,t)}\,(v\,,\lambda\,(\partial_t + \frac{1}{2}\,e^{t/2}\,z)\,) \\ &= d\,(e^t\alpha)_{(m,t)}\,(v\,,\,\lambda\,\partial_t) \\ &= -\,\lambda\;e^t\;\alpha_m(v) \\ (\omega_1)_{(m,z,t)}\,(v\,,\,\lambda\,\partial_t) &= e^t\,(dt \wedge \alpha)_{(m,z,t)}\,(v\,,\,\lambda\,\partial_t) \\ &= -\,\lambda\;e^t\;\alpha_m(v) \end{aligned}$$

Proof of equation (6)*:* Put $\hat{z} := \hat{x} + i\hat{y} := \hat{z}_1$ and $\lambda := \lambda_2$.

$$\begin{aligned} (H^*\omega_2)_{(m,z,t)}\,(\hat{z}\,,\,\lambda\,\partial_t) &= (2\,dx \wedge dy)_{e^{t/2}\,z}\;(e^{t/2}\,\hat{z}\,,\,\lambda\,(\partial_t + \frac{1}{2}\,e^{t/2}\,z)\,) \\ &= e^t\,(dx \wedge dy)_{e^{t/2}\,z}\;(\hat{z}\,,\,\lambda\,z) \\ &= \lambda\,e^t\;\det\begin{pmatrix} \hat{x} & x \\ \hat{y} & y \end{pmatrix} \\ &= \lambda\,e^t\,(\hat{x}y - \hat{y}x) \\ (\omega_1)_{(m,z,t)}\,(\hat{z}\,,\,\lambda\,\partial_t) &= e^t\,(dt \wedge (x\,dy - y\,dx)\,)_{(m,z,t)}\,(\hat{z}\,,\,\lambda\,\partial_t) \\ &= -\,\lambda\,e^t\;(\,x\,dy_z(\hat{z}) - y\,dx_z(\hat{z})\,) \\ &= \lambda\,e^t\,(\hat{x}y - \hat{y}x) \end{aligned}$$

$\square$

A Appendix

A.1 Regarding Chapter 2

A ***quotient map*** $q : X \to Y$ between the topological spaces X and Y is a surjective map such that the given topology on Y coincides with the quotient topology induced by q. Obviously, if q is a quotient map, it is continuous.

Lemma A.1.1 (see [Lee13, Exercise A.36])**.** *Let $q : X \to Y$ be an open quotient map. Then*

$$Y \text{ is Hausdorff} \iff \mathcal{R} := \{(x_1, x_2) \mid q(x_1) = q(x_2)\} \text{ is closed in } X \times X$$

Proof.
"$\Rightarrow$": Suppose that Y is Hausdorff. Let $(x_1, x_2) \in (X \times X)\backslash\mathcal{R} = \mathcal{R}^C$ be arbitrary. Since Y is Hausdorff, there exist U_1 , U_2 open neighborhoods of $q(x_1)$, $q(x_2)$ respectively with $U_1 \cap U_2 = \varnothing$ (here we have used $q(x_1) \neq q(x_2)$). Then $q^{-1}(U_1) \times q^{-1}(U_2)$ is an open neighborhood of (x_1, x_2) contained in $\mathcal{R}^C$ (for $(\hat{x}_1, \hat{x}_2) \in q^{-1}(U_1) \times q^{-1}(U_2)$ we have $q(\hat{x}_i) \in U_i$, $i = 1, 2$ and since U_1 and U_2 are disjoint, $q(\hat{x}_1) \neq q(\hat{x}_2)$). Because $(x_1, x_2) \in \mathcal{R}^C$ was arbitrary, $\mathcal{R}^C$ is open.
"$\Leftarrow$": Suppose $y_1 \neq y_2 \in Y$. As quotient map q is surjective, so we can find x_1 , $x_2 \in X$ such that $q(x_i) = y_i$, $i = 1, 2$. Thus, (x_1, x_2) is in $\mathcal{R}^C$ which is open by assumption. Therefore, there exist V_1 , V_2 open in X with $(x_1, x_2) \in V_1 \times V_2 \subseteq \mathcal{R}^C$. Let $U_i := q(V_i)$, then U_i are open (since q is open) neighborhoods of y_i and $U_1 \cap U_2 = \varnothing$. □

Lemma A.1.2 (see [Lee13, Theorem A.57])**.** *Let $F : X \to Y$ be a continuous and proper map between two topological spaces. Suppose further that Y is a locally compact Hausdorff space, i.e. a Hausdorff space such that every point has a compact neighborhood. Then F is closed.*

Proof. Let C be a closed subset in X. Suppose $(c_\alpha)_{\alpha\in A}$ is a net in C and $F(c_\alpha) \to y \in Y$. We want to show that $y \in F(C)$. Then $F(C)$ is closed in Y since every limit point of some net in $F(C)$ is itself in $F(C)$. Choose a compact neighborhood K of y. There exists a convergent subnet $(F(c_\alpha))_{\alpha\in B}$ with $F(c_\alpha) \in K \quad \forall\alpha \in B \subseteq A$ and limit y in K. Since F is proper, $F^{-1}(K)$ is compact and therefore also $C \cap F^{-1}(K)$ is compact (it is closed in the compact set $F^{-1}(K)$). Because F is continuous, $F(C \cap F^{-1}(K)) = F(C) \cap K$ is compact and hence closed in Y (Y is Hausdorff). Since $(F(c_\alpha))_{\alpha\in B}$ is a net in $F(C) \cap K$ and $F(c_\alpha) \to y$, $\alpha \in B$, it follows that $y \in F(C) \cap K \subseteq F(C)$. □

Lemma A.1.3. *Let X be a first countable Hausdorff space and $(p_n)_{n\in\mathbb{N}}$ a sequence in X. Suppose $(p_n)_{n\in\mathbb{N}}$ has a convergent subnet with limit point $p \in X$. Then it also has a convergent subsequence with limit p.*

Proof. Let $(p_{h(i)})_{i\in I}$ be a convergent subnet with limit p, where $(I, \succeq)$ is a directed set and the map $h : I \to \mathbb{N}$ is monotone and final, i.e. has cofinal image.
First suppose $\#\{n \in \mathbb{N} \mid p_n = p\} = \infty$. Then, obviously, there exists a convergent subsequence of (p_n) with limit p.
Now assume, there is an $n_0 \in \mathbb{N}$, such that $p_n \neq p$ for $n \geq n_0$. Since $h(I)$ is cofinal in $\mathbb{N}$, we can choose $i_0 \in I$ with $h(i_0) \geq n_0$. Then $(p_{h(i)})_{i\in J}$, where $h|_J : J := \{i \in I \mid i \succeq i_0\} \to \mathbb{N}$,

is a subnet of $(p_n)_{n\in\mathbb{N}}$ with limit p and $p_{h(i)} \neq p \quad \forall i \in J$. So, without loss of generality we may assume that $p_{h(i)} \neq p$ for all $i \in I$. Since X is first countable, we can choose a neighborhood basis $(U_n)_{n\in\mathbb{N}}$ of p such that $U_{n+1} \subseteq U_n \ \forall n \in \mathbb{N}$.
Choose i_1, such that $p_{h(i_1)} \in U_1$. Now choose $(i_n)_{n\in\mathbb{N}}$ inductively: Suppose we have chosen $i_1, \dots, i_n \in I$ with $p_{h(i_k)} \in U_k$ and $i_k \succeq i_{k-1}$ for $2 \leq k \leq n$. Then there exists $i \in I$ with $p_{h(j)} \in U_{n+1} \quad \forall j \succeq i$. Since I is a directed set, we can choose $i_{n+1} \in I$ with $i_{n+1} \succeq i, i_n$.
Because we have $p_{h(i_n)} \in U_n \ \forall n \in \mathbb{N}$ and the fact that $(U_n)_n$ is a neighborhood basis of p, it follows that $p_{h(i_n)} \to p$. By assumption X is Hausdorff and $p_{h(i)} \neq p$, so for every $n \in \mathbb{N}$ there is some open neighborhood W_n of p with $W_n \cap \{p_{h(i_1)}, \dots, p_{h(i_n)}\} = \varnothing$. Thus, $\#\{h(i_n) \mid n \in \mathbb{N}\} = \infty$. In addition, since h is monotone, $h(i_n) \leq h(i_{n+1})$. Consider $j : \mathbb{N} \xrightarrow{\sim} \{h(i_n) \mid n \in \mathbb{N}\}$ the enumeration of $\{h(i_n) \mid n \in \mathbb{N}\}$ in ascending order. Then $(p_{j(n)})_{n\in\mathbb{N}}$ is a subsequence of $(p_n)_{n\in\mathbb{N}}$ with limit p. □

Lemma A.1.4. *Let $\theta : G \times X \to X$ be a continuous action of a topological group G on the Hausdorff space X. For any two compact sets K_1, $K_2 \subseteq X$, the set*

$$((K_1, K_2)) := \{g \in G \mid (g \cdot K_1) \cap K_2 \neq \varnothing\}$$

is closed in G.

Proof. Suppose (g_α) is an arbitrary net in $((K_1, K_2))$ with limit point $g \in G$. Closeness follows if we prove $g \in ((K_1, K_2))$. Choose a net (p_α) in K_1 with $g_\alpha \cdot p_\alpha \in K_2$. By compactness of K_i we can take a convergent subnet of (p_α), say (p_{α_β}), and a convergent subnet of $(g_{\alpha_\beta} \cdot p_{\alpha_\beta})$, so we may instead assume (p_α) and $(g_\alpha \cdot p_\alpha)$ converge (and $g_\alpha \to g$ holds still). Say $p_\alpha \to p \in K_1$ and $g_\alpha \cdot p_\alpha \to q \in K_2$. Since θ is continuous, $g_\alpha \cdot p_\alpha \to g \cdot p$ and, because limits in Hausdorff spaces are unique, it follows that $g \cdot p = q \in K_2$. Thus, $g \in ((K_1, K_2))$. □

Suppose we have G-spaces X and Y.
A map $f : X \to Y$ is called ***(G-)equivariant*** if

$$f(g \cdot x) = g \cdot f(x) \quad \forall\, g \in G\,,\, x \in X.$$

If the G-action on Y is the trivial one, we also say that f is ***G-invariant***.

Theorem A.1.5 (Equivariant Rank Theorem)**.** *Given a smooth G-equivariant map $f : M \to N$ between the transitive smooth G-space M and the smooth G-space N, then f has constant rank (i.e.* $\mathrm{rk}(d_p f) = const.$*).*

Proof. Let p, $q \in M$ be arbitrary. Since the G-action on M is transitive, there is some $g \in G$ with $g \cdot p = q$. Let θ_M, θ_N denote the actions on M resp. N. Then $(\theta_M)_g$ and $(\theta_N)_g$ are diffeomorphisms. Since f is G-equivariant, we have the following commuting diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "(\theta_M)_g"'] & N \arrow[d, "(\theta_N)_g"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Differentiation at p and the chain rule give the commutative diagram below:

$$\begin{tikzcd} T_pM \arrow[r, "d_pf"] \arrow[d, "d_p(\theta_M)_g"'] & T_{f(p)}N \arrow[d, "d_{f(p)}(\theta_N)_g"] \\ T_qM \arrow[r, "d_qf"'] & T_{f(q)}N \end{tikzcd}$$

Since the differentials of $(\theta_M)_g$ respectively $(\theta_M)_g$ are isomorphisms, we conclude $\mathrm{rk}(d_pf) = \mathrm{rk}(d_qf)$. □

The following two lemmata are standard results in set theoretic topology and can be found in any textbook.

Lemma A.1.6. *Let X and Y be topological spaces, $x \in X$ and $K \subseteq Y$ a compact subset. Suppose U is open in $X \times Y$ and $\{x\} \times K \subseteq U \subseteq X \times Y$. Then there are open subsets $V_x \subseteq X$, $V_K \subseteq Y$ such that $\{x\} \times K \subseteq V_x \times V_K \subseteq U$.*

Proof. For each $k \in K$ choose open $W_k \subseteq X$ and $V_k \subseteq Y$ such that $(x,k) \in W_k \times V_k \subseteq U$. This is possible because a basis for the product topology is given by products of open sets. Since K is compact, we can find $k_1, \ldots, k_n \in K$ with $K \subseteq \bigcup_{i=1}^n V_{k_i} =: V_K$. Let $V_x := \bigcap_{i=1}^n W_{k_i}$ which contains x and is open as finite intersection. Now, if $(a,b) \in V_x \times V_K$, then $b \in V_{k_i}$ for some $i = 1, \ldots, n$. Then $a \in W_{k_i}$ and thus $(a,b) \in W_{k_i} \times V_{k_i} \subseteq U$. □

Note that the symmetric statement with a compact subset in the first component also holds, for example by the next lemma:

Lemma A.1.7. *Let X and Y be topological spaces and $K_X \subseteq X$, $K_Y \subseteq Y$ be compact subspaces. Suppose U is open in $X \times Y$ and $K_X \times K_Y \subseteq U \subseteq X \times Y$. Then there are open subsets $V_X \subseteq X$ and $V_Y \subseteq Y$ such that $K_X \times K_Y \subseteq V_X \times V_Y \subseteq U$.*

Proof. For every $x \in K_X$ choose open sets $W_1(x) \subseteq X$ and $W_2(x) \subseteq Y$ such that $\{x\} \times K_Y \subseteq W_1(x) \times W_2(x) \subseteq U$. By compactness of K_X we can find $x_1, \ldots, x_n \in K_X$ such that the $W_1(x_i)$ cover K_X. Then put $V_X := \bigcup_{i=1}^n W_1(x_i)$ and $V_Y := \bigcap_{i=1}^n W_2(x_i)$. □

Of course, Lemma A.1.7 is a generalization of Lemma A.1.6 since points are always compact.

Corollary A.1.8. *If $X_1, \ldots, X_n$ are topological spaces and $K_i \subseteq X_i$, $i = 1, \ldots, n$, are compact subspaces and $K_1 \times \ldots \times K_n \subseteq U \subseteq X_1 \times \ldots \times X_n$ is open, then there are open $V_i \subseteq X_i$, $i = 1, \ldots, n$, such that $K_1 \times \ldots \times K_n \subseteq V_1 \times \ldots \times V_n \subseteq U$.*

Proof. We may assume that all K_i are nonempty. Proof by induction. $n = 1$ is trivial. Assume the statement holds for n. Then $= K_1 \times \ldots \times K_n$ is compact as product of compact sets. By Lemma A.1.7 we can choose $W \subseteq X_1 \times \ldots \times X_n$ and $V_{n+1} \subseteq X_{n+1}$ open such that $K_1 \times \ldots \times K_n \times K_{n+1} \subseteq W \times V_{n+1} \subseteq U$. By induction hypothesis there are open $V_i \subseteq X_i$ for $i = 1, \ldots, n$ with $K_1 \times \ldots \times K_n \subseteq V_1 \times \ldots \times V_n \subseteq W$. Then $K_1 \times \ldots \times K_n \times K_{n+1} \subseteq V_1 \times \ldots \times V_n \times V_{n+1} \subseteq U$. □

Corollary A.1.9. *Let K be a compact topological space and X a topological space with point $x \in X$. Assume that $K \times \{y\}$ is contained in the open subset $U \subseteq K \times X$. Then there is an open neighborhood V of x such that $K \times \{x\} \subseteq K \times V \subseteq U \subseteq K \times X$.*

Proof. Apply Lemma A.1.6. □

Corollary A.1.10. *Let $f : K \times X \to Y$ be a continuous function, where K is a compact space and X , Y are arbitrary topological spaces, and $x \in X$ be a fixed point. Suppose $U \subseteq Y$ is an open subset with the property that $f(K \times \{x\}) \subseteq U$. Then there exists some open neighborhood $V \subseteq X$ of x with $f(K \times V) \subseteq U$.*

Proof. $f^{-1}(U)$ is open by continuity and $K \times \{x\} \subseteq f^{-1}(U)$. Now apply Corollary A.1.9, we get an open neighborhood V of x with $K \times V \subseteq f^{-1}(U)$. □

A.2 Regarding Chapter 3

The following two lemmata recap basic facts about submanifolds. They are both trivial, so we will omit a proof.

Lemma A.2.1. *If M_1 , M_2 are manifolds and $N_1 \subseteq M_1$, $N_2 \subseteq M_2$ are submanifolds (respectively immersed submanifolds), then the product manifold $N_1 \times N_2$ is a submanifold (respectively an immersed submanifold) of $M_1 \times M_2$.*

Lemma A.2.2. *Let M be a manifold and $N \subseteq M$ a submanifold (respectively an immersed submanifold) of M. Then for any subset $S \subseteq N$, endowed with the subspace topology and some smooth structure (respectively an arbitrary topology and a smooth structure), the following are equivalent:*

(i) S is a submanifold (respectively an immersed submanifold) of M

(ii) S is a submanifold (respectively an immersed submanifold) of N.

The next lemma is a general topological fact and is easy to see as well.

Lemma A.2.3. *If $\pi : X \to Y$ is a surjective continuous map between two topological spaces X, Y, then the following are equivalent:*

(i) π is a quotient map.

(ii) For every open subset $U \subseteq X$ with $\pi^{-1}(\pi(U)) = U$ the image $\pi(U)$ is open in Y.

Lemma A.2.4 (see [Lee13, Theorem A.27])**.** *Suppose $\pi : X \to Y$ is a quotient map. Let $U \subseteq X$ be a* ***saturated subset****, i.e. $\pi^{-1}(\pi(U)) = U$, which is open or closed in X. Then the map $\pi|_U^{\pi(U)} : U \to \pi(U)$ is a quotient map if $\pi(U)$ carries the subspace topology. In particular, the subspace topology on $\pi(U)$ and the quotient topology, induced by $\pi|_U^{\pi(U)}$, coincide.*

Proof. Since U is saturated, we obtain $(\pi|_U^{\pi(U)})^{-1}(A) = \pi^{-1}(A) \subseteq U$ for any subset $A \subseteq \pi(U)$. Let τ_{sub} and τ_{quot} denote the supspace topology respectively the quotient topology on $\pi(U)$. We have to show $\tau_{\text{sub}} = \tau_{\text{quot}}$. Assume $W \in \tau_{\text{sub}}$. Then $W = \pi(U) \cap W'$ for some open set $W' \subseteq Y$. Since U is saturated, we get $(\pi|_U^{\pi(U)})^{-1}(W) = U \cap \pi^{-1}(W) = U \cap \pi^{-1}(\pi(U)) \cap \pi^{-1}(W') = U \cap \pi^{-1}(W')$, so $(\pi|_U^{\pi(U)})^{-1}(W)$ is open in U, and therefore $W \in \tau_{\text{quot}}$.
To prove the converse inclusion, we have to consider two cases: First suppose U is open. For arbitrary $W \in \tau_{\text{quot}}$ the set $(\pi|_U^{\pi(U)})^{-1}(W) = \pi^{-1}(W)$ is open in U. Thus, since U is open, the subset $\pi^{-1}(W)$ is open in X and, because π is a quotient map, W is open in Y, in particular it is open in $\pi(U)$.
Now assume, U is closed. Let $C \subseteq \pi(U)$ be an arbitrary closed subset with respect to the quotient topology on $\pi(U)$. We will show that C is also closed with respect to the subspace topology. Because C is closed with respect to the quotient topology, the preimage $(\pi|_U^{\pi(U)})^{-1}(C) = \pi^{-1}(C)$ is closed in U. Since U is closed, $\pi^{-1}(C)$ is closed in X, and therefore, using that π is a quotient map, C is closed in Y. Thus, C is closed in $\pi(U)$ with respect to the subspace topology. $\square$

Proposition A.2.5 (cf. [Lee13, Theorem 21.10])**.** *Suppose the Lie group G of dimension k acts smoothly, freely and properly on the manifold M of dimension m. Put $n := m - k$. Then for every $p \in M$ there are smooth charts (W_M, φ_M) for M, containing p, and $(W_{M/G}, \varphi_{M/G})$ for M/G, containing $G \cdot p$, such that the following conditions hold:*

(i) The orbit projection, expressed in coordinates,

$$\begin{aligned}&\varphi_{M/G} \circ \Pi_M \circ \varphi_M^{-1} :\\ &\mathbb{R}^k \times \mathbb{R}^n \supset U' \times U'' := \varphi_M(W_M) \to \varphi_{M/G}(W_{M/G}) = U''\end{aligned}$$

is the projection onto the second factor.

(ii) Write $(x, y) := (x_1, \ldots, x_k, y_1, \ldots, y_n) := \varphi_M$ for the coordinates. For each orbit $\mathcal{O}$ that intersects W_M there exists an unique $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $\mathcal{O} \cap W_M = \{y_1 = a_1, \ldots, y_n = a_n\}$.

(iii) $a = (0, \ldots, 0) \in \mathbb{R}^n$ for the orbit through p.

The pair $(\varphi_M, \varphi_{M/G})$ *is called* ***adapted to the*** G***-action*** *and* φ_M *is a* G***-adapted chart***.

Proof. By the Quotient Manifold Theorem, (M, Π_M) is a smooth G-principal bundle. As we have shown in the proof of Theorem 2.4.4, one can find a slice S at p (fulfilling the conditions in Theorem 2.3.15) such that $\eta = \Pi_M|_S^V$ is a diffeomorphism, where $V := \Pi_M(S)$ is open. Put mult : $G{\times}S \to G{\cdot}S$ for the multiplication map (see equation (2.17)). Then $\phi := (\mathrm{id}_G \times \eta) \circ \mathrm{mult}^{-1} : G{\cdot}S \to G{\times}V$ is a G-equivariant diffeomorphism such that $\Pi_M = \mathrm{pr}_2 \circ \phi$. Choose smooth charts (W_G, φ_G) for G, containing $e \in G$, and $(W_{M/G}\,, \varphi_{M/G})$ for V, where $(W_{M/G}\,, \varphi_{M/G})$ is centered at $\Pi_M(p)$ (i.e. $\Pi_M(p) \in W_{M/G}$ and $\varphi_{M/G}(\,\Pi_M(p)\,) = 0 \in \mathbb{R}^n$). Now define

$$\begin{aligned} &\varphi_M := (\,\varphi_G \times \varphi_{M/G}\,) \circ \phi : \\ &W_M := \phi^{-1}(W_G \times W_{M/G}) \to \varphi_G(W_G) \times \varphi_{M/G}(W_{M/G}) =: U' \times U'' \end{aligned}$$

Clearly, (W_M, φ_M) is a smooth chart for M, containig p. Then condition (i) follows immediately:

$$\varphi_{M/G} \circ \Pi_M \circ \varphi_M^{-1} = \varphi_{M/G} \circ \mathrm{pr}_2 \circ (\,\varphi_G \times \varphi_{M/G}\,)^{-1} = \mathrm{pr}_2^{U' \times U'' \twoheadrightarrow U''} \;.$$

Now let us check condition (ii). Suppose $\mathcal{O}$ is an arbitrary orbit such that the intersection with W_M is not empty. Choose $q \in W_M$ with $G \cdot q = \mathcal{O}$. Put $a := \varphi_{M/G}\,(\mathcal{O}) = \varphi_{M/G}\,(G \cdot q)$. Let $\phi(q) =: (g, y)$. Then $q = g \cdot s$ for the unique element $s \in S$ with $\eta(s) = \Pi_M(s) = y$. Hence, $y = G \cdot s = G \cdot q$ and we conclude $\phi(q) = (g, G \cdot q)$.
Consider an arbitrary $\hat{q} \in G \cdot q \,\cap\, W_M$. Then we have $\phi(\hat{q}) = (\hat{g}, G \cdot \hat{q})$ for some $\hat{g} \in G$. It follows

$$\varphi_M(\hat{q}) = (\varphi_G \times \varphi_{M/G})(\,(\hat{g}, G \cdot \hat{q})\,) = (\varphi_G \times \varphi_{M/G})(\,(\hat{g}, G \cdot q)\,) = (\varphi_G(\hat{g}), a)\,.$$

To show the converse inclusion, suppose $\hat{q}$ is in W_M with $\varphi_M(\hat{q}) = (\hat{x}, a)$. Let $\hat{g}$ denote the element in G with $\phi(\hat{q}) = (\hat{g}, G \cdot \hat{q})$. Thus, $(\hat{g}, G \cdot \hat{q}) = \phi(\hat{q}) = (\,\varphi_G^{-1}(\hat{x}), (\varphi_{M/G})^{-1}(a)\,) = (\,\varphi_G^{-1}(\hat{x}), G \cdot q\,)$, in particular $G \cdot \hat{q} = G \cdot q$, which shows $\hat{q} \in G \cdot q \,\cap\, W_M$.
This proves $\mathcal{O} \cap W_M = \{y_1 = a_1, \ldots, y_n = a_n\}$. The uniqueness of such an a is trivial.
(iii) follows from $a = \varphi_{M/G}\,(G \cdot p) = 0 \in \mathbb{R}^n$. □

The next corollary will be appear again in Chapter 4 but the approach of the proof will be different.

Corollary A.2.6. *G is a Lie group, acting smoothly, properly and freely on the manifold M. For $p \in M$ the following holds:*

$$\ker\,(d_p\Pi_M) = T_p\,(G \cdot p)\;. \tag{A.1}$$

Proof. By the preceding Proposition A.2.5, we can choose a pair of charts $(\varphi_M, \varphi_{M/G})$, adapted to the G-action, where (W_M, φ_M) contains p and
$(W_{M/G}, \varphi_{M/G})$ contains $G \cdot p$. Put $(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_n) = \varphi_M$. By condition (i) in Proposition A.2.5, we obtain $\ker\,(d_p\Pi_M) = \mathrm{span}\,(\partial_{x^1}, \ldots, \partial_{x^k})$. By condition (ii) we also get $T_p\,(G \cdot p) = \mathrm{span}(\partial_{x^1}, \ldots, \partial_{x^k})$, thus proving the statement. □

Proposition A.2.7. *Let $\theta : G \times M \to M$ be a smooth, proper and free action of the Lie group G on the manifold M. Suppose the submanifold $N \subseteq M$ is G-invariant, i.e. $g \cdot q \in N \;\forall g \in G\,,\, q \in N$, and open or closed in M. Then the orbit space N/G, endowed with the smooth structure from the Quotient Manifold Theorem, is a submanifold of M/G.*

Proof. By the Quotient Manifold Theorem, the space M/G carries a canonical smooth structure. Since N is G-invariant, the action restricts to a smooth action $\theta|_{G \times N}^N$ on N (using the characteristic property of embeddings). Clearly, this restricted action is free. It is also proper by Proposition 2.1.10. Thus, by the Quotient Manifold Theorem, the orbit space

N/G is indeed a manifold.

Let $\iota_N : N \hookrightarrow M$ and $\iota_{N/G} : N/G \hookrightarrow M/G$ denote the inclusion maps. Since by assumption N is G-invariant and open or closed in M, the quotient topology and the subspace topology (with respect to the superset M/G) on N/G coincide (see Lemma A.2.4). Thus, the inclusion map $\iota_{N/G}$ is a topological embedding between the manifolds N/G and M/G. Because we have

$$\iota_{N/G} \circ \Pi_N = \Pi_M \circ \iota_N \ ,$$

where Π_N , Π_M denote the projections onto the orbit, the map $\iota_{N/G}$ is smooth by the characteristic property of surjective submersions. If, in addition, it is also an immersion, then N/G is indeed a submanifold of M/G.

Let $\bar{v} \in T_{G\cdot q}N/G$ with $d_{G\cdot q}\,\iota_{N/G} \cdot \bar{v} = 0 \in T_{G\cdot q}M/G$ be arbitrary. Since Π_N is a submersion, there exists a $v \in T_qN$ such that $d_q\Pi_N \cdot v = \bar{v}$. Applying the chain rule, we get

$$0 = d_{\Pi_N(q)}\,\iota_{N/G} \ \circ \ d_q\,\Pi_N \ \cdot \ v = d_{\iota_N(q)}\,\Pi_M \ \circ \ d_q\,\iota_N \ \cdot \ v \ .$$

By Corollary A.2.6 we reason that $d_q\,\iota_N \cdot v \in T_q\,(G\cdot q)$. Recall that, if we write '$T_q\,(G\cdot q)$' and mean a linear subspace of T_qM, the formal expression is in fact '$d_q\,\iota_{G\cdot q,M}\,(T_q\,(G\cdot q))$', where $\iota_{G\cdot q,M}$ denotes the inclusion $G\cdot q \hookrightarrow M$. If $\iota_{G\cdot q,N} : G\cdot q \hookrightarrow N$ is the inclusion, we obtain $d_q\,\iota_N \cdot v \in d_q\,\iota_{G\cdot q,M}\,(T_q\,(G\cdot q)) = d_q\,\iota_N\,(\,d_q\,\iota_{G\cdot q,N}\,(\,T_q\,(G\cdot q)\,)\,)$ and, because ι_N is an immersion, this leads to $v \in d_q\,\iota_{G\cdot q,N}\,(T_q\,(G\cdot q))$. Again, using Corollary A.2.6, this means $v \in \ker(d_q\,\Pi_N)$, which demonstrates $\bar{v} = 0$. In conclusion, $d_{G\cdot q}\,\iota_{N/G}$ is injective, which finishes the proof. □

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