Ruprecht-Karls-Universität Heidelberg

Bachelor thesis

The n-queens completion problem

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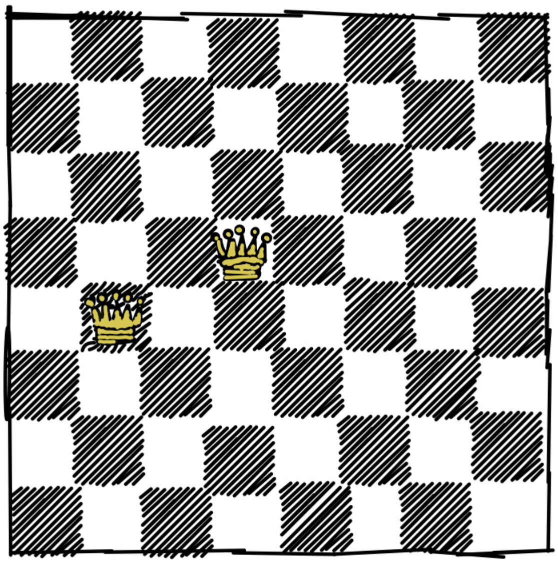

8-queens puzzle drawn by my little brother Milan

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The n-queens completion problem

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Danielle Eitelmann

Abstract

A well-known classic chess problem is the so-called n-queens problem, where n queens have to be placed on a $n \times n$ chessboard such that no two queens attack each other according to the classical chess rules. Such a placement of queens is called an *n-queens configuration*. This problem motivated countless other questions. This thesis focuses on the n-queens completion problem, which asks whether a set of non-attacking queens on the $n \times n$ chessboard can be extended to an n-queens configuration. More precisely, what is the maximum integer $\mathrm{qc}(n)$, such that any arrangement of at most $\mathrm{qc}(n)$ non-attacking queens is always completable? Progress on this question was obtained by Glock, Correia, and Sudakov in 2022 using graph theoretic arguments. In this work, I was able to improve their lower bound of $\mathrm{qc}(n) \geq \frac{n}{60}$ to $\mathrm{qc}(n) \geq \frac{n}{52}$.

Contents

1 Introduction

Almost everyone has heard of chess, a classic strategic board game with over a thousand years of history. In the Middle Ages, mastering chess was one of the seven skills of a knight. And to this day, interest in the game has not waned. The game structure offers various points of contact with mathematics. Several combinatorial and topological problems have emerged from chess. One example is the knight's tour. In this problem, a route is searched for the knight on an empty chess board so that the knight visits each square exactly once, moving according to the classical chess rules. Another example is the n-queens problem, which will be discussed in more detail. And do not worry, being a good chess player is no requirement for this thesis.

The n-queens problem is the task of placing n-queens on an $n \times n$ chessboard. It is a generalization from the 8-queens problem published in 1848 by the chess composer Bezzel [1]. The challenge here is that the queens should not be placed anywhere on the chessboard, but in such a way that they do not attack each other. This extended to the n-queens completion problem. In August 2017 a group of three mathematicians proved that the n-queens completion problem is NP-complete [2]. And in 2018 Mikhailovskii conjectured that it is solvable in polynomial time [3]. If this proves to be true this would solve one of the remaining Millenium Problems stated by the Clay Mathematics Institute in 2000.

Overall, this problem is classically viewed as a theoretical one, however, not only is it a simple but also a nontrivial problem. And therefore very well suited as a basis for a benchmark problem in various programming techniques, artificial intelligence, and combinatorial optimization.

After the first appearance of the n-queens problem, initial approaches were made by Nauck in 1850 [1]. Since then a lot of great mathematicians, including Carl Friedrich Gauss, have worked in this field. The first question that arises in this context is whether it is always possible to place n non-attacking queens on an $n \times n$ chessboard, called n-queens configuration. And indeed this is true for all natural numbers except 2 and 3 [4].

The n-queens problem provokes numerous other questions. Let us start with the question of how many different solutions there are. You can think of a trivially upper bound by considering, that every n-queens configuration is also an n-rooks configuration. And the rook arrangments correspond to permutation matrices, and here the exact number is known, actually $n!$. So this is an upper bound for the solutions by considering queens. But the precise number is also known for all $n \leq 27$. To determine the accurate number of solutions backtracking algorithms are used, though they are very inefficient. So the computing power is too big for higher numbers. In 2022 Simkin [5] proved that there is a constant $1,94 < \alpha < 1,9449$ such that the number of solutions of the n-queens problem is $((1 \pm o(1))ne^{-\alpha})^n$. Thus inconceivably large. For $n = 8$ there are precisely 92 different n-queens configurations whereas there are over 39 billion distinct solutions for $n = 20$. Consequently the linear increase in n corresponds to an exponential growth of the number of different n-queens configurations. The fact that the precise number of solutions is not known for arbitrary n is also consistent with the result of Hsiang, Hsu and Shieh [1]. Finding all the solutions for the n-queens problem is beyond the $\#P$-class.

But what happens if we consider other pieces besides the queen? Does the problem get easier? We have already taken a brief look at the situation with rooks. Since rooks can only move along rows and columns, a maximum of n rooks can be placed. And as already discussed, there are $n!$ possibilities for such configurations. So let us continue with kings. For them, we divide the board into 2×2 squares and place the kings at equivalent positions in each block. In this way, we are able to place $\frac{\left(2\cdot\lceil\frac{n}{2}\rceil\right)^2}{4}$ kings on an $n \times n$ chessboard. Next, have a quick look at the knights. First, we consider an 8×8 chessboard. Here we can place 32 knights. We can place one on each square of a given colour since they move only to the opposite colour. And so in general, on an $n \times n$ chessboard we are able to place $\lceil\frac{n^2}{2}\rceil$ non-attacking knights. With odd numbers, one should choose the more frequent colour to place them on. In particular, we have two possibilities to place them if n is even and only one if n is odd.

To make this situation on the other hand more complicated we can study the problem in higher dimensional chess spaces. In 2006 Barr and Rao [6] established a first lower bound of how many queens are at least necessary to attack all positions in a d-dimensional chess space of size n, where $d \geq 3, n \geq 4$. Furthermore, they showed that there are higher-dimensional chess spaces in which not all positions can be attacked by n^k queens. For a 3-dimensional chessboard Chakiat, Sudhakaran, Nair, and Venkatesh establish a backtracking algorithm to find maximal $3d$-solutions by using the solutions from 2-dimensions [7]. There is a fundamental difference between the two models because in Barr and Rao's approach queens can also attack each other through different $2d$-chessboard levels, while in the second model, they can only block if they are in the same chessboard level or right on top of each other.

Last but not least, let us look at toroidal chessboards. The toroidal chessboard arises from the standard board by gluing the first and last column together and the first and last row. This construction is mainly in use in the study of chess compositions. For example, on the toroidal chessboard, it is impossible to checkmate the king with a queen and a king. In 1918 Pólya [8] showed that one could place n queens on the chess torus of size $n \times n$ exactly when n is relatively prime to 6. In 2021 Bowtell and Keevash [9] extended this by specifying the number of solutions. They proved that if n is relatively prime to 6, then the number of different solutions on the toroidal chessboard is $\left((1+o(1))\frac{n}{e^3}\right)^n$. The structure of this expression is quite similiar to the number of n-queens configurations on the standard chessboard stated by Simkin.

That connects to our main point, namely the n-queens completion problem. It discusses the question, given a set of each wise non-attacking queens, can we extend it to an n-queens configuration? The n-queens completion problem is NP-complete and $\#P$-complete [2]. However, Mikhailovskii conjectured in 2021 that this problem is solvable in polynomial time. And this is the link to the Millenium P versus NP problem. So if it turns out that Mikhailovskii is right, this would solve the problem. But so far, no algorithms are solving the n-queens completion problem in polynomial time. Its simplicity but not triviality is why it has become a benchmark problem in artificial intelligence and other programming techniques.

My subsequent work is based on the paper from Glock, Correia, and Sudakov from 2022 [10]. Precisely I will focus on the following problem:

How many queens can always be placed on the chessboard such that they can be completed to an n-queens configuration?

To work on this issue more formally, we will translate it into a graph problem. Because then we can apply a lot of different tools. But first we need a few basic definitions.

Definition 1. An **n-queens configuration** is a set of n queens on an $n \times n$ chessboard such that no two are in the same row, column, or diagonal.

A **partial n-queens configuration** is a set Q' of k queens, where $k \leq n$, such that every row, column, and diagonal contains at most one element of Q'. Especially an **n-queens configuration** is a **partial n-queens configuration** of size n.

A partial n-queens configuration Q' is **completable** if there is an n-queens configuration Q with $Q' \subset Q$.

Now the **n-queens completion problem** asks if a given partial n-queens configuration is completable.

More precisely as stated before, we want to determine the number of non-attacking queens one can always place on the chessboard such that we can extend it to an n-queens configuration. For that, let us introduce the following parameter.

Definition 2. Define $\mathrm{qc}(n)$ as the maximum integer with the property that any partial n-queens configuration of size at most $\mathrm{qc}(n)$ is completable. $\mathrm{qc}(n)$ is called the **n-queens completion threshold**.

Now we can formulate the main theorem bounding the n-queens completion threshold stated by Correia, Glock and Sudakov in 2022.

Theorem 3 (Correia, Glock and Sudakov [10])**.** For all sufficiently large n, we have $\frac{n}{60} \leq \mathrm{qc}(n) \leq \frac{n}{4}$.

In this thesis I was able to improve the lower bound to $\frac{n}{52} \leq \mathrm{qc}(n)$ by generalising their approach.

2 Sketch of the proof of the lower bound

We want to establish a lower bound for the number of queens you can always place on an $n \times n$ chessboard such that you can extend them to an n-queens configuration. In this section I sketch the proof for the lower bound of Theorem 3 and discuss my main contribution in Section 2.4.

2.1 Translation into a graph theory problem

Construction 4. To formulate the problem more formally, we represent the chessboard by the 2-dimensional grid $[n] \times [n]$. For each $i \in [n]$, define

$$R_i = \{(i, j) : j \in [n]\}$$
$$C_i = \{(j, i) : i \in [n]\}.$$

Let $\mathcal{R} = \{R_i : i \in [n]\}$ denote the set of all rows and $\mathcal{C} = \{C_i : i \in [n]\}$ the set of all columns. For $k \in \{-(n-1), ..., n-1\}$, define

$$D_k^+ = \{(i, j) \in [n] \times [n] : i + j - (n + 1) = k\},$$
$$D_k^- = \{(i, j) \in [n] \times [n] : i - j = k\}.$$

Observe that D_0^+ and D_0^- are the two main diagonals of size n. Let $\mathcal{D} = \{D_k^+, D_k^- : k \in \{-(n-1), ..., n-1\}\}$ denote the set of all diagonals. Finally, let $\mathcal{L}_n = \mathcal{R} \cup \mathcal{C} \cup \mathcal{D}$ be the set of all **lines**.

To prove the lower bound of Theorem 3, we translate the problem into a graph problem. For that we need the notion of a complete bipartite graph.

Definition 5. A **bipartite graph** is a graph whose vertex set partitions into two disjoint and independent partition classes. Thus every edge has a vertex in each partition class.

A bipartite graph is **complete** if all possible edges, that is all edges between the two partition classes are present.

Let $\mathcal{G}$ be the complete bipartite graph on the vertex partition classes $\mathcal{R}$ and $\mathcal{C}$. Observe one vertex represents a row or column and one edge represents a square of our $n \times n$ chessboard.

With this reformulation, an n-queens configuration corresponds to a particular set of edges. Because these represented squares cannot be in the same row or column, this transfers to the condition that no two edges share an endpoint because those represent a row or a column in $\mathcal{G}$. We also want the configuration to have maximal size n, which means each row and each column must contain one queen, and this corresponds to every vertex in our graph should be incident to exactly one edge. Now this leads to the concept of perfect matchings.

Definition 6. A **matching** of a graph $\mathcal{G}$ is a set of pairwise non-adjacent edges, which means no two edges share a common vertex.

A matching is **perfect** if every vertex is incident to an edge of the matching.

With that, the problem transfers to finding a perfect matching.

Finally, we also have to consider the diagonals. Therefore we assign two colours to each edge, representing the two diagonals containing this edge. In the language of perfect matchings, we do not accept edges sharing the same colour because this would mean we put two queens in the same diagonal on the chessboard. Now we are at the concept of perfect rainbow matchings.

Definition 7. A **rainbow matching** of an edge-coloured graph $\mathcal{G}$ is a matching of $\mathcal{G}$ such that all the edges in the matching have distinct colours.

Especially a partial n-queens configuration on our chessboard corresponds to a rainbow matching in $\mathcal{G}$ and an n-queens configuration on our chessboard corresponds to a perfect rainbow matching in $\mathcal{G}$.

Now the question of whether we can complete a partial n-queens configuration translates to whether we are able to find a perfect rainbow matching in $\mathcal{G}$ which contains the edges fixed by our partial n-queens configuration. This perfect rainbow matching corresponds to an n-queens configuration, where we can embed the given partial n-queens configuration.

2.2 The rainbow matching lemma

We will now state a lemma that gives us sufficient conditions for the existence of a perfect rainbow matching.

For this, we need the notion of a proper, linear 2-fold edge colouring.

Definition 8. A **2-fold edge colouring** assigns two colours to each edge of a graph $\mathcal{G}$.

Such a colouring is **proper**, if all edges at a given vertex have pairwise disjoint colour pairs. And it is **linear** if every colour pair belongs to at most one edge.

Lemma 9 (rainbow matching lemma [10])**.** For any $\alpha > 0$, there exists $\epsilon > 0$ and n_0 such that the following is true for any $n \geq n_0$. Let $\mathcal{G}$ be a bipartite graph with parts A, B of size n with a proper, linear 2-fold edge colouring. Assume that the following conditions are satisfied for some d:

(i) every vertex has degree $(1 \pm \epsilon)d$

(ii) every colour has degree at most $(1 - \alpha)d$

(iii) any two sets $A' \subset A$ and $B' \subset B$ of size at least $(1 - \alpha)d$ have at least αn^2 edges between them.

Then $\mathcal{G}$ has a perfect rainbow matching.

Here the first condition means that all vertices in our graph should have roughly the same degree d. So our graph should be almost d-regular. Condition (ii) restricts the number of edges coloured with one colour. With that the number of edges of each colour is bounded from above by d. This way, we ensure that using one colour does not ban too many other edges. The last condition states that we have a certain amount of edges between sufficiently large subsets. Thus it guarantees that we have enough edges available.

2.3 Application of the rainbow matching lemma

With this model established, we are now well-equipped to prove the lower bound of Theorem 3. We start with a partial n-queens configuration $\mathcal{Q}'$ and define the graph $\mathcal{G}$ as the complete bipartite graph missing all edges attacked by one of the queens from $\mathcal{Q}'$. It remains to show that $\mathcal{Q}'$ is completable. And this is equivalent to showing that $\mathcal{G}$ has a perfect rainbow matching. Because perfect rainbow matchings of $\mathcal{G}$ correspond to placements of queens who do not attack each other and do not affect the queens from $\mathcal{Q}'$. To apply Lemma 9 to our graph $\mathcal{G}$, we need to ensure that it satisfies all the conditions. Now $\mathcal{G}$ might be irregular, and the degrees of some colours might be significantly larger than those of the vertices. Therefore we consider an appropriate spanning subgraph $\mathcal{G}' \subseteq \mathcal{G}$. Note that a perfect rainbow matching of $\mathcal{G}'$ is also a perfect rainbow matching of $\mathcal{G}$. Because only edges already contained in $\mathcal{G}$ will be used (it is a subgraph), and all vertices in $\mathcal{G}'$ will be incident to some edge (it is spanning). We obtain this graph by choosing each edge from $\mathcal{G}$ independent with a certain probability. So we need some probabilistic arguments and tools like this inequality.

Lemma 10 (Chernoff-Hoeffding bound)**.** Let X be the sum of n independent Bernoulli random variables. Then for any $\lambda \geq 0$, we have

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \lambda] \leq 2\exp\left(\frac{-2\lambda^2}{n}\right)$$

To define a suitable probability we will use edge-weightings. With that we will be able to regularise the number of edges of one colour.

Definition 11. An **edge-weighting** is a function $\omega : E(\mathcal{G}) \to X$, where $X \subset \mathbb{R}$.

But to fulfill condition (i), the edge-weighting needs to be adjusted, such that all vertices have roughly the same degree. Therefore we will use the following technical proposition.

Proposition 12 ([10])**.** Let $c, d' > 0$ and let $\mathcal{G}$ be a bipartite graph with parts A, B of size n where any two vertices in the same part have at least c common neighbors. Let $\omega_0 : E(\mathcal{G}) \to [0,1]$ be an edge weighting such that $\sum_{e \ni u} \omega_0(e) = \overline{d} \pm d'$ for all vertices u, where $\overline{d} = \frac{1}{n}\sum_{e \in E(\mathcal{G})} \omega_0(e)$. Then there exists an edge weighting $\omega : E(\mathcal{G}) \to \mathbb{R}$ such that the total weight of edges at every vertex is $\overline{d}$, and $|\omega(e) - \omega_0(e)| \leq \frac{2d'}{c}$ for all $e \in E(\mathcal{G})$.

Now we can check all conditions from Lemma 9. We will show that $\mathcal{G}'$ fulfills all conditions with high probability. And therefore, we can conclude that $\mathcal{G}'$ contains a perfect rainbow matching corresponding to the completion of $\mathcal{Q}'$ to an n-queens configuration with positive probability.

2.4 Generalisations and the resulting conditions for β

To adapt $\mathcal{G}'$ appropriately, we need to define an edge-weighting. In [10] they work with a fixed weighting. Whereas I will use the following refined definition of edge-weightings in my improved version of the proof.

Definition 13. Let $0 < t_0 \leq m \leq s \leq t_1 \leq 1$. An (s, m, t_0, t_1)**-edge-weighting** is a function

$$\omega : [n] \times [n] \to [t_0, t_1], \tag{1}$$

such that

$$\omega(R_i) = \omega(C_i) = sn + \mathcal{O}(1) \ \forall i \in [n]$$
$$\omega(D_k^+), \omega(D_k^-) \leq mn + \mathcal{O}(1) \ \forall k \in \{-(n-1), ..., n-1\}.$$

We need the following inequality during the proof that the second condition of Lemma 9 holds,

$$m + \mu < s - 3\beta.$$

Note that βn with $0 \leq \beta < 1$ will be the size of our partial n-queens configuration. We are interested in upper bounds on β.

This inequality is equivalent to

$$\beta^2 + \left(\frac{m}{3} - \frac{t_1}{3} + \frac{t_0}{9} - \frac{s}{3} - \frac{1}{6}\right)\beta + \frac{s-m}{18} > 0$$

by using the weighting of Definition 13. Which especially gives an upper bound for β.

$$\beta < \left(-\frac{m}{6} + \frac{t_1}{6} - \frac{t_0}{18} + \frac{s}{6} + \frac{1}{12}\right) - \sqrt{\left(\frac{m}{6} - \frac{t_1}{6} + \frac{t_0}{18} - \frac{s}{6} - \frac{1}{12}\right)^2 - \frac{s-m}{18}}. \tag{2}$$

Observe that this equation is only increasing in terms of $s - m$. We get one more constraint such that our probability is well-defined.

$$\beta < \frac{t_0}{6t_1 + 4t_0}. \tag{3}$$

And by checking the third condition of Lemma 9 we also need that β is small in terms of s.

$$\beta \leq \frac{s}{10}. \tag{4}$$

Note that this third upper bound is a very rough one. But optimising it would not bring any improvment since (2) is the main limiting bound.

3 Detailed and generalized version of the proof of the lower bound

Theorem 14. Given $0 < t_0 \leq m \leq s \leq t_1 \leq 1$, let $\beta > 0$ such that β satisfies (2), (3) and (4). Then there exists an $n_0 \in \mathbb{N}$ such that

$$\beta n \leq \mathrm{qc}(n) \qquad \forall n \geq n_0.$$

Proof. Given $0 < t_0 \leq m \leq s \leq t_1 \leq 1$ and β as in the statement, let α be small in terms of t_0, s, m, t_1 and β. Let ϵ be small in terms of t_0, s, m, t_1, β and α and let then n_0^{-1} be small in terms of all these parameters. Now let $n \geq n_0$.

We use the notation from Construction 4. In particular, recall $\mathcal{R}$ denotes the set of all rows and $\mathcal{C}$ of all columns. Now consider the complete bipartite graph $\mathcal{G}_n$ on the vertex set $\mathcal{R} \cup \mathcal{C}$, consisting of the n rows and n columns from the chess board. Define a 2-fold edge colouring by colouring each edge $(R_i, C_j) \in \mathcal{R} \times \mathcal{C}$ with the two colours representing the two diagonals containing the square of the chessboard corresponding to (R_i, C_j). That is, define

$$\begin{aligned}\varphi : E(\mathcal{G}_n) = \mathcal{R} \times \mathcal{C} &\to \mathcal{D}_n^2, \\ (R_i, C_j) &\mapsto (D^+_{i+j-(n+1)}, D^-_{i-j}).\end{aligned}$$

This colouring should be **proper**, and **linear**. Since each row or column intersects with every diagonal at most once it is proper, and because any two diagonals intersect in at most one square, it is linear.

Let $\mathcal{Q}' \subseteq [n] \times [n]$ be an arbitrary partial n-queens configuration of size βn. Let $\mathcal{R}' \subseteq \mathcal{R}$ be the rows, and $\mathcal{C}' \subseteq \mathcal{C}$ be the columns not containing a queen from $\mathcal{Q}'$. Let $\mathcal{G}$ be the subgraph of $\mathcal{G}_n$ induced by $\mathcal{R}'$ and $\mathcal{C}'$ after deleting all edges (R_i, C_j) for which $\mathcal{Q}'$ has a queen on $D^+_{i+j-(n+1)}$ or D^-_{i-j}. So now we are only left with edges not attacked by a queen of $\mathcal{Q}'$.

Let ω_0 be an (s, m, t_0, t_1)-edge-weighting. Note that the weight of a line is just the sum of the weights of all squares on that line. Now, we restrict ω_0 to the edges of $\mathcal{G}$. Slightly abusing notation, this restricted weighting will still be called ω_0. Note that it is still true that

$$\begin{aligned}&\omega_0(R_i) = \omega_0(C_i) \leq sn + \mathcal{O}(1) \;\forall i \in [n] \\ &\omega_0(D^+_k), \omega_0(D^-_k) \leq mn + \mathcal{O}(1) \;\forall k \in \{-(n-1), ..., n-1\}\end{aligned}$$

by definition.

For applying Lemma 9 the vertex degrees need to be regularized by maintaining a slight gap between the degrees of vertices and colours. The average weight of a vertex is

$$\overline{d} := \frac{1}{(1-\beta)n} \sum_{(R_i, C_j) \in E(\mathcal{G})} \omega_0((R_i, C_j)).$$

Indeed, recall that $(1-\beta)n$ is the number of rows and columns in $\mathcal{G}$, respectively. So, in particular, it has the size of the perfect matching which we need to construct. Note that every queen attacks at most 3 squares in each row or column, and the maximum weight of these squares is t_1. This gives the lower bound on the remaining weighting in a row or column. Otherwise each queen attacks at least one square with weight at least t_0. This is because each of the queens block a unique square belonging to the row or column their in, and each row intersects with each column at least once and vice versa. And thus each of the βn queens blocks at least one field in each row and column. Consequently we get the following bounds:

$$sn + \mathcal{O}(1) - 3t_1\beta n \leq \overline{d} \leq sn + \mathcal{O}(1) - t_0\beta n. \tag{5}$$

For the same reason, any two vertices of $\mathcal{G}$ in the same part have at least $n-6\beta n$ common neighbors. So by applying Proposition 12 with $d' = (3t_1-t_0)\beta n+\mathcal{O}(1)$ and $c = (1-6\beta)n$, there is a weighting $\omega : E(\mathcal{G}) \to \mathbb{R}$ such that every vertex has total weight $\overline{d}$, that is

$$\forall u \in V(\mathcal{G}): \quad \omega(u) = \sum_{uv\in E(\mathcal{G})} \omega(uv) = \overline{d} \tag{6}$$

and the weight of ω_0 is changed by at most

$$\mu := \frac{2d'}{c} = \frac{2\cdot(3t_1-t_0)\beta}{(1-6\beta)} + \mathcal{O}\left(\frac{1}{n}\right) \overset{(3)}{<} t_0 - \epsilon. \tag{7}$$

Where the inequality holds since (3) is equivalent to

$$\frac{2\cdot(3t_1-t_0)\beta}{(1-6\beta)} < t_0,$$

and n is large in terms of ϵ.

Now, we randomly sparsify the graph $\mathcal{G}$ to obtain a subgraph $\mathcal{G}'$ which is approximately regular. For this, define

$$p_e := \frac{\omega(e)}{1+\mu} \tag{8}$$

for each edge $e \in E(\mathcal{G})$. Note that $1 \geq p_e \geq \frac{\epsilon}{2} > 0$. Because

$$p_e = \frac{\omega(e)}{1+\mu} \leq \frac{\omega_0(e)+\mu}{1+\mu} \overset{\omega_0(e)\in[t_0,t_1]}{\leq} \frac{t_1+\mu}{1+\mu} \leq \frac{1+\mu}{1+\mu} = 1$$

and

$$p_e = \frac{\omega(e)}{1+\mu} \geq \frac{\omega_0(e)-\mu}{1+\mu} \overset{\omega_0(e)\in[t_0,t_1]}{\geq} \frac{t_0-\mu}{1+\mu} \overset{(7)}{\geq} \frac{\epsilon}{2}. \tag{9}$$

Now include every edge in $\mathcal{G}$ with probability p_e. We will now show that $\mathcal{G}'$ satisfies the properties (i)-(iii) of Lemma 9.

Let $u \in V(\mathcal{G})$. The expected degree of u in $\mathcal{G}'$ is

$$\begin{aligned} d := \mathbb{E}[d(u)] &= \sum_{uv \in E(\mathcal{G}')} p_{uv} \\ &\overset{(8)}{=} \frac{1}{1+\mu} \sum_{uv \in E(\mathcal{G}')} \omega(e) \overset{(6)}{=} \frac{\overline{d}}{1+\mu} \\ &\overset{(5)}{\geq} \frac{sn + \mathcal{O}(1) - 3t_1\beta n}{1+\mu} \\ &\geq \frac{(s-3\beta)n}{2} + \mathcal{O}(1) \overset{(4)}{\geq} \epsilon n. \end{aligned} \tag{10}$$

Therefore, Lemma 10 implies that

$$\mathbb{P}[|d_{\mathcal{G}'}(u) - d| \geq \epsilon d] \leq \mathbb{P}\left[|d_{\mathcal{G}'}(u) - d| \geq \frac{\alpha}{2}\epsilon n\right] \leq 2\exp\left(-2 \cdot \frac{\alpha^2}{4}\epsilon^2 \frac{n^2}{n}\right) \leq \frac{1}{n^2}.$$

Thus, a union bound over at most $2n$ vertices gives

$$\mathbb{P}\left[\exists u \in V(\mathcal{G}') : (|d_{\mathcal{G}'}(u) - d| \geq \epsilon d)\right] \leq \sum_{u \in V(\mathcal{G}')} \mathbb{P}\left[|d_{\mathcal{G}'}(u) - d| \geq \epsilon d\right] \leq 2n \cdot \frac{1}{n^2} = o(1).$$

So with high probability, all vertices have degree $(1 \pm \epsilon)d$ and therefore condition (i) from Lemma 9 holds with high probability.

To check the second condition, consider any colour c. Since the weight of each edge has increased by at most μ and c appears on at most n edges (the longest diagonal has size n) and due to our probability p_e, the expected degree $d(c)$ in $\mathcal{G}'$ is

$$\begin{aligned} \mathbb{E}[d(c)] &= \sum_{e \in E(\mathcal{G}'), e \text{ is coloured } c} p_e = \sum_{e \in E(\mathcal{G}'), e \text{ is coloured } c} \frac{w(e)}{1+\mu} \\ &\leq \frac{mn + \mathcal{O}(1) + \mu n}{1+\mu} \end{aligned}$$

Note that (2) is equivalent to

$$m + \mu < s - 3\beta. \tag{11}$$

And because ϵ and α are chosen sufficiently small they fulfill

$$0 < 2\epsilon + (s - 3\beta - \epsilon)\alpha \leq s - 3\beta - (m + \mu).$$

So, in particular, that means that ϵ and α are sufficiently small to close the gap given by the strict inequality (11) but not conflict with it. And with that, the following holds

$$\begin{aligned} m + \mu + \epsilon &\leq (1-\alpha)(s - 3\beta - \epsilon) \\ &\leq (1-\alpha)(s - 3t_1\beta - \epsilon). \end{aligned}$$

And therefore

$$\frac{mn+\mu n+\epsilon n}{1+\mu} \leq \frac{1-\alpha}{1+\mu}(sn - 3t_1\beta n \underbrace{-n\cdot\epsilon}_{\leq \mathcal{O}(1)(1+\mu)}) \overset{(10)}{\leq} (1-\alpha)d. \tag{12}$$

Furthermore

$$\begin{aligned}
\frac{mn+\mu n+\epsilon n}{1+\mu} - \mathbb{E}[d(c)] &\geq \frac{mn+\mu n+\epsilon n}{1+\mu} - \frac{mn+\mathcal{O}(1)+\mu n}{1+\mu} \\
&= \frac{\epsilon n}{1+\mu} - \frac{\mathcal{O}(1)}{1+\mu} \\
&\overset{(7)}{>} \frac{\epsilon n}{2} - \frac{\epsilon n}{4} = \frac{\epsilon n}{4}.
\end{aligned}$$

So Lemma 10 implies

$$\begin{aligned}
\mathbb{P}[d(c) \geq (1-\alpha)d] &\overset{(12)}{\leq} \mathbb{P}\left[d(c) \geq \frac{m+\mu+\epsilon}{1+\mu}\cdot n\right] \\
&\leq \mathbb{P}\left[|d(c) - \mathbb{E}[d(c)]| \geq \frac{m+\mu+\epsilon}{1+\mu}\cdot n - \mathbb{E}[d(c)]\right] \\
&\leq \mathbb{P}\left[|d(c) - \mathbb{E}[d(c)]| \geq \frac{\epsilon n}{4}\right] \\
&\leq 2\exp\left(-2\frac{\left(\frac{\epsilon n}{4}\right)^2}{2n}\right) \\
&= 2\exp\left(-\frac{1}{16}\epsilon^2 n\right) \leq \frac{1}{n^2}.
\end{aligned}$$

Due to a union bound over the at most $2n$ colours

$$\mathbb{P}\left[\exists c \in \mathcal{D}_n^2 : d(c) \geq (1-\alpha)d\right] \leq \sum_c \mathbb{P}[d(c) \geq (1-\alpha)d] \leq 2n \cdot \frac{1}{n^2} = o(1).$$

And therefore, with high probability, all colours have a degree lower than $(1-\alpha)d$, so condition (ii) holds with high probability.

Last, but not least, it remains to show that the third condition of Lemma 9 holds. Therefore consider a set of rows $R'' \subseteq \mathcal{R}'$ and a set of columns $C'' \subseteq \mathcal{C}'$ each of size at least $(1-\alpha)d$. Note that each of the βn queens attacks at most 2 squares of each of the remaining $(1-\beta)n$ rows in the subchessboard induced by R'' and C''.

So there are at least

$$\begin{aligned}|R''||C''| - 2(1-\beta)n \cdot \beta n &\geq (1-\alpha)^2 d^2 - 2(1-\beta)\beta n^2 \\ &\geq \left(\frac{1-\alpha}{1+\mu}\right)^2 (sn + \mathcal{O}(1) - 3\beta n)^2 - 2\beta^2 n^2 \\ &\geq \frac{1}{16}(s-4\beta)^2 n^2 - 2\beta^2 n^2 \\ &\overset{(4)}{\geq} \frac{1}{400} s^2 n^2 \\ &\geq \epsilon n^2\end{aligned}$$

edges between R'' and C''. Since $p_e \overset{(9)}{\geq} \frac{\epsilon}{2} > 0$ for all $e \in \mathcal{G}$, it follows that

$$\mathbb{E}[e_{\mathcal{G}'}(R'', C'')] \geq \frac{\epsilon}{2} \cdot \epsilon n^2 > 2\alpha n^2.$$

And thus Lemma 10 implies

$$\begin{aligned}&\mathbb{P}\left[e_{\mathcal{G}'}(R'', C'') \leq \alpha n^2\right] \\ &\leq \mathbb{P}\left[|e_{\mathcal{G}'}(R'', C'') - \mathbb{E}[e_{\mathcal{G}'}(R'', C'')]| \geq \alpha n^2\right] \\ &\leq 2\exp\left(-2\frac{(\alpha n^2)^2}{n^2}\right) \\ &\leq 2\exp\left(-2\alpha^2 n^2\right).\end{aligned}$$

Note that there are at most 2^n choices for $R'' \subseteq \mathcal{R}'$ and similarly for $C'' \subseteq \mathcal{C}'$. Thus, a union bound over the at most 4^n choices for R'' and C'' yields that the probability, that there are sets $R'' \subseteq \mathcal{R}', C'' \subseteq \mathcal{C}'$ with $|R''|, |C''| \geq (1-\alpha)d$ and less or equal αn^2 edges between them, is less or equal to

$$4^n \cdot 2\exp\left(-2\alpha^2 n^2\right) \leq o(1).$$

And so with high probability there are at least αn^2 edges between R'' and C'' $\forall R'' \subseteq \mathcal{R}', C'' \subseteq \mathcal{C}'$ of size at least $(1-\alpha)d$, satisfying (iii) with high probability.

Hence by choosing each edge with probability p_e, we obtain a subgraph $\mathcal{G}' \subset \mathcal{G}$, which satisfies each of the required properties with high probability. Due to a union bound over the three conditions, $\mathcal{G}'$ satisfies (i), (ii), and (iii) simultaneously with positive probability. Finally, Lemma 9 can be applied to conclude that $\mathcal{G}'$ has a perfect rainbow matching corresponding to a completion of $\mathcal{Q}'$. $\square$

4 Main results

Proposition 15 ([10]). For all $n \in \mathbb{N}$, there exists a weighting $\omega : [n] \times [n] \to [\frac{1}{2}, 1]$ with the property that every row and column has total weight $\frac{5n}{6} + \mathcal{O}(1)$, but every diagonal has weight at most $\frac{2n}{3} + \mathcal{O}(1)$.

$\frac{3}{4}$	1	$\frac{3}{4}$
1	$\frac{1}{2}$	1
$\frac{3}{4}$	1	$\frac{3}{4}$

Figure 2: The weight function for $n = 3$.

Corollary 16. For n sufficiently large, we have $\mathrm{qc}(n) \geq 0.01926 \geq \frac{1}{52}n$.

Remark. We get an even better bound than in Theorem 3 basically because of our adaptation in equation (5).

Proof. Apply Theorem 14 with Proposition 15. $\square$

Construction 17. Another approach for improving the bound for β was to adjust the weight function. In my generalized version of the proof, the direct dependence of the lower bound - more precisely β - on the weighting becomes clear by equation (2). Therefore I thought I try to improve these weighting in terms of improving the distance between the weighting of the diagonals to the weight of the columns and rows. Because (2) is increasing with $s - m$. For this, I want to introduce a weighting on an $n \times n$ board with $n > 3$. Because, for $n = 3$, we can not improve the distance anymore. Consider for this

$$\omega(D_0^+) = 2 = \frac{2}{3} \cdot 3$$

the highest weighting of a diagonal and

$$\omega(R_1) = 2.5 = \frac{5}{6} \cdot 3$$

the weighting of all rows and columns. As soon as we change some weight on the diagonal to make the total weighting smaller, the weight of a row or column would decrease with the same amount. Therefore I decided to try higher values for n. My goal was to get a weighting such that

$$\omega(D_i^+), \omega(D_j^-) \leq m \cdot n < \frac{2}{3} \cdot n$$
$$\omega(R_i), \omega(C_i) = s \cdot n \geq \frac{5}{6} \cdot n.$$

I have reached this goal with linear optimization in Python for $n = 9$. With this, I got the following weighting.

0.778	0.472	1	1	1	1	1	0.472	0.778
0.472	0.778	1	1	1	1	1	0.778	0.472
1	1	0.412	0.977	0.722	0.977	0.412	1	1
1	1	0.977	0.509	0.528	0.509	0.977	1	1
1	1	0.722	0.528	1	0.528	0.722	1	1
1	1	0.977	0.509	0.528	0.509	0.977	1	1
1	1	0.412	0.977	0.722	0.977	0.412	1	1
0.472	0.778	1	1	1	1	1	0.778	0.472
0.778	0.472	1	1	1	1	1	0.472	0.778

Figure 3: The optimized weight function for $n = 9$.

With this weighting, we obtain the following weights for the diagonals, rows, and columns

$$\omega(D_i^+), \omega(D_j^-) \leq 5.96 = \underbrace{\frac{149}{225}}_{m} \cdot 9 \qquad \forall i, j \in \{-8, ..., 8\}$$

$$\omega(R_i), \omega(C_j) = 7.5 = \underbrace{\frac{5}{6}}_{s} \cdot 9 \qquad \forall i, j \in \{1, ..., 9\}.$$

Therefore the distance between s and m has been slightly improved

$$s - m = \frac{77}{450} > \frac{1}{6} = \frac{5}{6} - \frac{2}{3}.$$

Proposition 18. For all $n \in \mathbb{N}$, there exists a weighting $\omega : [n] \times [n] \to [0, 412, 1]$ with the property that every row and column has total weight $\frac{5n}{6} + \mathcal{O}(1)$, but every diagonal has weight at most $\frac{149n}{225} + \mathcal{O}(1)$.

Proof. Define a weight function $\omega : [n] \times [n] \to [0.412, 1]$ by defining a partition of $[0, 1]$ in nine subintervalls as follows

$$I_i = \left[\frac{i-1}{9}, \frac{i}{9}\right) \qquad \forall i \in \{1, ..., 8\}$$

$$I_9 = \left[\frac{8}{9}, 1\right].$$

This partition in subintervals gives a checkered division of our $[n] \times [n]$ grid. And now assign the values given in Fig. 3 to each square in a corresponding area.

Then we have

$$\omega(R), \omega(C) = 7.5 \cdot \frac{n}{9} + \mathcal{O}(1) = \frac{5}{6} \cdot n + \mathcal{O}(1) \qquad \forall C \in \mathcal{C}, R \in \mathcal{R}.$$

By symmetry it suffices now to consider D_k^- for fixed $k \in \{0, ..., n-1\}$. By the definition of ω, the weight of any such diagonal with $k \geq \frac{n}{3}$ is dominated by the one with $k = \lceil \frac{n}{3} \rceil$, since for larger k, the size of the diagonals and the weight of the diagonals are non-increasing as k increases. Finally, for $k \leq \lceil \frac{n}{9} \rceil$, we have

$$\begin{aligned}
\omega(D_k^-) &= 2 \cdot \left(\frac{n}{9} - k\right) \cdot 0.778 + 2 \cdot k \cdot 0.472 + 2 \cdot \left(\frac{n}{9} - k\right) \cdot 0.778 + 2 \cdot k \cdot 1 \\
&\quad + 2 \cdot \left(\frac{n}{9} - k\right) \cdot 0.412 + 2 \cdot k \cdot 0.977 + 2 \cdot \left(\frac{n}{9} - k\right) \cdot 0.509 + 2 \cdot k \cdot 0.528 \\
&\quad + \left(\frac{n}{9} - k\right) \cdot 1 + \mathcal{O}(1) \\
&= \frac{2977}{4500} n + \mathcal{O}(1) \\
&\leq \frac{149}{225} n + \mathcal{O}(1)
\end{aligned}$$

Analogous calculations are obtained for $\frac{n}{9} \leq k \leq \frac{2n}{9}$ and $\frac{2n}{9} \leq k \leq \lceil \frac{n}{3} \rceil$. And this proves the claim. $\square$

Corollary 19. For n sufficiently large, we have $\mathrm{qc}(n) \geq 0.019324n$.

Proof. Apply Theorem 14 with Proposition 18. $\square$

Remark. Maybe we can find an even better weighting considering higher n. However, the resulting improvement compared to Corollary 16 is so minimal that the cost is considerably high compared to. And the finer the weighting, the more complex it gets.

5 A special placement of the partial n-queens configuration

Another idea to achieve a better lower bound was to be more restrictive in the placement of our partial n-queens configuration on the chessboard. So in other words be more precise on where the partial n-queens configuration is placed. Since where a queen is placed, affects the amount of weights it is going to block. If we have another look at the proof, the main bound on β is determined by (11), which significantly depends on μ. In particular, if μ decreases the bound for β will increase. μ depends on d' which is the difference between the lower and upper bound on the average vertex weighting, see (5). And here we can be more precise if we know where our queens are placed on the chessboard. And that would decrease d', and so μ and finally this would lead to a higher upper bound on β.

5.1 The corner placement

To formulate this approach let us adapt our n-queens completion threshold.

Definition 20. Fix an $n \in \mathbb{N}$ and let $k \leq n$. Now $\hat{\mathrm{qc}}(n)$ is the maximum integer with the property that any partial n-queens configuration which is a k-queens configuration placed in the corner of the chessboard is completable. $\hat{\mathrm{qc}}(n)$ is the ***n*-queens corner completion threshold**.

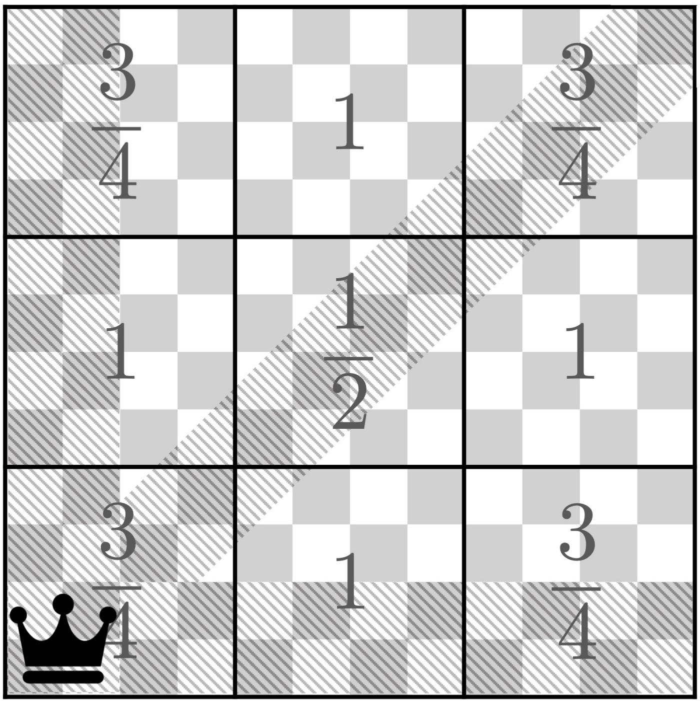

Figure 4: A partial n-queens configuration that is a βn-queens configuration placed in the corner of the chessboard.

Theorem 21. For n sufficiently large, we have $\hat{\text{qc}}(n) \geq 0.0277778n > \frac{n}{36}$.

Proof. Let $\mathcal{Q}'$ be a partial n-queens configuration of size βn that is a βn-queens configuration. Wlog we place $\mathcal{Q}'$ in the bottom left corner of our chessboard. We use the weighting given by Proposition 15. Now we can be more precise on the lower and upper bounds for the remaining weighting of a row or column. So instead of (5)

$$sn + \mathcal{O}(1) - 3t_1\beta n \leq \overline{d} \leq sn + \mathcal{O}(1) - t_0\beta n$$

we get

$$\frac{5}{6}n + \mathcal{O}(1) - 2\beta n \leq \overline{d} \leq \frac{5}{6}n + \mathcal{O}(1) - \frac{3}{4}\beta n. \tag{13}$$

Indeed, first note for the lower bound that by placing our complete partial n-queens configuration in one corner, all queens of $\mathcal{Q}'$ can block at most two squares per row and column instead of three (Fig. 4). As seen in Fig. 2, we can easily divide our chessboard in $\frac{n}{3}$ fractions. So let us have a look at each of these parts separately. It suffices to consider the rows because for the columns symmetric arguments hold. In the first and last thirds, each queen can attack at most two squares with weight $\frac{3}{4}$ or a square with weight 1 and a square with weight $\frac{3}{4}$. And in the middle, each queen can attack at most two squares with weight 1. So the maximum weight that is blocked is $2\beta n$.

If we now look for the minimum weight blocked by $\mathcal{Q}'$, observe that each queen blocks a column in the first third. And therefore each queen blocks at least one square of weight at least $\frac{3}{4}$.

With these new bounds on $\overline{d}$, we get

$$\mu = \frac{\frac{5}{2}\beta n}{(1-6\beta)n} + \mathcal{O}\left(\frac{1}{n}\right) < \frac{1}{2} - \epsilon$$

which is equivalent to

$$\beta < \frac{1}{11}.$$

And to fulfill

$$\frac{2}{3} + \mu < \frac{5}{6} - 3\beta,$$

β needs to satisfy

$$\beta^2 - \frac{13}{36}\beta + \frac{1}{108} > 0.$$

And this gives us the bound

$$\beta \leq \frac{1}{36} < 0.02778.$$

Note that these bounds imply that $\beta \leq \frac{\frac{5}{6}}{10} \approx 0.083$, so (4) is also satisfied.

Now all other steps are analoguous to the proof of Theorem 3. And therefore our partial n-queens configuration of size βn that is a βn-queens configuration in the corner of the chessboard is completable. $\square$

Thus, Theorem 21 shows that it is possible to get better bounds by restricting the placement of the partial n-queens configuration.

5.2 Further work

We could get even better bounds for the n-queens corner completion number, if we had tighter bounds than (13). And we could get them, if we knew more about the arrangement itself. As seen in Fig. 4 the weight blocked by a queen via the diagonal depends on where the queen is placed. For example, in the top third of the rows, a queen in the bottom corner can either block a square of weight $\frac{3}{4}$ or a square of weight 1 via the diagonal. In particular, I could not be precise on the weighting blocked by the diagonals.

To get more knowledge of the weight blocked via diagonals, it could be interesting to look at how the queens are allocated within an n-queens configuration. In particular, which amount of queens is on one side if we divide our chessboard on one of the main diagonals? With this, we might make the bound in (13) more accurate because we could also use the knowledge from diagonals.

For example, if it turns out that the queens are evenly distributed, say we have between $\frac{n}{2}$ and $\frac{n}{2}+1$ many queens on one side including the main diagonal. (This is motivated by analysing n-queens configurations for $n = 8, 9, 10$.) Then we would have $\hat{\text{qc}}(n) \geq 0.03411n$, which is notably bigger than the result of Theorem 21.

Another direction for further work is to study other special placements of the partial n-queens configuration. The approach will be the same as in Theorem 21. So it suffices to study the difference in weights blocked and then use the arguments of Theorem 3. This Theorem thus provides a good framework for studying this type of problems.

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