Student Research Projects - Topics

Research Station Geometry+Dynamics

Starting winter term 2024/25

1 Billiards in Higher Dimensions – Only ellipsoids have caustics. And symplectic?

Supervisor: Prof. Dr. Peter Albers

Prerequisits: LA/Ana, DiffGeo1 is certainly helpful but not necessary

Abstract: We consider a surface $\Sigma \subset \mathbb{R}^3$ bounding a convex compact body B. Then we can play (Euclidean) billiard inside B, i.e., a ray inside B hitting the boundary $\partial B = \Sigma$ is reflected according to angle of incidence = angle of reflection. A caustic is a (compact, convex, smooth) subset C of B with the property that if a ray is tangent to C then after every reflection on Σ is will again be tangent to C. It is a Theorem due to Berger and also Gruber that if B admits a caustic C then B is an ellipsoid and C is a confocal ellipsoid.

The goal of this project is to first understand the proof by Gruber and then to investigate if a similar assertion is true for symplectic billiards in $\mathbb{R}^4$ and higher dimensions.

Literature and further reading:

- Gruber, P., Only ellipsoids have caustics. Math. Ann., 303(1995), no. 2, 185–194.
- Tabachnikov, S., Geometry and Billiards. AMS Student Mathematical Library #30

2 Implementation of New Manifolds on manifolds.jl

Supervisor: Prof. Dr. Roland Herzog, Prof. Dr. Peter Albers, Prof. Dr. Ronny Bergmann (NTNU Norway)

Prerequisites: elementary differential geometry; programming in Julia is not a requirement, but some proficiency in the Python or Matlab (which share some similarity with Julia) is strongly desirable

Abstract: manifolds.jl is a Julia package that provides a library of manifolds and functions to work with them (such as geodesics, inner products etc.). In this project the emphasis is on getting to know the library and then add to it one or two additional manifolds that are not yet available. Development happens on GitHub and includes writing test cases.

3 Divisibility theory in Lean

Supervisor: Dr Florent Schaffhauser.

Prerequisites:

- Basic ring theory, principal ideal domains, notion of g.c.d.
- Experience in a functional programming language (e.g. Haskell) is desirable but not necessary.

Abstract: The notion of ideal in a commutative ring with unit provides an abstract generalisation of the theory of divisibility in usual arithmetics. And just like with numbers, it is possible to compute with ideals, and even generalise the notion of being decomposable as a product of prime factors. Understanding this is the basis for common identities such as $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$, that people who are not mathematicians find very strange. In this project, we set out to teach a computer how to do this computation and more. More precisely, our goal is to do this in a proof assistant and, under certain assumptions on the base ring, reach a *computable theory* of ideals, with the previous equality as our first benchmark. The preferred choice of proof assistant will be Lean but other options are also available.

Literature and further reading:

- J. Avigad, P. Massot. *Mathematics in Lean*, 2020. `https://leanprover-community.github.io/mathematics_in_lean/`
- A. Chambert-Loir. *(Mostly) commutative algebra*, 2021. Springer Universitext.

4 Hopf Fibration and Euclidean Embeddings of S^3

Supervisor: Prof. Dr. Christoph Schnörr

Prerequisits: LA/Ana, DiffGeo1 is certainly helpful but not necessary

Abstract: Established visualizations of the 3-sphere $S^3 \subset \mathbb{R}^4$ rely on the Hopf fibration and stereographic projection. The latter projection preserves topological structure and even is conformal. On the other hand, it suffers from distortion of metric structure.

The goal of this project, therefore, is to exploit established techniques for Euclidean embeddings of finite metric spaces, which approximately preserve given pairwise distances. This enables to turn subsets of S^3 in many ways into a finite metric space and to explore visually its structure by inspecting the 3-dimensional Euclidean embedding with minimal metric distortion.

Literature and further reading:

- P. Petersen. Riemannian Geometry. Springer, 3rd edition, 2016.
- H. K. Urbantke. The Hopf Fibration – Seven Times in Physics. J. Geometry and Physics, 46:125–150, 2003.
- N. Krislock and H. Wolkowicz. Euclidean Distance Matrices and Applications. In Handbook on Semidefinite, Conic and Polynomial Optimization, chapter 30. Springer, 2012.

5 The Inverted Pendulum: Simulation, Control and Machine Learning

Supervisor: Prof. Dr. Christoph Schnörr

Prerequisits: LA/Ana, basic programming skills

Abstract: The inverted pendulum on a cart is a classical example of an unstable dynamical system. The project starts with understanding, simulating and visualizing the system dynamics after perturbing the unstable position of rest. Next, the capabilities of classical techniques of closed-loop optimal control of linear dynamical systems are studied with respect to the stabilization of the nonlinear system for various perturbations. Additionally, depending on time and interest, more advanced aspects of either model-predictive control, of controlled Lagrangian systems or of learning feedback control, may be considered.

Literature and further reading: The list of references below merely reflects some of the above-mentioned aspects. It will be complemented once the project starts, depending on the particpants background and interests.

- D. Liberzon. Calculus of Variations and Optimal Control Theory. Princeton Univ. Press, 2012. (ch. 6)
- A. M. Bloch, N. E. Leonard, and J. E. Marsden. Controlled Lagrangians and the Stabilization of Mechanical Systems I: The First Matching Theorem. IEEE Trans. Automatic Control, 45(12):2253–2270, 2000.
- C. W. Anderson. Learning to Control and Inverted Pendulum using Neural Networks. IEEE Control Systems Magazine, 9(3):31–37, 1989.
- S. Israilov, L. Fu, J. Sánchez-Rodríguez, F. Fusco, G. Allibert, C. Raufaste, and M. Argentina. Reinforcement Learning Approach to Control and Inverted Pendulum: A General Framework for Educational Purposes. PLoS ONE, 18(2):e0280071, 2023.

6 Folded Galleries in Coxeter Groups

Supervisor: Prof. Dr. Petra Schwer

Prerequisits: LA, Ana, Algebra, experience in using computer algebra systems such as Sage or MAGMA; ideally some knowledge about Coxeter groups

Abstract: Coxeter groups are abstract versions of reflection groups given by presentations of the form $\langle s_1, s_2, \ldots, s_n | (s_i s_j)^{m_{ij}} \rangle$ where $m_{ii} = 1$ and $m_{ij} \in \mathbb{N}_{\geq 2} \cup \infty, m_{ij} = m_{ji}$ for all $i \neq j$. The combinatorics of Coxeter groups encodes many interesting algebraic phenomena. A powerful tool, which allows for example to compute dimensions of certain varieties, are the so called folded galleries and shadows. These objects correspond to "decorated"words in the standard generators s_i of the Coxeter group and allow to mix geometric, combinatorial and algebraic methods. See for example Schwer's survey listed below for further details on this story.

The main goal of this project would be to both enhance further the online-shadows-viewer developed by Henri Nikoleit (see link below) and to implement counting methods of various special instances of folded galleries satisfying certain extra conditions in e.g. Sage or Magma. Eventually this will lead to the computation of certain counting functions associated with group actions on flag varieties.

Literature and further reading:

- P. Schwer, *Shadows in the wild - folded galleries and their applications*, Jahresber. Dtsch. Math.-Ver. 124, No. 1, 3–41 (2022)
- M. Graeber and P. Schwer, *Ann. Comb.* 24, No. 1, 119–147 (2020)
- Shadows-viewer: `www.mathelabor.ovgu.de/shadows`

7 Finite index subgroups of reflection groups

Supervisor: Prof. Dr. Petra Schwer, Dr. Yuri Santos Rego (Lincoln, UK)

Prerequisits: LA, Ana, Algebra, experience in Sage, MAGMA, GAP or other computer algebra system, GGT is helpful but not necessary

Abstract: The main aim of this project is to explore ways to determine the finite index subgroups of a Coxeter group or more generally any group given by a finite presentation.

It is, in general, impossible to decide whether two given presentations $\langle S|R\rangle$ and $\langle S'|R'\rangle$ determine the same group. Sometimes, when groups are profinitely rigid, knowing the finite quotients of a group helps to answer this question.

The person on this project shall learn Conder's finite index method with the aim to explore (using computer algebra systems and own code) the subgroups of Coxeter groups of rank four. The ultimate goal would be to generalize the results from Santos-Rego and Schwer obtained for rank 3 Coxeter groups.

Literature and further reading:

- M. Conder and P. Dobcsanyi. *Applications and adaptations of the low index subgroups procedure*. Math. Comp. 74.249 (2005). doi: 10.1090/S0025-5718-04- 01647-3.
- M. Bridson, M. Conder, A. Reid, *Determining Fuchsian groups by their finite quotients.* Isr. J. Math. 214, 1?41 (2016). https://doi.org/10.1007/s11856-016-1341-6
- Y. S. Rego and P. Schwer, *The galaxy of Coxeter groups*, Journal of Algebra (2023). doi.org: 10.1016/j.jalgebra.2023.12.006.
- MAGMA library `https://magma.maths.usyd.edu.au/magma/handbook/text/869`

8 Visualizing posets of noncrossing partitions

Supervisors: Prof. Dr. Petra Schwer, Dr. José Pedro Quintanilha

Prerequisites: Algebra, linear algebra, knowledge about programming and developing visualizations. Background on geometric group theory or Coxeter groups is valued.

Abstract: A partition λ of the set $\{1, \dots, n\}$ is said to be **noncrossing** if there are no four elements $1 \leq a < b < c < d \leq n$ with a, c in one block of λ, and b, d in a different block. This can be visualized by regarding $1, \dots, n$ as points on a circle arranged in clockwise fashion, and considering the polygons obtained as convex hulls of each block of λ. Then, λ is noncrossing if and only if no two polygons intersect. We are interested in the lattice NC_n, consisting of the non-crossing partitions partially ordered by $\lambda' \leq \lambda$ if λ' refines λ. Besides having various remarkable combinatorial properties, noncrossing partitions are prominent in geometric group theory because the local geometry of a certain classifying space for the braid group on n strands $\mathrm{B}(n)$ can be directly inspected on a geometric incarnation of NC_n, called its order complex.

The aim of this project is to produce tools for visualizing features of the order complex of NC_n. In particular, we are interested in better understanding a certain embedding of its diagonal link into a spherical building of type A, which has been successfully applied in proving that braid groups on up to 7 strands are CAT(0). Other possible directions to explore would focus on generalizations of non-crossing partitions that arise when studying Coxeter systems of types other than A.

Literature and further reading:

- T. Brady, J. McCammond, *Braids, posets and orthoschemes*, Algebraic & Geometric Topology 10 (2010), 2277-2314
- T. Haettel, D. Kielak, P. Schwer, *The 6-strand braid group is CAT(0)*, Geom Dedicata (2016) 182, 263-286
- J. Heller, *Structural properties of non-crossing partitions from algebraic and geometric perspectives*, doctoral dissertation (2018)

9 Visualizing the saddle connection graph

Supervisor: Anja Randecker

Prerequisits: Algebra I, basic graph theory, and some programming experience

Abstract: When considering some polygons in the plane and identifying pairs of edges that are parallel and have the same length, we obtain a translation surface. This surface is naturally equipped with a metric and all the geometric information is actually encoded in an (infinite) graph. For the simplest translation surface (a flat torus), the graph is in fact a very nice object: the Farey graph.

The goal of the project is to write a program that can build approximations of this graph for a certain type of translation surfaces. These approximations can then be investigated and conjectures can be made, especially in comparison to the Farey graph (e.g. can the saddle connection graph be planar?).

Literature and further reading:

- D. Davis, S. Lelièvre. Periodic paths on the pentagon, double pentagon and golden L. `https://arxiv.org/abs/1810.11310`
- J. Boulanger, E. Lanneau, D. Massart. Algebraic intersection in regular polygons. `https://arxiv.org/abs/2110.14235`
- V. Disarlo, A. Randecker, R. Tang. Rigidity of the saddle connection complex. J. Topol. 15, 2022. `https://arxiv.org/abs/1810.00961`